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Domain Decomposition Via the Sinc-Galerkin Method for Second Order Differential Equations

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ABSTRACT. The solution of elliptic problems using domain decomposition techniques has been of great interest in recent years. Sinc basis functions form a desirable basis to use in approaching domain decomposition for elliptic problems because they are especially well-suited for problems with boundary singularities, and both the Sinc-Galerkin and sinc-collocation methods converge exponentially. This paper deals with overlapping and patching domain decomposition used in conjunction with the Sinc-Galerkin method for both the two-point boundary value problem and Poisson's equation on a rectangle.

1. Introduction

Sinc methods for differential equations were originally introduced in [**9**]. Since then they have become increasingly popular, and have been well-studied. Both the Sinc-Galerkin and the sinc-collocation methods converge exponentially, even in the presence of boundary singularities, as shown in [**1**], [**6**], and [**9**]. Both methods perform equally well in domain decomposition for the two-point boundary value problem, as seen in [**8**]. For Poisson's equation, the Sinc-Galerkin and sinc-collocation methods are identical. For this reason only the Sinc-Galerkin results will be presented.

Although elliptic problems are generally approached with patching domain decomposition methods, certain characteristics of the sinc basis functions make overlapping domain decomposition desirable. Because of this, both methods have been explored. See [**2**] for further details on these methods. Numerical results are presented for both overlapping and non-overlapping methods. These results exhibit nearly identical errors achieved with each method. A brief introduction to the Sinc-Galerkin method is given in §2. Domain decomposition for the two-point boundary value problem via the Sinc-Galerkin method is presented in §3. Similarly, domain decomposition for Poisson's equation via the Sinc-Galerkin method is discussed in

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§4. Both sections §3 and §4 include one example designed to highlight the Sinc-Galerkin method's ability to deal with boundary singularities. In each case, results for both the patching and overlapping methods are presented.

2. The Sinc-Galerkin Method

The two-point boundary value problem on the finite interval (a, b) is given by

$$\begin{aligned} \mathcal{L}u(x) &\equiv -u''(x) + p(x)u'(x) + q(x)u(x) \\ &= f(x) \;, \quad a < x < b \\ u(a) &= u(b) = 0 \;. \end{aligned} \tag{2.1}$$

The classical Sinc-Galerkin method for problems of this type is discussed in detail in [3–7],[9], and [12].

The sinc basis functions used in solving (2.1) are given by

$$S_j(x) \equiv S(j,h) \circ \phi(x) \equiv \text{sinc}\left(\frac{\phi(x) - jh}{h}\right)$$

where $h > 0$, j is an integer, $x \in (a, b)$, and

$$\text{sinc}(y) = \begin{cases} \frac{\sin(\pi y)}{\pi y} \,, & y \neq 0 \\ 1 \,, & y = 0 \end{cases} .$$

The conformal map

$$\phi(z) = \ln\left(\frac{z-a}{b-z}\right)$$

is used to define the basis functions on the finite interval (a, b). The sinc nodes x_k are chosen so that

$$x_k \equiv \psi(kh) = \phi^{-1}(kh) = \frac{a + be^{kh}}{e^{kh} + 1} \,.$$

The approximate solution is then given by

$$u_m(x) = \sum_{k=-M}^{N} u_k S_k(x) \;, \; m = M + N + 1 \quad . \tag{2.2}$$

Orthogonalization of the residual against each basis function

$$(\mathcal{L}u - f, S_j) = 0 \;, \; -M \leq j \leq N$$

uses the weighted inner product

$$(f, g) = \int_a^b f(x)g(x)(\phi'(x))^{-1/2} dx \,.$$

Integration by parts is used to remove all derivatives from u, and applying the sinc quadrature rule (found in [6] or [11]) yields the discrete Sinc-Galerkin system. The following theorem for the convergence of this method in the case $p(x) \equiv 0$ is proven in [9].

THEOREM 2.1. *Let the numbers* u_k $(k = -M, \ldots, N)$ *be determined from the discrete Sinc-Galerkin system, and let* $u_m(x)$ *be as defined in (2.2). Then under appropriate assumptions (see* [9]*), with* $h = \left(\pi/\sqrt{2M}\right)$*, and* $N = M$*, the estimate*

$$\|u_m - u\|_\infty \le CM^2 e^{-\sqrt{\pi d \alpha M}}$$

holds where u *is the solution of* (2.1) *with* $p(x) \equiv 0$*. The parameters* d *and* α *depend on the analyticity and rate of decay of the solution* u*.*

3. One-Dimensional Domain Decomposition

Both the overlapping and the patching methods of domain decomposition introduce matching conditions at the interface of the domains. This requires the addition of a boundary basis function to the approximation on each subdomain. In either case, appropriate fourth degree polynomial boundary functions are given in [8]. Similar situations arise in the solution of boundary value problems with nonhomogeneous Dirichlet boundary conditions. See [8] and [10] for more information.

The following examples have been chosen to show the rapid convergence achieved by using the Sinc-Galerkin method in conjunction with both patching and overlapping domain decomposition methods.

EXAMPLE 3.1. Consider

$$\begin{aligned} -u''(x) + u'(x) + u(x) &= f(x)\ , \ -1 < x < 4 \\ u(-1) = u(4) &= 0\ . \end{aligned}$$

In this example, $f(x)$ was chosen so that the true solution is given by

$$u(x) = \frac{\sqrt{x+1}(x-4)^2}{16}\ .$$

Split the domain $\Omega = [-1, 4]$ into two subdomains given by either $\Omega^1 = [-1, 1]$ and $\Omega^2 = [.9, 4]$ for the overlapping method, or $\Omega^1 = [-1, 1]$ and $\Omega^2 = [1, 4]$ for the patching method. The approximate solutions are shown in Fig. 1. The error results are given in Table 1. Here, the sinc error is given by

$$\|E_S\| = \max_{x \in S} |u(x) - u_m(x)|\ ,$$

where S is the set of all grid points x_k generated from the Sinc-Galerkin method in both subdomains. The uniform error is found by letting

$$\|E_U\| = \max_{y \in U} |u(y) - u_m(y)|$$

where $U = \{y_j = -1 + 5j/100 : 0 \le j \le 100\}$ is a uniform grid of mesh size 0.05.

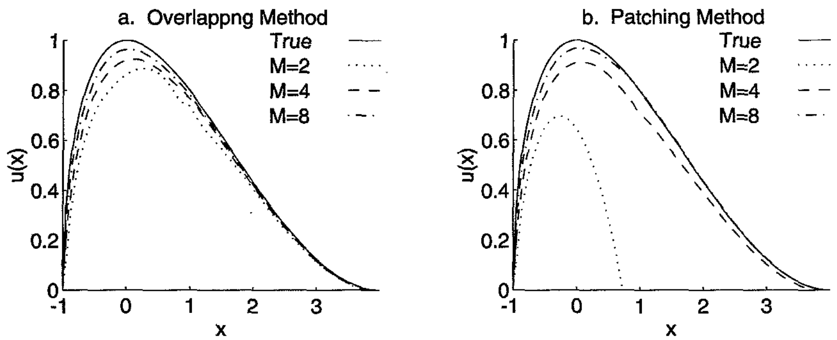

FIGURE 1. True vs. Approximate Solutions for Example 3.1. Fig. 1a shows the solution from the overlapping method. Fig. 1b shows the solution from the patching method.

TABLE 1. Error in overlapping and patching methods for Example 3.1.

	Overlapping		Patching	
$M = N$	$\|E_U\|$	$\|E_S\|$	$\|E_U\|$	$\|E_S\|$
2	$2.3394e-01$	$2.2852e-01$	$1.2803e+00$	$1.2811e+00$
4	$1.4258e-01$	$1.4880e-01$	$1.4278e-01$	$1.4888e-01$
8	$6.6033e-02$	$6.8459e-02$	$6.5930e-02$	$6.8455e-02$
16	$1.9821e-02$	$2.0531e-02$	$1.9811e-02$	$2.0531e-02$
32	$3.3660e-03$	$3.4897e-03$	$3.3652e-03$	$3.4897e-03$
64	$2.6173e-04$	$2.7141e-04$	$2.6159e-04$	$2.7141e-04$

4. Domain Decomposition for Poisson's Equation

The basis used to solve Poisson's equation is a product of sinc basis functions in the x and y directions. Let

$$S_{jk}(x,y) = S_j(x)S_k(y)\ ,\ -M_x \le j \le N_x\ ,\ -M_y \le k \le N_y.$$

The approximate solution takes the form

$$u_{m_x,m_y}(x,y) = \sum_{j=-M_x}^{N_x} \sum_{k=-M_y}^{N_y} u_{jk} S_{jk}(x,y)$$

where $m_x = M_x + N_x + 1$ and $m_y = M_y + N_y + 1$. As in the one-dimensional case, one orthogonalizes the residual against each basis function and perform integration by parts to reach the discrete Sinc-Galerkin system. Again, both the patching and overlapping methods for domain decomposition applied to Poisson's equation require the addition of boundary basis functions. In the example given, the domain is split only in the x direction, so there is no need for boundary basis functions in the y variable. The extra basis functions in the x variable are the same ones used in §3.

EXAMPLE 4.1. Consider the problem

$$-\Delta u(x,y) = f(x,y)\ ,\ (x,y) \in \Omega = (-1,4) \times (0,1)$$

$$u(x,y) = 0\ ,\ (x,y) \in \partial\Omega\ ,$$

where $f(x,y)$ is chosen so that the true solution is given by

$$u(x,y) = \sqrt{y}\sqrt{x+1}(x-4)^2(1-y)^2\ .$$

Here the domain $\Omega = [-1,4] \times [0,1]$ is split into the two overlapping subdomains $\Omega^1 = [-1,1] \times [0,1]$ and $\Omega^2 = [.9,4] \times [0,1]$ or the two non-overlappinng subdomains $\Omega^1 = [-1,1] \times [0,1]$ and $\Omega^2 = [1,4] \times [0,1]$. In this example $M \equiv M_x = M_y$ and $N \equiv N_x = N_y$. A graphical representation of the results is found in Fig. 2, while the numerical errors are reported in Table 2. The error columns are analogous to those reported in Example 3.1. In spite of the steep solution along the x and y axes the convergence is rapid.

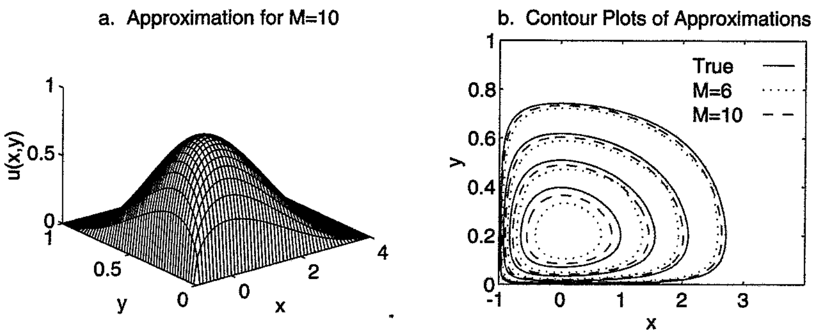

FIGURE 2. Patching results for Example 4.1. Fig. 2a shows the solution computed by the patching method for $M = 10$. Fig. 2b shows contour plots of the solutions computed by the patching method versus the true solution. The contour levels are .2, .4, .6, and .8.

TABLE 2. Error in overlapping and patching methods for Example 4.1.

	Overlapping		Patching	
$M = N$	$\|E_U\|$	$\|E_S\|$	$\|E_U\|$	$\|E_S\|$
2	$3.7729e-01$	$3.5832e-01$	$7.5272e-01$	$6.4327e-01$
4	$2.3657e-01$	$2.4320e-01$	$2.3659e-01$	$2.3423e-01$
6	$1.5294e-01$	$1.5612e-01$	$1.5282e-01$	$1.5610e-01$
8	$1.0619e-01$	$1.0813e-01$	$1.0619e-01$	$1.0813e-01$
10	$7.6184e-02$	$7.7340e-02$	$7.6181e-02$	$7.7340e-02$

5. Conclusion

The performance of the Sinc-Galerkin method in conjunction with both patching and overlapping domain decomposition methods is quite good in simple cases, as seen in Examples 3.1 and 4.1. In these test problems, the resulting systems were solved directly using MATLAB. The large size of the matrices in the two-dimensional case became prohibitive after $M = 10$. The results of Example 3.1 have been confirmed using FORTRAN on a CRAY Y-MP. Work is in progress to convert the codes for Poisson's equation to FORTRAN so that larger systems may be run for Example 4.1. An iterative method for solving these problems would be a logical next step.

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