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Schwarz Methods for Obstacle Problems with Convection-Diffusion Operators

YU.A. KUZNETSOV, P. NEITTAANMÄKI AND P. TARVAINEN

ABSTRACT. Multiplicative and additive Schwarz methods are applied to the algebraic problems arising from finite element or finite difference approximations of obstacle problems with convection-diffusion operators. We show that the methods are monotonically convergent in the subset of supersolutions. Moreover, we present a new technique, by which we obtain two-sided approximations for the mesh contact domain. Numerical experiments are included to illustrate the theoretical results.

1. Introduction

In this paper we consider the numerical solution of obstacle problems with convection-diffusion operators by Schwarz-type overlapping domain decomposition methods. We present here some theoretical and experimental results reported earlier in [7].

The motivation for studying the solution of obstacle-type variational inequalities by methods based on the ideas of domain decomposition is natural because of their complementarity property [10]: the solution of the obstacle problem decomposes the domain into two (possibly overlapping) subdomains: one, where the solution equals to a given obstacle function, and the other, where the solution satisfies linear equations. Several papers have been issued about overlapping domain decomposition methods for the obstacle problems, e.g. [2], [6], [9], but in those papers there are no considerations about taking advantage of the complementarity property in order to construct reasonable domain partitions. Moreover, the results of those papers are valid only for self-adjoint operators.

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It is clear that the multiplicative and additive Schwarz methods being applied to the mesh systems arising from finite difference or finite element discretizations of the differential problems are particular cases of block relaxation methods with overlapping groups of unknowns. The convergence of the block relaxation methods without overlapping applied to the algebraic obstacle problems was studied for the self-adjoint case in **[3]** and for the case of M-matrices in **[1]**, for instance.

This paper consists of two parts: In the first, we formulate the problem and give the convergence results for the multiplicative and additive Schwarz methods. We can show that methods are monotonically convergent in the subset of supersolutions. In the second part, we introduce a new technique to obtain two-sided approximations for the mesh contact domain. The technique can be used within the Schwarz methods to decompose the computational domain into overlapping subdomains with linear and obstacle subproblems. We include some numerical experiments to illustrate this technique.

Let Ω be an open bounded polygon in $\mathbb{R}^2$. We consider the following obstacle problem: Find $u \in K$ such that

$$a(u, v-u) \geq (f, v-u) \quad \forall v \in K, \tag{1}$$

where K is a closed, convex subset of $H_0^1(\Omega)$:

$$K = \{v \in H_0^1(\Omega) \mid \; v \geq \psi \text{ a.e. in } \Omega\}, \tag{2}$$

$\psi \in H^2(\Omega)$ is an obstacle function such that $\psi|_{\partial\Omega} \leq 0$, $f \in \mathrm{L}^2(\Omega)$,

$$(f, v) = \int_\Omega fv \; d\Omega, \quad v \in \mathrm{L}^2(\Omega), \tag{3}$$

and the bilinear form corresponds to the convection-diffusion differential operator:

$$a(u, v) = \int_\Omega [a \, \nabla u \cdot \nabla v + (\vec{b} \cdot \nabla u)v + c \; uv] \; d\Omega, \tag{4}$$

where the coefficients $a \geq \text{const} > 0$, $\vec{b} = (b_1, b_2)$ and $c \geq 0$ are piecewise smooth and bounded. Such kind of obstacle problems arise, for instance, in mathematical modelling of the continuous casting process **[10]** and some problems in mathematical economics **[1]**.

By using discretization techniques like finite element method with upwinding **[5]**, we obtain an algebraic problem, which can be represented either in terms of variational inequalities: find $u \in K = \{v \in \mathbb{R}^N \mid \; v \geq \psi\}$ such that

$$(Au, v-u) \geq (f, v-u) \quad \forall v \in K, \tag{5}$$

or in the complementarity form: find $u \in S = \{v \in K \mid \; (Av - f)_j \geq 0, j = 1, \ldots, N\}$ such that

$$(u-\psi)_j \cdot (Au - f)_j = 0, \quad j = 1, \ldots, N, \tag{6}$$

where $f,\ \psi \in \mathbb{R}^N$. The subset S is called the subset of supersolutions to the problem (5) or (6).

We assume that the matrix A is an M-matrix [11], not necessarily symmetric. Under the assumptions made it can be shown that the problem (5) has a unique solution [4].

2. Schwarz overlapping domain decomposition methods

Let Ω_h – the set of mesh nodes, or the mesh domain – be decomposed into m overlapping subdomains $\Omega_h^{(i)}$ such that $\Omega_h = \bigcup_{i=1}^m \Omega_h^{(i)}$, and every mesh node $x_j \in \Omega_h$ belongs to at least one subdomain $\Omega_h^{(i)}$. For given $w \in K$ and $\Omega_h^{(i)}$, $i = 1, \ldots, m$, we define the subset $K_i(w)$ of K by

$$K_i(w) = \{v \in K \mid\ v_j = w_j,\ x_j \notin \Omega_h^{(i)}\}, \tag{7}$$

and the operator $T_i : K \to K$ such that the solution of the subdomain problem: find $z \in K_i(w)$ such that

$$(Az, v - z) \geq (f, v - z) \quad \forall v \in K_i(w), \tag{8}$$

is given by

$$z = T_i(w), \quad i = 1, \ldots, m. \tag{9}$$

The multiplicative Schwarz method can be formulated in terms of the operators (9) in the following way: Let $u^0 \in S$ be given. Then for $k \geq 0$

$$u^{k+1} = T_m(T_{m-1}(\cdots(T_1(u^k))\cdots)). \tag{10}$$

Similarly, we can formulate the additive Schwarz method: Let $u^0 \in S$ be given, and choose parameters ω_i, $i = 1, \ldots, m$, such that

$$\sum_{i=1}^m \omega_i = 1. \tag{11}$$

Then, for $k \geq 0$,

$$u^{k+1} = \sum_{i=1}^m \omega_i T_i(u^k). \tag{12}$$

In [7] we have shown the following convergence results:

THEOREM 1. *The multiplicative and additive Schwarz methods are monotonically convergent in S.*

Here, we mean by the monotonic convergence in the subset S, that for all $u^0 \in S$ the algorithms generate a monotonically decreasing sequence $\{u^k\}$, $u^k \in S$, which converges to the solution of the problem (5).

3. Two-sided approximations for the mesh contact domain

The aim of this section is to construct two-sided approximations within the domain decomposition methods of Section 2 for the mesh contact domain G_h,

$$G_h = \{x_j \in \Omega_h |\ u_j^* = \psi_j\}, \tag{13}$$

where u^* is the solution of the problem (5).

Let a monotonically decreasing sequence $\{u^k\}$ be given such that $u^k \underset{k\to\infty}{\longrightarrow} u^*$ in S. If we define the mesh domains G_h^k, $k \geq 0$, by

$$G_h^k = \{x_j \in \Omega_h |\ u_j^k = \psi_j\}, \tag{14}$$

then it follows immediately that

$$G_h^k \subseteq G_h^{k+1}, \ k \geq 0. \tag{15}$$

To obtain outer approximations within the domain decomposition procedures, we have to solve some additional linear problems of the following form: Assume that G_h^k is given for some $k \geq 0$. Find $w^k \in \mathbb{R}^N$ such that

$$\begin{cases} w_j^k = \psi_j, & \text{if } x_j \in G_h^k, \\ (Aw^k - f)_j = 0, & \text{otherwise.} \end{cases} \tag{16}$$

It can be shown [7] that

$$w^k \leq u^* \leq u^k, \tag{17}$$

and if we define the mesh domains $\widehat{G}_h^k$, $k \geq 0$, by

$$\widehat{G}_h^k = \{x_j \in \Omega_h |\ w_j^k \leq \psi_j\}, \tag{18}$$

we can state the following conclusion [7]:

THEOREM 2. *Under the assumptions made*

$$G_h^k \subseteq G_h \subseteq \widehat{G}_h^k, \quad k = 0, 1, \ldots. \tag{19}$$

As a consequence, we notice that in each iteration step k the mesh domain can be divided into three subdomains with respect to the two-sided approximations: the contact subdomain G_h^k, the linear subdomain $\Omega_h \setminus \widehat{G}_h^k$ and the problematic subdomain $\widehat{G}_h^k \setminus G_h^k$. It follows from the above theory and the complementarity property, that the solution u^* satisfies:

$$\begin{aligned} u_j^* &= \psi_j, && \text{if } x_j \in G_h^k \ \text{(the contact subdomain)}, \\ (Au^* - f)_j &= 0, && \text{if } x_j \in \Omega_h \setminus \widehat{G}_h^k \ \text{(the linear subdomain)}, \end{aligned} \tag{20}$$

and only in $\widehat{G}_h^k \setminus G_h^k$ do we not know which condition of (20) the solution satisfies. Naturally, this information can be used to construct reasonable domain partitions for the Schwarz methods. Furthermore, these partitions can be modified within the domain decomposition procedure.

4. Numerical experiments

This section consists of two examples: in the first example we illustrate by means of figures the technique of two-sided approximations, and in the second example we compare execution times of an iterative procedure based on our approach and a traditional solution algorithm, the SOR method with projection.

Let Ω be the unit square and consider the obstacle problem (1) with the following data: $f \equiv -6$, $\psi(x) = -\text{dist}(x, \partial\Omega)$, $a \equiv 1$, $\vec{b} \equiv (5,5)$, $c \equiv 0$, where the function $\text{dist}(\cdot, \partial\Omega)$ means the distance from the boundary of Ω. Thus, we consider the obstacle problem with the nonself-adjoint operator. We have solved the problem by a multiplicative procedure, and Figure 1 demonstrates the behaviour of the algorithm. We have denoted by dots $(\cdot)$ the contact subdomains, by bullets $(\bullet)$ the problematic subdomains, and white regions denote the linear subdomains of each iteration step. It can be seen that the problematic subdomains are efficiently reduced by the two-sided approximations. Hence, Schwarz methods can be applied such that in the main part of the domain linear subdomain solvers are used.

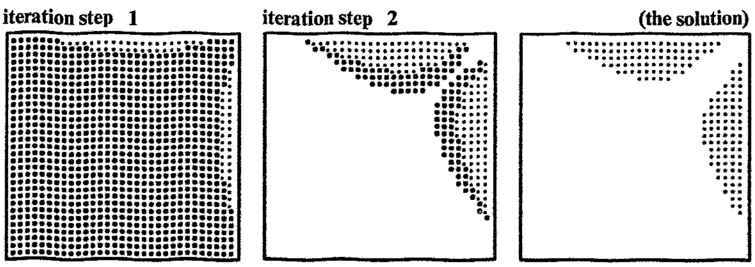

FIGURE 1. TWO-SIDED APPROXIMATIONS.

In the second example we consider the obstacle problem (1) in the unit square with the data: $a \equiv 1$, $\vec{b} \equiv (0,0)$, $c \equiv 0$, $\psi \equiv 0$, and

$$f(x) = \begin{cases} -2, & x \in (3/8, 5/8) \times (3/8, 5/8), \\ 1, & \text{otherwise.} \end{cases} \tag{21}$$

We have implemented the multiplicative Schwarz procedure, which makes use of information from the two-sided approximations in such a way that we decompose the computational domain into rectangular subdomains such that the problematic subdomain of each iteration step is included in one of the subdomains, and others are linear subdomains. Then we apply the multiplicative Schwarz method such that in the linear subdomains we use fast direct solvers based on the fast Fourier transform, and in the nonlinear subdomains we apply the line Gauss-Seidel method. The additional linear problems needed to construct two-sided approximations are solved by the fictitious domain method.

In Table 1 we see the execution times of the projected SOR-method [5] with the acceleration parameter $\omega = 1.7$ (PSOR) and the algorithm described above

(DDM). We notice that the domain decomposition algorithm based on the two-sided approximations works faster in all cases above. We emphasize, that this algorithm is only the simplest possible implementation, without any acceleration.

Execution times (sec.) – HP 9000/735		
$n \times n$	PSOR	DDM
15×15	0.05	0.05
31×31	0.33	0.17
63×63	3.37	1.53
127×127	42.35	19.23

TABLE 1. COMPARISON OF TWO SOLUTION ALGORITHMS.

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INSTITUTE OF NUMERICAL MATHEMATICS, RUSSIAN ACADEMY OF SCIENCES, LENINSKI PROSPECT 32-A, MOSCOW 117334, RUSSIA

E-mail address: labnumat@node.ias.msk.su

LABORATORY OF SCIENTIFIC COMPUTING, DEPARTMENT OF MATHEMATICS, UNIVERSITY OF JYVÄSKYLÄ, P.O. BOX 35, FIN-40351, JYVÄSKYLÄ, FINLAND

E-mail address: neittaanmaki@finjyu.bitnet

LABORATORY OF SCIENTIFIC COMPUTING, DEPARTMENT OF MATHEMATICS, UNIVERSITY OF JYVÄSKYLÄ, P.O. BOX 35, FIN-40351, JYVÄSKYLÄ, FINLAND

E-mail address: pht@math.jyu.fi