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ROBUST METHODS FOR HIGHLY NONSYMMETRIC PROBLEMS

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ABSTRACT. We consider the solution of linear systems arising from discretizations of problems which are highly convection dominated in some subregions and diffusion dominated in others. For such problems, *robustness*, meaning uniformity in the problem parameters for consistent discretizations, rather than asymptotic optimality in h, is of paramount importance. This report presents a particularly simple method which is provably robust for highly nonsymmetric and anisotropic problems.

1. Introduction

A large number of interesting problems in scientific applications involve operators that are both highly nonsymmetric and highly anisotropic. There continues to be a great deal of research into adapting optimal order solution methods from symmetric positive definite problems to these interesting ones so that their "robustness" is incrementally improved (see other articles in this proceedings). The authors have been attacking these problems from the opposite point of view, namely: developing optimally robust methods for the problems' singular limits and then attempting to decrease their serial or parallel complexity for (simpler) problems. This report gives an overview of some results to date, the basic algorithms developed and one interesting computational example.

Semiconductor problems and fluid dynamics problems lead to (linearized) convection diffusion equations which are highly convection dominated in some subregions and diffusion dominated in others. The involved velocity field typically has stagnation points and closed loops which do not permit "streamwise" solution strategies. With these difficulties in mind, consider the problem: seek $u(x,y)$ satisfying:

$$-\nabla \cdot (k(x,y)\nabla u) + \mathbf{b}(x,y) \cdot \nabla u + g(x,y)u = f(x,y) \quad \text{in } \Omega, \tag{1.1}$$

subject to $u = 0$ at $\partial\Omega$. The coefficients are assumed to satisfy:

$$0 \le k_{\min} \le k(x,y) \le k_{\max} \le 1, \; g - \frac{1}{2}\nabla \cdot \mathbf{b} \ge g_{\min} > 0, \tag{1.2}$$

and Ω is a planar polygonal domain. Accurate approximation of (1.1) usually requires adaptivity [14,15], nonlinear discretizations [9] or subgridscale modelling [4],

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and in all these approaches, (re)assembly can often dominate linear system solution costs [16]. Therefore, we consider an elementwise data-parallel finite element solution procedure from [11,12]. Massive parallelism [8, 17, 18], point adaptivity [7,8] and nonlinear discretizations [4,6] are handled trivially by the present procedure which uses *elements* as the logical units. The new results presented herein state that the method introduced in [11, 12] is *optimally robust* for (1.1), (1.2). Specifically, the solution procedure converges uniformly in $k(x,y)$ (the cell Reynolds or Péclet number), even for centered discretizations! In Theorem 1.1 the parameter δ interpolates between the usual (centered) Galerkin method ($\delta = 0$) and the streamline diffusion finite element method ($\delta = h$).

Let us postpone the presentation of the algorithm until Section 2. The following theorem, proven in [7] shows that the Parallel FEM algorithm converges uniformly in both the anisotropy ($k_{\max}/k_{\min}$) and the Péclet number ($k_{\max}$). The technical condition upon $k(x,y)$ states that $k(x,y)$ should not change from near zero to near 1 *inside* of elements, i.e., such drastic material discontinuities should be followed by meshlines. This is certainly good computational practice but we do not know if it is necessary or an artifact of our method of proof.

Theorem 1.1. *Assume that the finite element triangulation satisfies the usual minimum angle condition, and that in addition inside each triangle e either: $0 \le k_{\min,e} \le k(x,y) \le k_{\max,e} \le C_1 h$ or $C_2 h \le k_{\min,e} \le k(x,y) \le k_{\max,e} \le 1$. Let T denote the iteration operator of the Parallel FEM algorithm, $|\cdot|_2$ be the Euclidean matrix norm and let ρ be the algorithm acceleration parameter. Then, provided ρ is scaled as $\rho = \rho_0 h$,*

$$\sup_{\substack{0 \le k_{\min} \le k(x,y) \le k_{\max} \le 1 \\ 0 \le k_{\max}/k_{\min} \le \infty, 0 \le \delta \le h}} |T|_2 \le 1 - ch.$$

□

We emphasize that this holds even for centered discretizations so that robustness is not an artifact of a special discretization which implicitly increases the size of the matrices' symmetric part. Further, the basic methods and results have recently been extended to finite difference and finite volume discretizations in [1,2]. The parallel FEM algorithm also is attractive as a combination pre- and post-conditioner. This is described in Section 2 and developed fully in [5].

2. The Algorithm

Let $\Pi^h(\Omega)$ denote an edge-to-edge triangulation of the polygonal domain Ω, in our tests usually generated self-adaptively as in [14, 15]. Without loss of generalization, let S^h denote the span of the 3−node nonconforming linear element; all the results below extend directly to higher order elements as no M-matrix structure is used. Now, focus on a one parameter family of discretization methods ($k_{\max} \le O(h)$ and $\delta = O(h)$ gives streamline diffusion; $0 \le k(x,y) \le 1$ and $\delta = 0$ gives the usual Galerkin method), given by: seek $u^h \in S^h$ satisfying

$$\sum_{e \in \Pi^h(\Omega)} [a_e(u^h, v) - (f, v + \delta \mathbf{b} \cdot \nabla v)_e] = 0, \text{ for all } v \in S^h, \tag{2.1}$$

where $(f,v)_e = \int_e fv dx$, and $a_e(u,v) := \int_e \{k\nabla u \cdot \nabla v + \delta \mathbf{b} \cdot \nabla u \mathbf{b} \cdot \nabla v + (g - \frac{1}{2}\nabla \cdot (\mathbf{b} + \delta g \mathbf{b})uv + \frac{1}{2}(1 - \delta g)[\mathbf{b} \cdot \nabla uv - \mathbf{b} \cdot \nabla vu]\} dx$ is the usual elemental bilinear form, explicitly skew-symmetrized. Let N_j denote the nodes in the mesh (situated at the mid-edges) and ϕ_j the associated nodal basis functions (which have support on two triangles only). Color the triangles in $\Pi^h(\Omega)$ with two colors (Red and Black, traditionally). If the graph-degree of each interior vertex of $\Pi^h(\Omega)$ is even it is trivially possible to separate [2,3,10,11] the triangles (no two same-color triangles share an edge), otherwise the coloring algorithm presented in [3] is used; see [3] for details. We focus on the case of even degree interior vertices for compactness of presentation. Split the stiffness matrix A and right hand side vector f into $A = A_R + A_B$ and $f = f_R + f_B$, as follows:

$$[A_{R/B}]_{i,j} := \sum_{e \text{Red/Black}} a_e(\phi_j, \phi_i) \ , \quad [f_{R/B}]_j = \sum_{e \text{Red/Black}} (f, \phi_j + \delta \mathbf{b} \cdot \nabla \phi_j)_e.$$

One key observation is that the nodes can be ordered such that the 3 nodes in each Red (resp., Black) triangle are consecutively numbered causing A_R (resp., A_B) to be block diagonal with blocks of the 3×3 element matrices corresponding to Red (resp., Black) triangles. Let these two orderings be called the Red (resp., Black) ordering. Reordering corresponds to a local communication on a mesh connected array of processors [8, 17, 18].

Algorithm: Parallel FEM. *Given c_B^0 and $\rho > 0$ compute until satisfied*

I. Calculate $d_B := (\rho I + A_B)^{-1}[f_B - (A_B - \rho I)c_B^k]$

II. reorder $c_R^k \leftarrow d_B$

III. Calculate $d_R := (\rho I + A_R)^{-1}[f_R - (A_R - \rho I)c_R^k]$

IV. reorder $c_B^{k+1} \leftarrow d_R$

Upon convergence, set in consistent node ordering, $c = c_B + c_R$, then $Ac = f$. Note that as each $(\rho I \pm A_{R/B})$ is block 3×3 diagonal each step is embarassingly parallel. A good choice of ρ is $\rho = (|\lambda|_{\max}(A_e) \cdot |\lambda|_{\min}(A_e))^{1/2}$. This algorithm is precisely the D'Yakunov operator splitting method [13], with the new splitting of [11, 12].

The above algorithm is of the domain decomposition type in that each block in $(\rho I \pm A_{R/B})$ is associated with a subdomain (i.e., a single Red or Black element). There are neither overlap of subdomains nor interface conditions. Further, in cost per iteration, ability to parallelize and case of implementation it is more akin to simple relaxation methods. The next result follows immediately from the Algorithm.

Proposition 2.1. *The per iteration complexity of the Parallel FEM algorithm is as follows. (a) Computational complexity: per triangle one 3×3 matrix vector multiply, one 3×3 system solve and one 3-vector addition. (b) Communication complexity: two local communications on the physical mesh.* □

Discretization (2.1) leads to the usual linear system $Ac = f$ for the nodal degrees of freedom. The previous solution algorithm reduces local errors very quickly so it is natural to study it both in combination with Krylov subspace methods and as a smoother for multilevel methods, especially on serial machines. Since (2.1) may yield a highly non-normal system, reduction of the spectral radius of A is not sufficient to reduce $\text{cond}_2(A) = |A|_2|A^{-1}|_2$. This reduction is accomplished through a judicious combination of preconditioning with $(\rho I + A_R)^{-1}$ and postconditioning with $(\rho I + A_B)^{-1}$, leading to the system $Kd = g$, where

$$(3.1) \quad K := (\rho I + A_R)^{-1} A(\rho I + A_B)^{-1},\ d := (\rho I + A_B)c,\ g := (\rho I + A_R)^{-1} f.$$

The following basic result is given in [5] along with extensive experiments.

Theorem 3.1. *Under the assumption of Theorem 1.1, the pre- and postconditioned system (3.1) has condition number $O(h^{-1})$ uniformly in $k(x,y), k_{\max}/k_{\min}$ and δ:*

$$\sup_{\substack{0 \le k \le 1 \\ 0 \le k_{\max}/k_{\min} \le \infty \\ 0 \le \delta \le O(h)}} (|K|_2 |K^{-1}|_2) \le Ch^{-1}.$$

□

To date we have no experience using the parallel FEM algorithm as a smoother in a multilevel method.

4. An Illustrative Example

We have attempted to construct a "model" problem which captures some interesting features of internal flow problems. To this end, let $\Omega := (-1,1)^2$, $r^2 := x^2 + y^2$, and define $k(x,y) = 1$ if $r \ge 1$ while $k(x,y) = k_f$ for $r < 1$. Let the velocity field $\mathbf{b}$ be given by $\mathbf{b}(x,y) := [-y\phi(r), x\phi(r)]^T$ with ϕ a smooth function satisfying $\phi(r) \equiv 0$ for $r \ge 1$, here chosen to be $\phi(r) := 1 - r^2$ for $r \le 1$. Note that div $\mathbf{b}(x,y) \equiv 0$. We chose $g(x,y) \equiv 2$ and $f(x,y) = 1$ if $r \le 1/2$ and 0 if $r > 1/2$. The true solution is a rotational pulse $u \cong 1/2$ if $r < 1/2$ and $u \cong 0$ if $r > 1/2$ with an $O(\sqrt{k_f})$ transition layer, thus at least an asymptotic solution was available for comparisons. Note that this problem has: closed loops and stagnation points in the convection field, characteristic layers, neither inflow nor outflow boundaries and conductivities k varying from $O(1)$ to quite small. In [7], we have tracked the numbers of iterations as $k_f \to 0$, keeping h fixed, giving the result predicted by Theorem 1.1. Here, we present a slightly more interesting test, which also indirectly verifies robustness. For elements with polynomial degree r the range $O(h) \le k_f \le O(h^{r+1})$ is the critical one. Using nonconformity linear elements we fix $k_f = h^2/10$ and solve the linear system $Ac = f$ with varying meshwidths, taking $\rho = h$. Theorem 1.1 predicts $O(h^{-1})$ iterations exactly as observed in Table 1. If the method were *not* robust, for example if the number of iterations varied as $O(k_f^{-\beta} h^{-1})$ then with $k_f = h^2, O(h^{-(1+2\beta)})$ iterations would be observed. Thus Table 1 verifies both robustness and the complexity bound of $O(h^{-1})$ iterations.

h^{-1}	Centered Galerkin $\delta = 0$ case	Streamline Upwind $\delta = h$ case
64	167	111
32	104	72
16	60	46
8	30	23

Table 1. Number of Iterations of the PFEM Algorithm.

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