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PRECONDITIONING CELL-CENTERED FINITE DIFFERENCE EQUATIONS ON GRIDS WITH LOCAL REFINEMENT

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ABSTRACT. We consider cell-centered finite difference discretizations with local refinement for nonsymmetric boundary value problems. Preconditioners with mesh independent convergence properties for corresponding matrices are constructed. The method is illustrated with numerical experiments.

1. INTRODUCTION

This paper is devoted to construction of preconditioners of Bramble-Ewing-Pasciak-Schatz (**BEPS**) type [3] for solving nonsymmetric boundary value problems discretized by finite difference schemes on cell-centered grids with local refinement. Approximation properties of cell-centered finite difference schemes are investigated in [5], [11] for the symmetric problems, and in [7] for the nonsymmetric ones (see also [4] and [9]). The theory for two-level preconditioners is developed in [3],[6] and [8]. We extend the results obtained in [6] for nonsymmetric matrices without loss of optimality of the preconditioners, i.e., convergence rate is mesh independent.

We consider the following convection-diffusion boundary value problem:
find a function $u(x)$ which satisfies the following differential equation and boundary condition:

$$\left\{ \begin{array}{rcll} \mathbf{div}(-a(x)\nabla u(x) + \underline{b}(x)u(x)) & = & f(x) & \text{in } \Omega \\ u(x) & = & 0 & \text{on } \Gamma \end{array} \right. \tag{1.1}$$

where $\Omega \subset R^2$ is a bounded domain and $\Gamma = \partial\Omega$. The coefficients $a(x)$ and $\underline{b}(x) = (b_1(x), b_2(x))$ are supposed to fulfill for some constants a_0 and β_0, β_1 the conditions

(i) $a(x) \geq a_0 > 0$, $a(x) \in W^1_\infty(\Omega)$,

(ii) $| b_i(x) | \leq \beta_1$, $b_i \in W^1_\infty(\Omega)$,

and in order to obtain coercivity it is sufficient that

(iii) $(\nabla.\underline{b}(x)) \geq \beta_0 > 0$.

The function $f(x)$ is given in Ω and $f(x) \in L^2(\Omega)$.

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2. Description of the preconditioners

Suppose the domain Ω is divided in two parts Ω_1 and Ω_2, $\Omega = \Omega_1 \cup \Omega_2$, $\Omega_1^o \cap \Omega_2^o = \phi$, where Ω_1 is the nonrefined and Ω_2 is the refined subdomain. Let us consider the composite grid ω (see [7] for a detailed description) divided in the same way, i.e., $\omega = \omega_1 \cup \omega_2$, $\omega_1 \subset \Omega_1$, $\omega_2 \subset \Omega_2$. The nodes of ω can be partitioned into three groups. The first group consists of the nodes in the refined subdomain Ω_2, the second one consists of nodes in Ω_1 next to the interface boundary, denoted by γ, and in the last are the rest of the nodes from Ω_1. Correspondingly our finite difference matrix A [7] admits a three-by-three block structure, i.e., we have

$$A = \begin{bmatrix} A_{11} & A_{12} & 0 \\ A_{21} & A_{22} & A_{23} \\ 0 & A_{32} & A_{33} \end{bmatrix} \begin{array}{l} \}\omega_2 \\ \}\gamma \\ \}\omega_1 \backslash \gamma \end{array} .$$

We need also the s.p.d. coarse-grid matrix $\tilde{C}$, which is a approximation of matrix $\tilde{A}$ derived from the nonrefined finite difference scheme. We partition $\tilde{C}$ in the same manner as A into a three by three block structure on the nonrefined mesh $\tilde{\omega}$

$$\tilde{C} = \begin{bmatrix} \tilde{C}_{11} & \tilde{C}_{12} & 0 \\ \tilde{C}_{21} & \tilde{C}_{22} & \tilde{C}_{23} \\ 0 & \tilde{C}_{32} & \tilde{C}_{33} \end{bmatrix} \begin{array}{l} \}\tilde{\omega}_2 \\ \}\gamma \\ \}\omega_1 \backslash \gamma \end{array} ,$$

where $\tilde{\omega}_2 = \tilde{\omega} \cap \Omega_2$. Then the preconditioner (**BEPS**) is constructed as follows [3], [6].

Given a vector $v = [v_1\ v_2\ v_3]^T$, we perform the following steps
(i) solve in Ω_2

$$A_{11} y_1^F = v_1;$$

(ii) compute the defect

$$d = v - A \begin{bmatrix} y_1^F \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ v_2 - A_{21} y_1^F \\ v_3 \end{bmatrix} \}\omega_2 ;$$

(iii) approximate the coarse-grid correction

$$\tilde{C}\tilde{y} = \begin{bmatrix} 0 \\ v_2 - A_{21} y_1^F \\ v_3 \end{bmatrix} \}\tilde{\omega}_2 ;$$

(iv) find y_1^H in ω_2 such that

$$A_{11} y_1^H + A_{12} \tilde{y}_2 = 0 .$$

Then

$$y = B^{-1} v = \begin{bmatrix} y_1^F + y_1^H \\ \tilde{y}_2 \\ \tilde{y}_3 \end{bmatrix} .$$

In matrix notation

$$B = \begin{bmatrix} A_{11} & 0 \\ \begin{bmatrix} A_{21} \\ 0 \end{bmatrix} & \tilde{S}_c \end{bmatrix} \begin{bmatrix} I & \begin{bmatrix} A_{11}^{-1} A_{12} & 0 \end{bmatrix} \\ 0 & I \end{bmatrix} ,$$

where

$$\tilde{S}_c = \begin{bmatrix} \tilde{A}_{22}^c - \tilde{A}_{21}^c \tilde{A}_{11}^{c(-1)} \tilde{A}_{12}^c & \tilde{A}_{23}^c \\ \tilde{A}_{32}^c & \tilde{A}_{33}^c \end{bmatrix}$$

is the Schur complement of the coarse-grid matrix $\tilde{C}$.

Using the results in [7], we can easily prove the following auxiliary result.

Lemma 2.1. *There exists two positive constants $\hat{\gamma}_1$ and $\hat{\gamma}_2$ independent of h such that*

$$\hat{\gamma}_1 v_2^T \tilde{A} v_2 \leq v^T A v$$

$$v^T A y \leq \hat{\gamma}_2 (v_2^T \tilde{A} v_2)^{\frac{1}{2}} (y_2^T \tilde{A} y_2)^{\frac{1}{2}} ,$$

where $v = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \begin{matrix} \}\omega\backslash\tilde{\omega} \\ \}\omega \end{matrix}$.

Our task is to estimate the eigenvalues of the preconditioned matrix $B^{-1}A$. We have

$$B^{-1}A = \begin{bmatrix} I & * \\ 0 & (\tilde{S}_c)^{-1} S_2 \end{bmatrix}$$

and from

$$\lambda(B^{-1}A) = (1, \lambda((\tilde{S}_c)^{-1} S_2))$$

is clear that we have to evaluate the spectrum of $((\tilde{S}_c)^{-1} S_2)$.

We consider two cases for the matrices C and $\tilde{C}$, respectively:

1. $\tilde{C} = \frac{\tilde{A}+\tilde{A}^T}{2}$, $C = \frac{A+A^T}{2}$,

i.e., the symmetric part of $\tilde{A}$ and A, and

2. $\tilde{C} = \tilde{A}^{(1)}$, $C = A^{(1)}$,

i.e., the part arising from the approximation of the diffusion term.

For the first case we have $v^T \tilde{A} v = v^T \tilde{C} v$. Using the results in [7], we get for the second one

$$v_2^T \tilde{A}^{(1)} v_2 \leq v_2^T \tilde{A} v_2 \leq E v_2^T \tilde{A}^{(1)} v_2 .$$

Therefore, there exist constants γ_1 and γ_2 such that the following inequalities hold:

$$\gamma_1 v_2^T \tilde{C} v_2 \leq v^T A v \tag{2.2}$$

$$v^T A y \leq \gamma_2 (v_2^T \tilde{C} v_2)^{\frac{1}{2}} (y_2^T \tilde{C} y_2)^{\frac{1}{2}} . \tag{2.3}$$

We apply the technique proposed by Vassilevski [10] for the same problem and prove the auxiliary result.

Lemma 2.2. *Let the assumptions (2.2) and (2.3) be fulfilled. Then the following spectral equivalence relations hold:*

$$\gamma_1 v_2^{(2)^T} \tilde{S}_c v_2^{(2)} \leq v_2^{(2)^T} S_2 v_2^{(2)} \text{ for all } v_2^{(2)}, \tag{2.4}$$

$$w_2^{(2)^T} S_2 v_2^{(2)} \leq \frac{\gamma_2^2}{\gamma_1} (w_2^{(2)^T} \tilde{S}_c w_2^{(2)^T})^{\frac{1}{2}} (v_2^{(2)^T} \tilde{S}_c v_2^{(2)^T})^{\frac{1}{2}} \text{ for all } w_2^{(2)} \text{ and } v_2^{(2)} .$$

Now we are ready to prove our main result.

Theorem 2.1. *The spectrum of $B^{-1}A$ lies in the following rectangle*

$$\{z\,:\,Re\,z \geq \min(1,\gamma_1)\,,\quad Re\,z\,,\ |Im\,z| \leq \max(1,\gamma_2^2/\gamma_1)\}$$

Proof. We can rewrite (2.4) in the following way

$$\frac{v_2^{(2)^T} S_2 v_2^{(2)}}{v_2^{(2)^T} \tilde{S}_c v_2^{(2)}} = \frac{v_2^{(2)^T} \frac{1}{2}\left(S_2^T + S_2\right) v_2^{(2)}}{v_2^{(2)^T} \tilde{S}_c v_2^{(2)}} \geq \gamma_1\,.$$

Hence

$$\begin{aligned} Re\,\lambda[B^{-1}A] &\geq \min\left(1, Re\,\lambda\left[\tilde{S}_c^{-1}S_2\right]\right) \\ &\geq \min\left(1, \lambda\left[\tilde{S}_c^{-\frac{1}{2}}\frac{1}{2}\left(S_2^T+S_2\right)\tilde{S}_c^{-1\frac{1}{2}}\right]\right) \\ &\geq \min(1,\gamma_1)\,. \end{aligned}$$

For the other bound we have

$$|\lambda[B^{-1}A]| \leq \max\left(1, \left|\lambda\left[\tilde{S}_c^{-1}S_2\right]\right|\right)$$

and

$$\frac{v_2^{(2)^T} S_2 v_2^{(2)}}{v_2^{(2)^T} \tilde{S}_c v_2^{(2)}} \leq \frac{\gamma_2^2}{\gamma_1}\,.$$

Then

$$\left|\lambda\left[\tilde{S}_c^{-1}S_2\right]\right| = \left|\lambda\left[\tilde{S}_c^{-\frac{1}{2}}S_2\tilde{S}_c^{-\frac{1}{2}}\right]\right| \leq \frac{\gamma_2^2}{\gamma_1}\,.$$

□

Theorem 2.1 implies the following corollary.

Corollary 2.1. *The the preconditioned* ***GCG-LS*** *from Axelsson* [1], [2] *for solving the composite grid system with the preconditioner B will have rate of convergence independent of h and jumps of the coefficient $a(x)$.*

3. Numerical results

In this section we illustrate the convergence behavior of the two preconditioners on two model examples. We solve the problem (1.1) in the domain $\Omega = (0,1)\times(0,1)$ with the velocity field

$$b_1 = (1 + x\cos(\alpha))\cos(\alpha)\,,\quad b_2 = (1 + y\sin(\alpha))\sin(\alpha)\,, \tag{3.5}$$

where $\alpha = 15^0$. The refined subdomain is $\Omega_2 = \{0.5 \leq x_1 \leq 1\,,\ 0.5 \leq x_2 \leq 1\}$

Problem 1. *Consider a smooth solution $u(x)$ and a smooth coefficient $a(x)$,*

$$a(x) = \left[1 + 10(x_1^2 + x_2^2)\right]^{-1}\,,\quad u(x) = \phi_1(x_1)\psi_2(x_2)\,,$$

$$\phi_i(x_i) = \begin{cases} \sin^2\left(\pi\frac{x_i - d_i}{1-d_i}\right), & x_i \in (0.875, 1), \\ 0, & \textit{otherwise.} \end{cases}$$

TABLE 1. Preconditioner with $\tilde{C} = (\tilde{A} + \tilde{A}^T)/2$

n_c	h_c/h_f	Problem 1		Problem 2	
		$iter$	$arfac$	$iter$	$arfac$
	3	4	0.0264	3	0.0038
20	5	3	0.0098	3	0.0038
	7	4	0.0229	3	0.0036
	3	3	0.0070	3	0.0032
40	5	3	0.0055	3	0.0030
	7	2	0.0009	3	0.0029
80	3	2	0.0006	2	0.0009
	5	1	0.0001	2	0.0008

TABLE 2. Preconditioner with $\tilde{C} = \tilde{A}^{(1)}$

n_c	h_c/h_f	Problem 1		Problem 2	
		$iter$	$arfac$	$iter$	$arfac$
	3	4	0.0203	3	0.0038
20	5	3	0.0093	3	0.0038
	7	3	0.0082	3	0.0037
	3	2	0.0009	3	0.0032
40	5	2	0.0006	3	0.0030
	7	2	0.0005	3	0.0029
80	3	2	0.0004	2	0.0010
	5	1	0.0001	2	0.0008

We report the numbers of iterations for the preconditioned **GCG-LS** [1], [2]. The stopping criterion is $\|r_{last}\|/\|r_{first}\| < 10^{-6}$, $r = b - Ay$ where y is the current iteration, and $arfac = (\|r_{last}\|/\|r_{first}\|)^{1/iter}$. Our initial guess is found by constant interpolation of a coarse grid solution.

Problem 2. *Consider a piecewise continuous solution $u(x)$ and piecewise constant coefficient $a(x)$:*

$$u(x) = \left[\frac{(x_1 - b_1)(x_2 - b_2)}{a(x)}\right] \phi(x),$$

where

$$\phi(x) = \sin\left(\frac{\pi}{2}x_1\right) \sin\left(\frac{\pi}{2}x_2\right), \quad a(x) = \begin{cases} 1000, & x_i > (n_c + 3)h_c/2, \\ 1, & \textit{otherwise.} \end{cases}$$

The results in Tables 1 and 2 show that the convergence rate of the considered algorithms is independent of a mesh size h, jumps of the coefficient $a(x)$ and smoothness of the solution. Although each iteration is relatively expensive (it includes solution of two problems on a refined grid and one problem on a coarse grid), the overall algorithm is very efficient because we need only a few iteration.

The theoretical and numerical results are in accordance with the general theory of overlapping domain decomposition. In fact we have overlap of the whole refined subdomain and that explanes the very good numerical results we report.

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