Domain Decomposition for Elliptic Problems with Large Condition Numbers

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ABSTRACT. This paper suggests a technique for the construction of preconditioning operators for the iterative solution of systems of grid equations approximating elliptic boundary value problems with strong singularities in the coefficients. The technique suggested is based on the decomposition of the original domain into subdomains in which the singularity of coefficients is characterized by some parameter. The convergence rate of the preconditioned iterative process is independent of both the mesh size and the coefficients.

1. Introduction

In this paper, we design preconditioning operators for the system of grid equations approximating the following boundary value problem:

\[
\begin{align*}
- \sum_{i,j=1}^{2} \frac{\partial}{\partial x_i} a_{ij}(x) \frac{\partial u}{\partial x_j} + a_0(x) u &= f(x), & x \in \Omega, \\
u(x) &= 0, & x \in \Gamma
\end{align*}
\]

We assume that $\Omega$ is a bounded, polygonal region and $\Gamma$ is its boundary. Let $\Omega$ be a union of $n$ nonoverlapping subdomains $\Omega_i$,

\[\Omega = \bigcup_{i=1}^{n} \Omega_i, \quad \Omega_i \cap \Omega_j = \emptyset, \quad i \neq j,\]

where $\Omega_i$ are polygons and $\Gamma_i$ are their boundaries. Let $\Omega^h$

\[\Omega^h = \bigcup_{i=1}^{n} \Omega^h_i\]

be a regular triangulation of $\Omega$ which is characterized by a parameter $h$.

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Let us introduce the weighted Sobolev spaces $H^1_{\alpha}(\Omega)$ with the norms [11]

\[
\|u\|_{H^1_{\alpha}(\Omega)}^2 = \|u\|_{L^2(\Omega)}^2 + |u|_{H^1_{\alpha}(\Omega)}^2,
\]
\[
\|u\|_{L^2(\Omega)}^2 = \int_{\Omega} u^2(x) \, dx,
\]
\[
|u|_{H^1_{\alpha}(\Omega)}^2 = \int_{\Omega} \left( \frac{|\nabla u(x)|}{|\varphi(x)|^{\alpha/2}} \right)^2 \, dx.
\]

Here

$\alpha(x) \equiv \alpha_i = \text{const.}, \ x \in \Omega_i$

and $g(x)$ is the distance between the point $x \in \Omega_i$ and the boundary $\Gamma_i$ of the subdomain $\Omega_i$. We assume that

\[
|\alpha| \leq \frac{1}{2}.
\]

Denote by $H^1_{\alpha}(\Omega)$ the subspace of $H^1_{\alpha}(\Omega)$ with zero trace on $\Gamma$ and introduce the bilinear form

\[
a(u, v) = \int_{\Omega} \left( \sum_{i,j=1}^2 a_{ij}(x) \frac{\partial u}{\partial x_j} \frac{\partial v}{\partial x_i} + a_0(x) uv \right) \, dx
\]

and the linear functional

\[
l(v) = \int_{\Omega} f(x)v \, dx.
\]

We assume that the coefficients of the problem (1.1) are such that $a(u, v)$ is a symmetric, coercive and continuous form on $H^1_{\alpha}(\Omega) \times H^1_{\alpha}(\Omega)$, and that the linear functional $l(v)$ is continuous in $H^1_{\alpha}(\Omega)$.

Denote by $W$ a space of real-valued continuous functions linear on triangles of the triangulation $\Omega^h$. A weak formulation of (1.1) is: Find $u \in H^1_{\alpha}(\Omega)$ such that

\[
a(u, v) = l(v), \quad \forall v \in H^1_{\alpha}(\Omega).
\]

Using the finite element method, we can pass from (1.4) to the linear algebraic system

\[
Au = f.
\]

The condition number of the matrix $A$ depends on $h, \alpha$ and can be large. Our purpose is the design of a preconditioner $B$ for the problem (1.5) such that the following inequalities are valid:

\[
c_1(Bu, u) \leq (Au, u) \leq c_2(Bu, u), \quad \forall u \in R^N.
\]

Here $N$ is the dimension of $W$, the positive constants $c_1, c_2$ are independent of $h$ and $\alpha$, and the action of $B^{-1}$ on a vector can be realized at low cost.
2. Additive Schwarz method for singular problems

The construction of the preconditioner for the system (1.5) will be realized on the basis of the additive Schwarz method [1, 3, 4]. To design the preconditioning operator $B$, we follow [7, 10] and decompose the space $W$ into a sum of subspaces

$$W = W_0 + W_1.$$ 

To this end, divide the nodes of the triangulation $\Omega^h$ into two groups: those which lie inside of $\Omega^h_i$ and those which lie on boundaries of $\Omega^h_i$. The subspace $W_0$ corresponds to the first set. Let

$$S = \bigcup_{i=1}^{n} \partial \Omega^h_i,$$

$$W_0 = \left\{ u^h \in W \mid u^h(x) = 0, \ x \in S \right\},$$

$$W_{0,i} = \left\{ u^h \in W_0 \mid u^h(x) = 0, \ x \in \Omega^h_i \right\}, \ i = 1, 2, \ldots, n.$$ 

It is clear that $W_0$ is the direct sum of the orthogonal subspaces $W_{0,i}$ with respect to the scalar product in $H^1_0(\Omega)$:

$$W_0 = W_{0,1} \oplus \cdots \oplus W_{0,n}.$$ 

The subspace $W_1$ corresponds to the second group of nodes $\Omega^h$ and can be defined in the following way. First, define $V$ which is the space of traces of functions from $W$ on $S$:

$$V = \left\{ \varphi^h \mid \varphi^h(x) = u^h(x), \ x \in S, \ u^h \in W \right\}.$$ 

To define the subspace $W_1$, we need a norm preserving extension operator of functions given at $S$ into $\Omega^h$. The basis of the further construction is the following trace theorem for the weighted Sobolev spaces $H^1_{\alpha}(\Omega)$ [11]:

**Theorem 2.1.** Let $\Omega$ be a bounded domain with piecewise-smooth boundary $\Gamma$ from the class $C^2$ satisfying the Lipschitz condition and $\alpha$ is a constant such that $|\alpha| < \frac{1}{2}$. Then there exists a positive constant $c_1$ independent of $\alpha$, such that

$$\| \varphi \|_{H^{1/2 + \alpha}(\Gamma)} \leq c_1 \| u \|_{H^1_{\alpha}(\Omega)}$$

for any function $u \in H^1_{\alpha}(\Omega)$, where $\varphi \in H^{1/2 + \alpha}(\Gamma)$ is the trace of $u$ at the boundary $\Gamma$. Conversely, there exists a positive constant $c_2$, independent of $\alpha$, such that for any function $\varphi \in H^{1/2 + \alpha}(\Gamma)$ there exist $u \in H^1_{\alpha}(\Omega)$ such that

$$u(x) = \varphi(x), \ x \in \Gamma,$$

$$\| u \|_{H^1_{\alpha}(\Omega)}^2 \leq c_2 \| \varphi \|_{H^{1/2 + \alpha}(\Gamma)}^2.$$
Here $\|\varphi\|_{H^{\frac{1}{2}+\alpha}(\Gamma)}$ is the norm in the Sobolev space $H^{\frac{1}{2}+\alpha}(\Gamma)$:

\[
\|\varphi\|_{H^{\frac{1}{2}+\alpha}(\Gamma)}^2 = \|\varphi\|_{L^2(\Gamma)}^2 + |\varphi|_{H^{\frac{1}{2}+\alpha}(\Gamma)}^2,
\]

\[
|\varphi|_{H^{\frac{1}{2}+\alpha}(\Gamma)}^2 = \int_{\Gamma} \int_{\Gamma} \frac{(\varphi(x)-\varphi(y))^2}{|x-y|^{\frac{3}{2}+2\alpha}} \, dx \, dy.
\]

Denote

\[H(S) = \left\{ \varphi \mid \varphi\big|_{\Gamma_i} = \varphi_i, \quad \varphi_i \in H^{\frac{1}{2}+\alpha_i}(\Gamma_i) \right\},\]

\[\|\varphi\|_{H^2(S)}^2 = \sum_{i=1}^n \|\varphi\|_{H^{\frac{1}{2}+\alpha_i}(\Gamma_i)}^2.\]

To define the bounded extension operator for the finite element case from $V$ into $W$, we need mesh counterparts of the norms (1.2) and (2.1). To this end, let us split the triangles $T_j$ of the triangulation $\Omega^h$ into three groups. Denote by $M_1$ a set of such $T_j$ that $T_j$ do not have vertices on $S$, denote by $M_2$ a set of such $T_j$ that $T_j$ have only one vertex on $S$, and denote by $M_3$ a set of such $T_j$ that $T_j$ have more than one vertex on $S$. Set

\[
\|u^h\|_{H^1_{\alpha,h}(\Omega)}^2 = \sum_{i=1}^n \|u^h\|_{H^1_{\alpha,h}(\Omega_i)}^2,
\]

\[
\|u^h\|_{H^2_{\alpha,h}(\Omega)}^2 = \|u^h\|_{L^2(\Omega)}^2 + |u^h|_{H^2_{\alpha,h}(\Omega)}^2,
\]

\[
|u^h|_{H^2_{\alpha,h}(\Omega)}^2 = \sum_{T_j \in M_1 \cap \Omega_i} \frac{(u_{j1} - u_{j2})^2 + (u_{j2} - u_{j3})^2 + (u_{j3} - u_{j1})^2}{(\varrho(T_j, \Gamma_i))^{2\alpha_i}} + \sum_{T_j \in M_2 \cap \Omega_i} \frac{(u_{j1} - u_{j2})^2 + (u_{j2} - u_{j3})^2 + (u_{j3} - u_{j1})^2}{h^{2\alpha_i}} + \sum_{T_j \in M_3 \cap \Omega_i} \frac{(u_{j1} - u_{j2})^2 + (u_{j2} - u_{j3})^2 + (u_{j3} - u_{j1})^2}{(1-2\alpha_i)h^{2\alpha_i}}, \forall u^h \in W.
\]

Here $z_i$ are vertices of $\Omega^h$, $u_{j1}, u_{j2}, u_{j3}$ are values of $u^h$ at vertices of $T_j$, and $\varrho(T_j, \Gamma_i)$ is the distance between $T_j$ and $\Gamma_i$.

Using the natural order of nodes on $\Gamma_i$, let us put for each node $z_j \in \Gamma_i$ into correspondence the node $z_{j+1}$, which is a node neighboring upon $z_i$, and set
\[
\|\varphi^h\|_{H^1_h(S)}^2 = \sum_{i=1}^n \|\varphi^h\|_{H^{\frac{1}{2}+\alpha_i}(\Gamma_i)}^2, \\
\|\varphi^h\|_{H^{\frac{1}{2}+\alpha_i}(\Gamma_i)}^2 = \|\varphi^h\|_{L^2_h(\Gamma_i)}^2 + |\varphi^h|_{H^{\frac{1}{2}+\alpha_i}(\Gamma_i)}^2, \\
\|\varphi^h\|_{L^2_h(\Gamma_i)}^2 = \sum_{z_j \in \Gamma_i} (\varphi^h(z_j))^2 h, \\
|\varphi^h|_{H^{\frac{1}{2}+\alpha_i}(\Gamma_i)} = \sum_{z_j \in \Gamma_i} \sum_{z_k \in \Gamma_i} \frac{(\varphi^h(z_j) - \varphi^h(z_k))^2}{|z_j - z_k|^{2+2\alpha_i}} + \sum_{z_j \in \Gamma_i} \frac{(\varphi^h(z_j) - \varphi^h(z_{j+1}))^2}{(1 - 2\alpha_i)h^{2\alpha_i}}, \forall \varphi^h \in V.
\]

The following lemmas are valid.

**Lemma 2.1.** There exist positive constants \(c_3\) and \(c_4\), independent of \(\alpha\) and \(h\), such that

\[
c_3 \|u^h\|_{H^1_h(\Omega_i)} \leq \|u^h\|_{H^{\frac{1}{2}+\alpha_i}(\Omega_i)} \leq c_4 \|u^h\|_{H^1_h(\Omega_i)}, \quad \forall u^h \in W_i, \quad i = 1, 2, \ldots, n.
\]

**Lemma 2.2.** There exist positive constants \(c_5\) and \(c_6\), independent of \(\alpha\) and \(h\), such that

\[
c_5 \|\varphi^h\|_{H^{\frac{1}{2}+\alpha_i}(\Gamma_i)} \leq \|\varphi^h\|_{H^{\frac{1}{2}+\alpha_i}(\Gamma_i)} \leq c_6 \|\varphi^h\|_{H^{\frac{1}{2}+\alpha_i}(\Gamma_i)}, \quad \forall \varphi^h \in V, \quad i = 1, 2, \ldots, n.
\]

To define \(W_1\), let us use the explicit extension operator

\[
t^h : V \to W_i, \quad \text{(2.2)}
\]

which was suggested for regular elliptic second order problems. The definition and the realization algorithm were done in [5, 6, 8] and briefly can be described in the following way. Let us introduce the near–boundary coordinate system \((s, n)\) which is defined in a \(\delta–\)neighborhood of \(\Gamma_i\). Here \(s\) defines a point \(P\) at \(\Gamma_i\) and \(n\) is the distance between the given point and \(\Gamma_i\) along the internal pseudonormal, whose direction at the angular points coincides with the bisectrix of the angle and along the smooth part the vector changes, for example, linearly. Set

\[
t : H(S) \to H^1_h(\Omega), \quad \text{(2.3)}
\]

\[
t^h \varphi = u, \quad u(s, n) = \left(1 - \frac{n}{\delta}\right) \int_s^{s+n} \frac{\varphi(t)}{n} \, dt,
\]

where the function \(u\) is extended by zero in the rest of \(\Omega\). Using the auxiliary mesh, which is topologically equivalent to a uniform rectangular mesh, we can define the finite–element analogue \(t^h\) (2.2) of the operator \(t\) from (2.3). The following theorem is valid.
THEOREM 2.2. There exists a positive constant $c_7$, independent of $\alpha$ and $h$, such that
\[ \|u^h\|_{H_{h,\alpha}^1(\Omega)} = \|t^h \varphi^h\|_{H_{h,\alpha}^1(\Omega)} \leq c_7 \|\varphi^h\|_{H_{h}^{1+\alpha}(\tilde{S})}, \quad \forall \varphi^h \in V. \]

Remark 2.1. The cost of the actions of $t^h$ and $(t^h)^*$ on vectors is $O(h^{-2})$ arithmetic operations (see [5] for details).

At last, we can define subspace $W_1$
\[ W_1 = \{ u^h \mid u^h = t^h \varphi^h, \quad \varphi^h \in V \}. \]
It is obvious that
\[ W = W_0 + W_1 \]
and this decomposition of the space $W$ is regular in the following sense.

THEOREM 2.3. There exists a positive constant $c_8$, independent of $\alpha$ and $h$, such that for any function $u^h \in W$ there exist $u_0 \in W_i, i = 0, 1$, such that
\[ u_0 + u_1 = u, \]
\[ \|u_0\|_{H_0^2(\Omega)} + \|u_1\|_{H_0^2(\Omega)} \leq c_8 \|u\|_{H_0^2(\Omega)}. \]

According to [8], we can construct a preconditioner for the subspace $W_1$ of the following form
\[ B_1^+ = t^h \Sigma^{-1}(t^h)^*, \]
where $\Sigma$ has to satisfy
\[ c_9 \|\varphi^h\|^2_{H(\tilde{S})} \leq (\Sigma \varphi, \varphi) \leq c_{10} \|\varphi^h\|^2_{H(\tilde{S})}, \quad \forall \varphi^h \in V. \]

Here the components of the vector $\varphi$ are equal to the values of the function $\varphi^h$ in corresponding nodes. The constants $c_9, c_{10}$ should be independent of $\alpha$ and $h$. The construction of the easily invertible operator (matrix) $\Sigma$ can be done, using [4, 6, 7]. The cost of the action $B_1^+$ on vectors is $O(h^{-2})$ arithmetic operations.

3. Preconditioning operator for $W_0$.

The goal of this section is the design of the preconditioning operator for the space $W$ which was defined in Section 2. Since $W_0$ is the direct sum of the orthogonal subspaces $W_{0,i}, i = 1, 2, \ldots, n$ which correspond to the subdomains $\Omega_i$, we can design preconditioners independently for each subdomain $\Omega_i$ with the boundary $\Gamma_i$. For the sake of simplicity, we omit the subscript $i$. To construct the preconditioner, we use the additive Schwarz method. Let us decompose the domain $\Omega$ into two overlapping parts
\[ \Omega = \Omega_{in} \cup \Omega_b, \]
\[ \Omega_b = \{ x = (s, n) \in \Omega \mid 0 \leq s \leq L, \quad 0 < n < 2\delta \}, \]
\[ \text{dist} (\Gamma, \partial \Omega_{in}) = \delta. \]

Here $(s, n)$ is the near-boundary coordinate system; $\delta = O(1)$ is independent of $h$; for the sake of simplicity, we assume that $\Omega$ is the simply connected domain and $L$ is
the length of $\Gamma$. Then the triangulation $\Omega^h$ can be decomposed into two overlapping parts

$$\Omega^h = \Omega^h_{in} \cup \Omega^h_b,$$

$$\Omega^h_{in} = \bigcup_{T_j \subset \Omega^h_{in}} T_j,$$

$$\Omega^h_b = \bigcup_{T_j \subset \Omega^h_b} T_j$$

and the finite element space $W_0$ can be decompose into two overlapping subspaces

$$W_0 = W_{in} + W_b,$$

$$W_{in} = \{ u^h \in W_0 \mid u^h(x) = 0, \ x \in \Omega^h_{in} \},$$

$$W_b = \{ u^h \in W_0 \mid u^h(x) = 0, \ x \in \Omega^h_b \}.$$

Using (3.1), it is easy to see that there exists a positive constant $c_1$, independent of $\alpha$ and $h$, such that for any $u^h \in W_0$ there exists $u^h_{in} \in W_{in}$ and $u^h_b \in W_b$ such that

$$u^h_{in} + u^h_b = u^h,$$

$$\|u^h_{in}\|_{H^1_0(\Omega)} + \|u^h_b\|_{H^1_0(\Omega)} \leq c_1 \|u\|_{H^2_0(\Omega)}.$$

Then, according to [3], we can define the preconditioner in the following form

$$B^{-1} = B_{in}^+ + B_b^+,$$

where $B_{in}$ and $B_b$ are such that

$$B_{in} : W \rightarrow W_{in},$$

$$c_2 \|u^h\|_{H^1_0(\Omega)}^2 \leq (B_{in} u, u) \leq c_3 \|u^h\|_{H^1_0(\Omega)}^2, \ \forall u \in W_{in},$$

$$B_b : W \rightarrow W_b,$$

$$c_2 \|u^h\|_{H^1_0(\Omega)}^2 \leq (B_b u, u) \leq c_3 \|u^h\|_{H^1_0(\Omega)}^2, \ \forall u \in W_b.$$

Here $c_2$ and $c_3$ are independent of $\alpha$ and $h$. From (3.1) we have that there exist positive constants $c_4$ and $c_5$, independent of $\alpha$ and $h$, such that

$$c_4 \|u^h\|_{H^1(\Omega)} \leq \|u^h\|_{H^2_0(\Omega)} \leq c_5 \|u^h\|_{H^1(\Omega)}, \ \forall u \in W_{in}.$$

This implies that the construction of $B_{in}$ is equivalent to the construction of preconditioners for regular elliptic problems. For instance, using combinations of the domain decomposition and fictitious domain methods, the construction of effective preconditioners was studied in [4, 5, 10]. A new element of the construction of the preconditioner $B$ is the construction of $B_b$. To this end, let us decompose $\Omega_b$ into overlapping parts

$$\Omega_b = \bigcup_{i=L}^{t} D_i,$$

$$D_i = \left\{ x = x(s, n) \in \Omega_b \mid (i - 1) \frac{L}{l} < s < (i + 1) \frac{L}{l}, 0 < n < \delta \right\},$$
where \( l = O(1), \) i.e. the number of subdomains is fixed, and \( x(L + s, n) = x(s, n). \) Then the triangulation \( \Omega^n_0 \) can be decomposed into overlapping parts

\[
\Omega^n_0 = \bigcup_{i=1}^{l} D_i^h,
\]

\[
D_i^h = \bigcup_{T_j \subset D_i} T_j
\]

and the space \( W^n_0 \) can be decomposed into overlapping subspaces

\[
W^{in} = \sum_{i=1}^{l} U_i,
\]

\[
U_i = \{ u^h \in W^{in} \mid u^h(x) = 0, x \in D_i^h \}.
\]

The following lemma is valid.

**Lemma 3.1.** Let \( \Omega \) be a rectangular domain

\[
\Omega = \{(x_1, x_2) \mid -1 < x_1 < 1, 0 < x_2 < 1\}
\]

and \( \Omega^n_0 \) be a regular triangulation of \( \Omega. \) Denote by \( W \) a space of real-valued continuous functions linear on triangles of the triangulation \( \Omega^n_0. \) Then, there exists a positive constant \( c_0, \) independent of \( \alpha \) and \( h, \) such that \( \forall u^h \in W
\]

\[
\int_{-1}^{1} \int_{0}^{1} \left( \frac{1}{x_2^2} \left| \nabla u^h \right|^2 + (u^h)^2 \right) dx_2 dx_1 \leq c_0 \int_{-1}^{1} \int_{0}^{1} \left( \frac{1}{x_2^2} \left| \nabla u^h \right|^2 + (u^h)^2 \right) dx_2 dx_1,
\]

for any constant \( \alpha : |\alpha| < 1/2. \) Here the function \( \tilde{u}^h \in W \) is defined in the following way:

\[
\tilde{u}^h(z_i) = \begin{cases} u^h(x_i, z_i) \in [0, 1] \times [0, 1], \\ (1 + z_i) u^h(\bar{z}_i), z_i \in [-1, 0] \times [0, 1]. \end{cases}
\]

Here \( \bar{z}_i \) is a node of \( \Omega^n \) which is the nearest for the point with the coordinates \((-x_{11}, x_{2i}).\)

Using Lemma 3.1, it is easy to see that there exists a positive constant \( c_7, \) independent of \( \alpha \) and \( h, \) such that for any \( u^h \in W^{in} \) there exists \( u_i^h \in U_i :\)

\[
u^h_i + \cdots + u^h_i = u^h, \]

\[
c_7 \left( \| u_i^h \|_{H^1_0(\Omega)} + \cdots + \| u^h_i \|_{H^1_0(\Omega)} \right) \leq \| u^h \|_{H^1_0(\Omega)}.\]

According to [8] and (3.2), to define the operator \( B_{in}, \) we can define the easily invertible norms for subspaces \( V_i, i = 1, 2, \ldots, l. \) To this end, we use the fictitious space lemma [5, 9].

To design the easily invertible norm in \( U_i, \) we consider an auxiliary topologically uniform mesh. For the sake of simplicity, we omit the subscript \( i.\)

Let us assume that the domain \( D \) in the near-boundary coordinate system \((s, n)\) has the following representation

\[
D = \{ z = (s, n) \mid 0 < s < \xi, \ 0 < n < \delta \}.
\]
Introduce in the domain \( D \) the auxiliary mesh \( Q^h \) with the mesh size \( h_0 \) and the nodes \( z_{ij} \)

\[
z_{ij} = (s_i, n_j), \quad s_i = i \cdot h_0, \quad n_j = j \cdot h_0
\]

\[
i = 0, 1, \ldots, n, \quad j = 0, 1, \ldots, m,
\]

\[
n \cdot h_0 = \xi, \quad m \cdot h_0 = \delta.
\]

Assume that \( h_0 \leq h_{\text{min}}/2 \), where \( h_{\text{min}} \) is the length of the minimal side of triangles of the triangulation \( D^h \). Denote the cells of the mesh \( Q^h \) by \( Q_{ij} \)

\[
Q_{ij} = \{ z = (s, n) | s_i \leq s < s_{i+1}, \quad n_j \leq n < n_{j+1}, \quad i = 0, 1, \ldots, n - 1, \quad j = 0, 1, \ldots, m - 1 \}.
\]

On the mesh \( Q^h \) we will consider the mesh function \( V(z_{ij}) \) vanishing at the boundary nodes of the mesh \( Q^h \). We will identify the mesh \( Q^h \) and the triangulation of \( D \) with the nodes \( z_{ij} \). Denote by \( F \) a space of real-valued continuous functions \( V^h \) linear on triangles of the triangulation \( Q^h \).

Using the tensor product of matrices, introduce

\[
B = A \otimes J + I_{n-1} \otimes T,
\]

where the tridiagonal matrix \( A \) of order \( n - 1 \) approximates the second derivative and \( J \) is a diagonal matrix of order \( m - 1 \)

\[
J = \text{diag} \left( \frac{1}{h_0^{2\alpha}}, \frac{1}{(2h_0)^{2\alpha}}, \ldots, \frac{1}{((m-1)h_0)^{2\alpha}} \right);
\]

The matrix \( I_{n-1} \) is the identity matrix of the order \( n - 1 \); the tridiagonal matrix \( T \) of the order \( m - 1 \):

\[
\begin{bmatrix}
\frac{1}{(1-2\alpha)h_0^{2\alpha}} + \frac{1}{(2h_0)^{2\alpha}} & \frac{1}{(2h_0)^{2\alpha}} & \frac{1}{(3h_0)^{2\alpha}} & \cdots & \frac{1}{((m-1)h_0)^{2\alpha}} \\
\frac{1}{(2h_0)^{2\alpha}} & \frac{1}{(3h_0)^{2\alpha}} & \cdots & \cdots & \frac{1}{((m-1)h_0)^{2\alpha}} \\
\frac{1}{(3h_0)^{2\alpha}} & \cdots & \cdots & \cdots & \frac{1}{((m-1)h_0)^{2\alpha}} \\
\vdots & \ddots & \ddots & \ddots & \ddots \\
\frac{1}{((m-1)h_0)^{2\alpha}} & \frac{1}{((m-1)h_0)^{2\alpha}} & \cdots & \cdots & \frac{1}{(mh_0)^{2\alpha}}
\end{bmatrix}
\]

The following Lemma is valid.

**Lemma 3.2.** There exist positive constants \( c_8, c_9 \), independent of \( \alpha \) and \( h \), such that

\[
c_8 \| V^h \|^2_{H^1(D)} \leq (BV, V) \leq c_9 \| V^h \|^2_{H^1_h(D)}, \quad \forall V^h \in F
\]

where the components of the vector \( V \) are equal to the values of the function \( V^h \) in the corresponding nodes.
Remark 3.1. Using the spectral decomposition of the matrix $A$

$$A = QAQ^T,$$

we can invert the matrix $B$:

$$B^{-1} = Q \otimes I_{m-1}(A^{-1} \otimes J^{-1} + I_{n-1} \otimes T^{-1})Q^T \otimes I_{m-1}$$

Then the multiplication of vectors by $B^{-1}$ can be performed in $O(h^{-2} \log h^{-1})$ arithmetic operations using the Fast Fourier Transform. To define the easily invertible norm in the space $V$, using [9], we need to define operators $R$ and $T$:

$$R: F \rightarrow V, \quad T: V \rightarrow F.$$ 

Let us define the operator $R$ which puts into the correspondence to each function $V^h(z_{ij}) \in F$ a function $u^h \in V$ in the following way. Let $z_L$ be a node of the triangulation $D^h$ and let $z_0 \in Q_{ij}$. Set

$$u^h(z_l) = V^h(z_{ij}).$$

Note that by the assumption on $h_0$ only one node $z_l$ of the triangulation $D^h$ belonging to the cell $Q_{ij}$ can exist. Then the operator $T$ is defined as follows. If the cell $Q_{ij}$ contains a node $z_l$ of the triangulation $D^h$, we set

$$V^h(z_{ij}) = u^h(z_l).$$

At other nodes of the mesh $Q^h$ the function $V^h(z_{ij})$ can be defined in a sufficiently arbitrary way, for instance, as follows. Let the node $z_{ij}$ belong to the triangle $T_t$ of the triangulation $D^h$ with the vertices $z_1, z_2$ and $z_3$. Set

$$V^h(z_{ij}) = \frac{1}{3}(u^h(z_1) + u^h(z_2) + u^h(z_3)).$$

It is easy to see that the above-defined operators $R$ and $T$ satisfy the hypothesis of Lemma 4.3 while the constants $c_{10}$ and $c_{11}$ are independent of $\alpha$ and $h$:

$$\| RV^h \|_{H^2(D)}^2 \leq c_{10} (BV, V), \quad \forall V^h \in F,$n

$$\langle BTu^h, Tu^h \rangle \leq c_{11} \| u^h \|_{H^2(D)}, \quad \forall u^h \in V.$$ 

Then, the following theorem is valid:

**Theorem 3.1.** There exist positive constants $c_{12}$ and $c_{13}$, independent of $\alpha$ and $h$, such that

$$c_{12} \| u^h \|_{H^2(D)}^2 \leq (C^{-1}u, u) \leq c_{13} \| u^h \|_{H^2(D)}^2, \quad \forall u^h \in V,$n

$$C = RB^{-1}RT,$$

where the components of the vector $u$ are equal to the values of the functions $u^h$ in the corresponding nodes.
REFERENCES


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