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Outflow Boundary Conditions and Domain Decomposition Method

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ABSTRACT. We consider an advection-diffusion problem. We write a Schur type formulation by using outflow boundary conditions on the interfaces. The condensed problem is solved by either a Jacobi algorithm (equivalent to an additive Schwarz method), GMRES, or BiCGstab. The use of outflow boundary conditions and of general iterative methods gives much better results than the original Schwarz method.

1. Introduction

Let Ω be an bounded open set of $\mathbb{R}^2$, we want to solve the following convection-diffusion problem:

$$\begin{aligned} \mathcal{L}(u) = \frac{u}{\Delta t} + a(x,y)\,\frac{\partial u}{\partial x} + b(x,y)\,\frac{\partial u}{\partial y} - \nu \Delta u &= f \text{ in } \Omega \\ \mathcal{C}(u) &= \tilde{g} \text{ on } \partial\Omega \end{aligned} \tag{1.1}$$

where $\vec{a} = (a, b)$ is the velocity field, ν is the viscosity, f and $\tilde{g}$ are given functions, $\mathcal{C}$ is a linear operator. Δt is a constant which could correspond for instance to a time step for a backward-Euler scheme for the time dependent convection-diffusion equation.

In [**4**], Hagstrom et al. write a substructuring method for the convection-diffusion equation based on the exact outflow boundary conditions. The method is thus limited to a constant coefficient operator and makes use of nonlocal boundary conditions. In [**1**], Despres writes a substructuring method for the Helmholtz equation based on the radiation boundary condition of order 0. In both previous works, only non overlapping domains were considered. In this paper, we consider the convection-diffusion equation with variable coefficients and a decomposition into possibly overlapping subdomains. We use local outflow boundary conditions of order 0 and 2 (see also [**7**]).

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The paper is organized as follows: in § 2, we write the substructuring formulation at the continuous level. In § 3, we discretize this formulation and we compare three solvers of the condensed problem.

2. A substructuring formulation

Let Ω be a bounded open set of $\mathbb{R}^2$. Let $\Omega_{i,\,1\le i\le N}$ be a finite sequence of sets embedded in Ω such that $\bar\Omega = \cup_{i=1}^N \bar\Omega_i$. Let $\Gamma = \partial\Omega$, $\Gamma_i = \partial\Omega_i - \Gamma$. The outward normal from Ω_i is $\vec{n}_i$ and $\vec{\tau}_i$ is a tangential unit vector. Let $\mathcal{B}_{i,\,1\le i\le N}$ be a sequence of operators leading to well posed boundary value problems (see equation (2.1) below). We assign to each subdomain i an operator S_i. Let f be a function from Ω_i to $\mathbb{R}$ and g a function from Γ_i to $\mathbb{R}$. $S_i(f, g, \tilde{g})$ is the solution v of the following boundary value problem:

$$
\begin{aligned}
\mathcal{L}(v) &= f(x), && x \in \Omega_i \\
\mathcal{B}_i(v) &= g(x), && x \in \Gamma_i \\
\mathcal{C}(v) &= \tilde{g}, && x \in \partial\Omega_i \cap \Gamma
\end{aligned}
\tag{2.1}
$$

We introduce a sequence (η_i^j), $1 \le i \le N$, $1 \le j \le N$, $i \ne j$ of functions defined on the boundaries of the subdomains which satisfy:

$$
\begin{aligned}
&i)\ \ \eta_i^j : \partial\Omega_i \longrightarrow [0,1] \\
&ii)\ \ \eta_i^j = 0 \text{ on } \partial\Omega_i - \bar\Omega_j \\
&iii)\ \sum_{j, j\ne i} \eta_i^j(x) = 1, \quad x \in \partial\Omega_i
\end{aligned}
$$

REMARK 1. η_i^j is zero if $\partial\Omega_i \cap \bar\Omega_j = \emptyset$.

It is now possible to write a substructuring formulation. Let u be the solution to (1.1) and $u_i = u_{|\Omega_i}$. We write a system for $\mathcal{B}_i(u_i)$:

$$
\begin{aligned}
\mathcal{B}_i(u_i) &= \sum_{j,j\ne i} \eta_i^j \mathcal{B}_i(u_i) = \sum_{j,j\ne i} \eta_i^j \mathcal{B}_i(u_j) \\
&= \sum_{j,j\ne i} \eta_i^j \mathcal{B}_i(S_j(f_{|\Omega_j}, \tilde{g}, \mathcal{B}_j(u_j))) \\
&= \sum_{j,j\ne i} \eta_i^j \mathcal{B}_i(S_j(f_{|\Omega_j}, \tilde{g}, 0)) + \sum_{j,j\ne i} \eta_i^j \mathcal{B}_i(S_j(0, 0, \mathcal{B}_j(u_j)))
\end{aligned}
$$

Thus, $(\mathcal{B}_i(u_i))_{1\le i\le N}$ solves the following linear system:

$$
\mathcal{B}_i(u_i) - \sum_{j,j\ne i} \eta_i^j \mathcal{B}_i(S_j(0,0,\mathcal{B}_j(u_j))) = \sum_{j,j\ne i} \eta_i^j \mathcal{B}_i(S_j(f_{|\Omega_j}, \tilde{g}, 0)) 1 \le i \le N
\tag{2.2}
$$

Let $U = (U_i)_{1\leq i\leq N}$ and $G = (G_i)_{1\leq i\leq N}$ be the vectors

$$U = \begin{bmatrix} \mathcal{B}_1(u_1) \\ \vdots \\ \mathcal{B}_N(u_N) \end{bmatrix} \text{ and } G = \begin{bmatrix} \sum_{j,j\neq 1} \eta_1^j \mathcal{B}_1(S_j(f_{|\Omega_j}, \tilde{g}, 0)) \\ \vdots \\ \sum_{j,j\neq N} \eta_N^j \mathcal{B}_N(S_j(f_{|\Omega_j}, \tilde{g}, 0)) \end{bmatrix}$$

and $\mathcal{T}$ be the linear operator defined by

$$\mathcal{T}(U) = \begin{bmatrix} \sum_{j,j\neq 1} \eta_1^j \mathcal{B}_1(S_j(0,0,\mathcal{B}_j(u_j))) \\ \vdots \\ \sum_{j,j\neq N} \eta_N^j \mathcal{B}_N(S_j(0,0,\mathcal{B}_j(u_j))) \end{bmatrix}$$

System (2.2) may now be written in the following compact form:

$$(Id - \mathcal{T})(U) = G \tag{2.3}$$

We shall consider three possibilities for $\mathcal{B}_i$:

$$\mathcal{B}_i^I = Id \tag{2.4}$$

or

$$\mathcal{B}_i^0 = \frac{\partial}{\partial \vec{n_i}} - \frac{\vec{a}.\vec{n_i} - \sqrt{(\vec{a}.\vec{n_i})^2 + \frac{4\nu}{\Delta t}}}{2\nu} \tag{2.5}$$

($\vec{a}$ is the velocity field (a,b)) or

$$\mathcal{B}_i^2 = \frac{\partial}{\partial \vec{n_i}} - \frac{\vec{a}.\vec{n_i} - \sqrt{(\vec{a}.\vec{n_i})^2 + \frac{4\nu}{\Delta t}}}{2\nu} + \frac{\vec{a}.\vec{\tau_i}}{\sqrt{(\vec{a}.\vec{n_i})^2 + \frac{4\nu}{\Delta t}}} \frac{\partial}{\partial \vec{\tau_i}}$$
$$- \frac{\nu}{\sqrt{(\vec{a}.\vec{n_i})^2 + \frac{4\nu}{\Delta t}}} \left(1 + \frac{(\vec{a}.\vec{\tau_i})^2}{(\vec{a}.\vec{n_i})^2 + \frac{4\nu}{\Delta t}}\right) \frac{\partial^2}{\partial \vec{\tau_i}^2}$$

The boundary conditions $\mathcal{B}_i^0$, or $\mathcal{B}_i^2$ are far field boundary conditions (also called Outflow B.C., Absorbing B.C., Artificial B.C., Radiation B.C.,etc , see **[2]**, **[5]**) of order 0 and 2.

3. Discretization and numerical results

In order to illustrate the validity of the method, a 2D test problem has been performed. We solve the following problem:

$$\begin{cases} \dfrac{u}{\Delta t} + a(x,y)\dfrac{\partial u}{\partial x} + b(x,y)\dfrac{\partial u}{\partial y} - \nu \Delta u = 0, & 0 \leq x \leq 1,\ 0 \leq y \leq 1 \\ u(0,y) = 1, \quad 0 < y < 1 \\ \dfrac{\partial u}{\partial y}(x,1) = 0, \quad 0 < x < 1 \\ \dfrac{\partial u}{\partial x}(1,y) = 0, \quad 0 < y < 1 \\ u(x,0) = 0, \quad 0 < x < 1 \end{cases}$$

The operator $\mathcal{L}$ is discretized by a standard upwind finite difference scheme of order 1 (see [**3**]) and $\mathcal{B}_{i,\,1\le i\le N}$ be a finite difference approximation. We used a rectangular finite difference grid. The mesh size is denoted by h. The unit square is decomposed into overlapping rectangles. This leads to a discretization of the operator $\mathcal{T}$ and thus of system (2.3):

$$(Id - \mathcal{T}_h)(U_h) = G_h \tag{3.1}$$

REMARK 2. Any other discretization could be used as well.

From the definition of $\mathcal{T}_h$, we see that the computation of $\mathcal{T}_h$ applied to some vector U_h amounts to the solving of N independent boundary value subproblems (one subproblem in each subdomain) which can be solved in parallel. We have considered three algorithms in order to solve (3.1): GMRES(∞), BiCGStab and a Jacobi algorithm:

$$U_h^{n+1} = \mathcal{T}_h(U_h^n) + G_h \tag{3.2}$$

The choice of the last algorithm is due to the fact that it is equivalent to an additive Schwarz method (ASM) whose convergence has been studied in [**6**] and [**1**] for Fourier interface conditions and in [**8**] for outflow boundary conditions. In Tables 1 and 2, we give the number of subproblems solved so that the maximum of the error is smaller than 10^{-6}. One iteration of GMRES(∞) or of ASM counts for one solution in each subdomain and one iteration of BiCGStab counts for two solutions in each subdomain. In the tables, Id corresponds to the use of the Id as interface condition, OBC0 to $\mathcal{B}^0$ (see (2.5)) and OBC2 to $\mathcal{B}^2$ (see (2.6)).

Table 1: Computational cost vs. interface conditions and solvers

Boundary Cond.	ASM	BiCGStab	GMRES
Id	> 200	88	61
OBC0	86	38	33
OBC2	46	28	24

Table 1 corresponds to the following parameters:
8×1 subdomains, 21×120 points in each subdomain, overlap $= 2h$, $\nu = 0.1$, $\Delta t = 10^{40}$, $a = y$, $b = 0$.

Table 2: Computational cost vs. interface conditions and solvers

Boundary Cond.	ASM	BiCGStab	GMRES
Id	479	64	50
OBC0	27	22	19
OBC2	18	16	16

Table 2 corresponds to the following parameters:
4×4 subdomains, 35×35 points in each subdomain, overlap $= 2h$, $\nu = 0.1$, $\Delta t = 1$, $a = y$, $b = 0$.

The use of outflow boundary conditions leads to a significant improvement, whatever iterative solver is used. BiCGStab and GMRES give similar results with an advantage to GMRES in terms of computational cost and to BiCGStab in terms of storage requirements, since only two directions have to be stored.

4. Conclusion

The interest of using outflow boundary conditions as interface conditions is clear. We have considered here the scalar convection-diffusion equation. The same strategy can be applied to systems of PDE's.

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