Contemporary Mathematics
Volume 180, 1994

Domain Decomposition for Elliptic Problems with Large Condition Numbers

S. V. NEPOMNYASCHIKH

ABSTRACT. This paper suggests a technique for the construction of preconditioning operators for the iterative solution of systems of grid equations approximating elliptic boundary value problems with strong singularities in the coefficients. The technique suggested is based on the decomposition of the original domain into subdomains in which the singularity of coefficients is characterized by some parameter. The convergence rate of the preconditoned iterative process is independent of both the mesh size and the coefficients.

1. Introduction

In this paper, we design preconditioning operators for the system of grid equations approximating the following boundary value problem:

$$\begin{cases} -\sum\limits_{i,j=1}^{2} \frac{\partial}{\partial x_i} a_{ij}(x) \frac{\partial u}{\partial x_j} + a_0(x)u = f(x), & x \in \Omega, \\ u(x) = 0, \quad x \in \Gamma \end{cases} \tag{1.1}$$

We assume that Ω is a bounded, polygonal region and Γ is its boundary. Let Ω be a union of n nonoverlapping subdomains Ω_i,

$$\overline{\Omega} = \bigcup_{i=1}^{n} \overline{\Omega}_i, \quad \Omega_i \bigcap \Omega_j = \emptyset, \quad i \neq j,$$

where Ω_i are polygons and Γ_i are their boundaries. Let Ω^h

$$\Omega^h = \bigcup_{i=1}^{n} \Omega_i^h$$

be a regular triangulation of Ω which is characterized by a parameter h.

1991 *Mathematics Subject Classification.* Primary 65N55, 65N30.
This paper is in final form and no version of it will be submitted for publication elsewhere.

Let us introduce the weighted Sobolev spaces $H^1_\alpha(\Omega)$ with the norms [11]

$$\begin{aligned} \|u\|^2_{H^1_\alpha(\Omega)} &= \|u\|^2_{L_2(\Omega)} + |u|^2_{H^1_\alpha(\Omega)}, \\ \|u\|^2_{L_2(\Omega)} &= \int_\Omega u^2(x)dx, \qquad (1.2) \\ |u|^2_{H^1_\alpha(\Omega)} &= \int_\Omega \left(\frac{|\nabla u(x)|}{(\varrho(x))^{\alpha(x)}}\right)^2 dx. \end{aligned}$$

Here

$$\alpha(x) \equiv \alpha_i = \text{ const}, \; x \in \Omega_i$$

and $\varrho(x)$ is the distance between the point $x \in \Omega_i$ and the boundary Γ_i of the subdomain Ω_i. We assume that

$$|\alpha| < \frac{1}{2}. \tag{1.3}$$

Denote by $\overset{\circ}{H}{}^1_\alpha(\Omega)$ the subspace of $H^1_\alpha(\Omega)$ with zero trace on Γ and introduce the bilinear form

$$a(u,v) = \int_\Omega \left(\sum_{i,j=1}^{2} a_{ij}(x) \frac{\partial u}{\partial x_j} \frac{\partial v}{\partial x_i} + a_0(x)uv \right) dx$$

and the linear functional

$$l(v) = \int_\Omega f(x)vdx.$$

We assume that the coefficients of the problem (1.1) are such that $a(u,v)$ is a symmetric, coercive and continuous form on $\overset{\circ}{H}{}^1_\alpha(\Omega) \times \overset{\circ}{H}{}^1_\alpha(\Omega)$, and that the linear functional $l(v)$ is continuous in $\overset{\circ}{H}{}^1_\alpha(\Omega)$.

Denote by W a space of real–valued continuous functions linear on triangles of the triangulation Ω^h. A weak formulation of (1.1) is: Find $u \in \overset{\circ}{H}{}^1_\alpha(\Omega)$ such that

$$a(u,v) = l(v), \quad \forall v \in \overset{\circ}{H}{}^1_\alpha(\Omega). \tag{1.4}$$

Using the finite element method, we can pass from (1.4) to the linear algebraic system

$$Au = f. \tag{1.5}$$

The condition number of the matrix A depends on h, α and can be large. Our purpose is the design of a preconditioner B for the problem (1.5) such that the following inequalities are valid:

$$c_1(Bu,u) \le (Au,u) \le c_2(Bu,u), \quad \forall u \in R^N. \tag{1.6}$$

Here N is the dimension of W, the positive constants c_1, c_2 are independent of h and α, and the action of B^{-1} on a vector can be realized at low cost.

2. Additive Schwarz method for singular problems

The construction of the preconditioner for the system (1.5) will be realized on the basis of the additive Schwarz method [**1, 3, 4**]. To design the preconditioning operator B, we follow [**7, 10**] and decompose the space W into a sum of subspaces

$$W = W_0 + W_1.$$

To this end, divide the nodes of the triangulation Ω^h into two groups: those which lie inside of Ω_i^h and those which lie on boundaries of Ω_i^h. The subspace W_0 corresponds to the first set. Let

$$S = \bigcup_{i=1}^{n} \partial\Omega_i^h,$$

$$W_0 = \left\{u^h \in W \;\middle|\; u^h(x) = 0, x \in S\right\},$$

$$W_{0,i} = \{u^h \in W_0 \;\Big|\; u^h(x) = 0, \quad x \overline{\in} \Omega_i^h\}, \quad i = 1, 2, \ldots, n.$$

It is clear that W_0 is the direct sum of the orthogonal subspaces $W_{0,i}$ with respect to the scalar product in $\overset{\circ}{H}{}^1_\alpha(\Omega)$:

$$W_0 = W_{0,1} \oplus \cdots \oplus W_{0,n}.$$

The subspace W_1 corresponds to the second group of nodes Ω^h and can be defined in the following way. First, define V which is the space of traces of functions from W on S:

$$V = \{\varphi^h | \quad \varphi^h(x) = u^h(x), \quad x \in S, \quad u^h \in W\}.$$

To define the subspace W_1, we need a norm preserving extension operator of functions given at S into Ω^h. The basis of the further construction is the following trace theorem for the weighted Sobolev spaces $H^1_\alpha(\Omega)$ [**11**]:

THEOREM 2.1. *Let Ω be a bounded domain with piecewise–smooth boundary Γ from the class C^2 satisfying the Lipschitz condition and α is a constant such that $|\alpha| < \frac{1}{2}$. Then there exists a positive constant c_1 independent of α, such that*

$$\|\varphi\|_{H^{\frac{1}{2}+\alpha}(\Gamma)} \leq c_1 \|u\|_{H^1_\alpha(\Omega)}$$

for any function $u \in H^1_\alpha(\Omega)$, where $\varphi \in H^{\frac{1}{2}+\alpha}(\Gamma)$ is the trace of u at the boundary Γ. Conversely, there exists a positive constant c_2, independent of α, such that for any function $\varphi \in H^{\frac{1}{2}+\alpha}(\Gamma)$ there exist $u \in H^1_\alpha(\Omega)$ such that

$$\begin{aligned} u(x) &= \varphi(x), \quad x \in \Gamma, \\ \|u\|^2_{H^1_\alpha(\Omega)} &\leq c_2 \|\varphi\|^2_{H^{\frac{1}{2}+\alpha}(\Gamma)}. \end{aligned}$$

Here $\|\varphi\|_{H^{\frac{1}{2}+\alpha}(\Gamma)}$ is the norm in the Sobolev space $H^{\frac{1}{2}+\alpha}(\Gamma)$:

$$\begin{aligned} \|\varphi\|^2_{H^{\frac{1}{2}+\alpha}(\Gamma)} &= \|\varphi\|^2_{L^2(\Gamma)} + |\varphi|^2_{H^{\frac{1}{2}+\alpha}(\Gamma)}, \\ \|\varphi\|^2_{L^2(\Gamma)} &= \int_\Gamma \varphi^2(x)dx, \\ |\varphi|^2_{H^{\frac{1}{2}+\alpha}(\Gamma)} &= \int_\Gamma \int_\Gamma \frac{(\varphi(x)-\varphi(y))^2}{|x-y|^{2+2\alpha}}\, dxdy. \end{aligned} \tag{2.1}$$

Denote

$$H(S) = \left\{\varphi| \quad \varphi\Big|_{\Gamma_i} = \varphi_i, \quad \varphi_i \in H^{\frac{1}{2}+\alpha_i}(\Gamma_i)\right\},$$

$$\|\varphi\|^2_{H(S)} = \sum_{i=1}^n \|\varphi\|^2_{H^{\frac{1}{2}+\alpha_i}(\Gamma_i)}.$$

To define the bounded extension operator for the finite element case from V into W, we need mesh counterparts of the norms (1.2) and (2.1). To this end, let us split the triangles $\mathcal{T}_j$ of the triangulation Ω^h into three groups. Denote by M_1 a set of such $\mathcal{T}_j$ that $\mathcal{T}_j$ do not have vertices on S, denote by M_2 a set of such $\mathcal{T}_j$ that $\mathcal{T}_j$ have only one vertex on S, and denote by M_3 a set of such $\mathcal{T}_j$ that $\mathcal{T}_j$ have more than one vertex on S. Set

$$\begin{aligned} \|u^h\|^2_{H^1_{\alpha,h}(\Omega)} &= \sum_{i=1}^n \|u^h\|^2_{H^1_{\alpha_i,h}(\Omega_i)}, \\ \|u^h\|^2_{H^1_{\alpha_i,h}(\Omega_i)} &= \|u^h\|^2_{L_{2,h}(\Omega_i)} + |u^h|^2_{H^1_{\alpha_i,h}(\Omega_i)}, \\ \|u^h\|^2_{L_{2,h}(\Omega_i)} &= \sum_{z_j\in\Omega_i^h} (u^h(z_j))^2 h^2, \\ |u^h|^2_{H^1_{\alpha_i,h}(\Omega_i)} &= \sum_{\mathcal{T}_j\in M_1\cap\Omega_i} \frac{(u_{j1}-u_{j2})^2+(u_{j2}-u_{j3})^2+(u_{j3}-u_{j1})^2}{(\varrho(\mathcal{T}_j,\Gamma_i))^{2\alpha_i}} + \\ &+ \sum_{\mathcal{T}_j\in M_2\cap\Omega_i} \frac{(u_{j1}-u_{j2})^2+(u_{j2}-u_{j3})^2+(u_{j3}-u_{j1})^2}{h^{2\alpha_i}} + \\ &+ \sum_{\mathcal{T}_j\in M_3\cap\Omega_i} \frac{(u_{j1}-u_{j2})^2+(u_{j2}-u_{j3})^2+(u_{j3}-u_{j1})^2}{(1-2\alpha_i)h^{2\alpha_i}}, \forall u^h \in W. \end{aligned}$$

Here z_i are vertices of $\Omega^h, u_{j1}, u_{j2}, u_{j3}$ are values of u^h at vertices of $\mathcal{T}_j$, and $\varrho(\mathcal{T}_j, \Gamma_i)$ is the distance between $\mathcal{T}_j$ and Γ_i.

Using the natural order of nodes on Γ_i, let us put for each node $z_j \in \Gamma_i$ into correspondence the node z_{j+1}, which is a node neighboring upon z_i, and set

$$\begin{aligned}
\|\varphi^h\|^2_{H_h(S)} &= \sum_{i=1}^{n} \|\varphi^h\|^2_{H_h^{\frac{1}{2}+\alpha_i}(\Gamma_i)}, \\
\|\varphi^h\|^2_{H_h^{\frac{1}{2}+\alpha_i}(\Gamma_i)} &= \|\varphi^h\|^2_{L_{2,h}(\Gamma_i)} + |\varphi^h|^2_{H_h^{\frac{1}{2}+\alpha_i}(\Gamma_i)}, \\
\|\varphi^h\|^2_{L_{2,h}(\Gamma_i)} &= \sum_{z_j \in \Gamma_i} (\varphi^h(z_j))^2 h, \\
|\varphi^h|_{H_h^{\frac{1}{2}+\alpha_i}(\Gamma_i)} &= \sum_{\substack{z_j \in \Gamma_i \\ j \neq k}} \sum_{z_k \in \Gamma_i} \frac{(\varphi^h(z_j) - \varphi^h(z_k))^2}{|z_j - z_k|^{2+2\alpha_i}} h^2 + \\
&\quad + \sum_{z_j \in \Gamma_i^h} \frac{(\varphi^h(z_j) - \varphi^h(z_{j+1}))^2}{(1-2\alpha_i)h^{2\alpha_i}}, \quad \forall \varphi^h \in V.
\end{aligned}$$

The following lemmas are valid.

LEMMA 2.1. *There exist positive constants c_3 and c_4, independent of α and h, such that*

$$c_3 \|u^h\|_{H^1_{\alpha_i}(\Omega_i)} \leq \|u^h\|_{H^1_{\alpha_i,h}(\Omega_i)} \leq c_4 \|u^h\|_{H^1_{\alpha_i}(\Omega_i)}, \quad \forall u^h \in W, \quad i = 1, 2, \ldots, n.$$

LEMMA 2.2. *There exist positive constants c_5 and c_6, independent of α and h, such that*

$$c_5 \|\varphi^h\|_{H^{\frac{1}{2}+\alpha_i}(\Gamma_i)} \leq \|\varphi^h\|_{H_h^{\frac{1}{2}+\alpha_i}(\Gamma_i)} \leq c_6 \|\varphi^h\|_{H^{\frac{1}{2}+\alpha_i}(\Gamma_i)}, \ \forall \varphi^h \in V, \quad i = 1, 2, \ldots, n.$$

To define W_1, let us use the explicit extension operator

$$t^h : V \to W, \tag{2.2}$$

which was suggested for regular elliptic second order problems. The definition and the realization algorithm were done in [**5, 6, 8**] and briefly can be described in the following way. Let us introduce the near–boundary coordinate system (s, n) which is defined in a δ–neighborhood of Γ_i. Here s defines a point P at Γ_i and n is the distance between the given point and Γ_i along the internal pseudonormal, whose direction at the angular points coincides with the bisectrix of the angle and along the smooth part the vector changes, for example, linearly. Set

$$\begin{gathered}
t : H(S) \to H^1_\alpha(\Omega), \\
t\varphi = u, \\
u(s,n) = \left(1 - \tfrac{n}{\delta}\right) \int\limits_s^{s+n} \tfrac{\varphi(t)}{n}\, dt,
\end{gathered} \tag{2.3}$$

where the function u is extended by zero in the rest of Ω. Using the auxiliary mesh, which is topologically equivalent to a uniform rectangular mesh, we can define the finite–element analogue t^h (2.2) of the operator t from (2.3). The following theorem is valid.

THEOREM 2.2. *There exists a positive constant c_7, independent of α and h, such that*

$$\|u^h\|_{H^1_{\alpha,h}(\Omega)} = \|t^h\varphi^h\|_{H^1_{\alpha,h}(\Omega)} \le c_7\|\varphi^h\|_{H_h^{\frac{1}{2}+\alpha}(S)}, \quad \forall\varphi^h \in V.$$

Remark 2.1. The cost of the actions of t^h and $(t^h)^*$ on vectors is $O(h^{-2})$ arithmetic operations (see [5] for details).

At last, we can define subspace W_1

$$W_1 = \{u^h| \quad u^h = t^h\varphi^h, \quad \varphi^h \in V\}.$$

It is obvious that

$$W = W_0 + W_1$$

and this decomposition of the space W is regular in the following sense.

THEOREM 2.3. *There exists a positive constant c_8, independent of α and h, such that for any function $u^h \in W$ there exist $u_i \in W_i, i = 0, 1$, such that*

$$u_0 + u_1 = u,$$

$$\|u_0\|_{H^1_\alpha(\Omega)} + \|u_1\|_{H^1_\alpha(\Omega)} \le c_8\|u\|_{H^1_\alpha(\Omega)}.$$

According to [8], we can construct a preconditioner for the subspace W_1 of the following form

$$B_1^+ = t^h\Sigma^{-1}(t^h)^*,$$

where Σ has to satisfy

$$c_9\|\varphi^h\|^2_{H(S)} \le (\Sigma\varphi, \varphi) \le c_{10}\|\varphi^h\|^2_{H(S)}, \quad \forall\varphi^h \in V.$$

Here the components of the vector φ are equal to the values of the function φ^h in corresponding nodes. The constants c_9, c_{10} should be independent of α and h. The construction of the easily invertible operator (matrix) Σ can be done, using [4, 6, 7]. The cost of the action B_1^+ on vectors is $O(h^{-2})$ arithmetic operations.

3. Preconditioning operator for W_0.

The goal of this section is the design of the preconditioning operator for the space W which was defined in Section 2. Since W_0 is the direct sum of the orthogonal subspaces $W_{0,i}, i = 1, 2, \ldots, n$ which correspond to the subdomains Ω_i, we can design preconditioners independently for each subdomain Ω_i with the boundary Γ_i. For the sake of simplicity, we omit the subscript i. To construct the preconditioner, we use the additive Schwarz method. Let us decompose the domain Ω into two overlapping parts

$$\begin{gathered} \Omega = \Omega_{in} \bigcup \Omega_b, \\ \Omega_b = \{x = (s, n) \in \Omega| \quad 0 \le s \le L, \quad 0 < n < 2\delta\}, \\ \text{dist } (\Gamma, \partial\Omega_{in}) = \delta. \end{gathered} \tag{3.1}$$

Here (s, n) is the near-boundary coordinate system; $\delta = 0(1)$ is independent of h; for the sake of simplicity, we assume that Ω is the simply connected domain and L is the length of Γ. Then the triangulation Ω^h can be decomposed into two overlapping parts

$$\Omega^h = \Omega^h_{in} \bigcup \Omega^h_b,$$
$$\Omega^h_{in} = \bigcup_{\mathcal{T}_j \subset \Omega_{in}} \mathcal{T}_j,$$
$$\Omega^h_b = \bigcup_{\mathcal{T}_j \subset \Omega_b} \mathcal{T}_j$$

and the finite element space W_0 can be decompose into two overlapping subspaces

$$W_0 = W_{in} + W_b,$$
$$W_{in} = \left\{u^h \in W_0 | \quad u^h(x) = 0, \quad x \overline{\in} \Omega^h_{in}\right\},$$
$$W_b = \left\{u^h \in W_0 | \quad u^h(x) = 0, \quad x \overline{\in} \Omega^h_b\right\}.$$

Using (3.1), it is easy to see that there exists a positive constant c_1, independent of α and h, such that for any $u^h \in W_0$ there exists $u^h_{in} \in W_{in}$ and $u^h_b \in W_b$ such that

$$u^h_{in} + u^h_b = u^h,$$
$$\|u^h_{in}\|_{H^1_\alpha(\Omega)} + \|u^h_b\|_{H^1_\alpha(\Omega)} \leq c_1 \|u\|_{H^1_\alpha(\Omega)}.$$

Then, according to [**3**], we can define the preconditioner in the following form

$$B^{-1} = B^+_{in} + B^+_b,$$

where B_{in} and B_b are such that

$$B_{in} : W \to W_{in},$$
$$c_2 \|u^h\|^2_{H^1_\alpha(\Omega)} \leq (B_{in} u, u) \leq c_3 \|u^h\|^2_{H^1_\alpha(\Omega)}, \quad \forall u^n \in W_{in},$$
$$B_b : W \to W_b,$$
$$c_2 \|u^h\|^2_{H^1_\alpha(\Omega)} \leq (B_b u, u) \leq c_3 \|u^h\|^2_{H^1_\alpha(\Omega)}, \quad \forall u^n \in W_b.$$

Here c_2 and c_3 are independent of α and h. From (3.1) we have that there exist positive constants c_4 and c_5, independent of α and h, such that

$$c_4 \|u^h\|_{H^1(\Omega)} \leq \|u^h\|_{H^1_\alpha(\Omega)} \leq c_5 \|u^h\|_{H^1(\Omega)}, \quad \forall u^n \in W_{in}.$$

This implies that the construction of B_{in} is equivalent to the construction of preconditioners for regular elliptic problems. For instance, using combinations of the domain decomposition and fictitious domain methods, the construction of effective preconditioners was studied in [**4, 5, 10**]. A new element of the construction of the preconditioner B is the construction of B_b. To this end, let us decompose Ω_b into overlapping parts

$$\Omega_b = \bigcup_{i=L}^{l} D_i,$$

$$D_i = \left\{x = x(s,n) \in \Omega_b | \quad (i-1)\frac{L}{l} < s < (i+1)\frac{L}{l}, 0 < n < \delta\right\},$$

where $l = O(1)$, i.e. the number of subdomains is fixed, and $x(L+s,n) = x(s,n)$. Then the triangulation Ω_b^h can be decomposed into overlapping parts

$$\Omega_b^h = \bigcup_{i=1}^{l} D_i^h,$$
$$D_i^h = \bigcup_{\mathcal{T}_j \subset D_i} \mathcal{T}_j$$

and the space W_b can be decomposed into overlapping subspaces

$$W_{in} = \sum_{i=1}^{l} U_i,$$
$$U_i = \{u^h \in W_{in} | \quad u^h(x) = 0, x \bar{\in} D_i^h\}.$$

The following lemma is valid.

LEMMA 3.1. *Let Ω be a rectangular domain*

$$\Omega = \{(x_1, x_2)| \quad -1 < x_1 < 1, 0 < x_2 < 1\}$$

and Ω^h be a regular triangulation of Ω. Denote by W a space of real-valued continuous functions linear on triangles of the triangulation Ω^h. Then, there exists a positive constant c_6, independent of α and h, such that $\forall u^h \in W$

$$\int_{-1}^{1}\int_{0}^{1}\left(\frac{1}{x_2^{2\alpha}}|\nabla\tilde{u}^h|^2 + (\tilde{u}^h)^2\right)dx_2dx_1 \le c_6 \int_{0}^{1}\int_{0}^{1}\left(\frac{1}{x_2^{2\alpha}}|\nabla u^h|^2 + (u^h)^2\right)dx_2dx_1,$$

for any constant $\alpha : |\alpha| < 1/2$. Here the function $\tilde{u}^h \in W$ is defined in the following way:

$$\tilde{u}^h(z_i) = \tilde{u}^h(x_{1i}, x_{2i}) = \begin{cases} u^h(z_i), z_i \in [0,1] \times [0,1], \\ (1 + x_{1i})u^h(\tilde{z}_i), z_i \in [-1,0] \times [0,1]. \end{cases}$$

Here $\tilde{z}_i$ is a node of Ω^n which is the nearest for the point with the coordinates $(-x_{1i}, x_{2i})$.

Using Lemma 3.1, it is easy to see that there exists a positive constant c_7, independent of α and h, such that for any $u^h \in W_{in}$ there exists $u_i^h \in U_i$:

$$(3.2) \qquad \begin{aligned} u_1^h + \cdots + u_l^h = u^h, \\ c_7\left(\|u_1^h\|_{H_\alpha^1(\Omega)} + \cdots + \|u_l^h\|_{H_\alpha^1(\Omega)}\right) \le \|u^h\|_{H_\alpha^1(\Omega)}. \end{aligned}$$

According to [**3**] and (3.2), to define the operator B_{in}, we can define the easily invertible norms for subspaces $V_i, i = 1, 2, \ldots, l$. To this end, we use the fictitious space lemma [**5, 9**].

To design the easily invertible norm in U_i, we consider an auxiliary topologically uniform mesh. For the sake of simplicity, we omit the subscript i.

Let us assume that the domain D in the near-boundary coordinate system (s, n) has the following representation

$$D = \{x = (s,n)| \quad 0 < s < \xi, \quad 0 < n < \delta\}.$$

Introduce in the domain D the auxiliary mesh Q^h with the mesh size h_0 and the nodes z_{ij}

$$z_{ij} = (s_i, n_j), \quad s_i = i \cdot h_0, \quad n_j = j \cdot h_0$$

$$i = 0, 1, \dots, n, \quad j = 0, 1, \dots, m,$$

$$n \cdot h_0 = \xi, \quad m \cdot h_0 = \delta.$$

Assume that $h_0 \le h_{min}/2$, where h_{min} is the length of the minimal side of triangles of the triangulation D^h. Denote the cells of the mesh Q^h by Q_{ij}

$$\begin{aligned} Q_{ij} &= \{x = (s, n) | \quad s_i \le s < s_{i+1}, \ n_j \le n < n_{j+1}\}, \\ & \quad i = 0, 1, \dots, n-1 \\ & \quad j = 0, 1, \dots, m-1. \end{aligned}$$

On the mesh Q^h we will consider the mesh function $V(z_{ij})$ vanishing at the boundary nodes of the mesh Q^h. We will identify the mesh Q^h and the triangulation of D with the nodes z_{ij}. Denote by F a space of real-valued continuous functions V^h linear on triangles of the triangulation Q^h.

Using the tensor product of matrices, introduce

$$B = A \otimes J + I_{n-1} \otimes T,$$

where the tridiagonal matrix A of order $n-1$ approximates the second derivative and J is a diagonal matrix of order $m-1$

$$J = \text{diag}\left(\frac{1}{h_0^{2\alpha}}, \frac{1}{(2h_0)^{2\alpha}}, \dots, \frac{1}{((m-1)h_0)^{2\alpha}}\right);$$

The matrix I_{n-1} is the identity matrix of the order $n-1$; the tridiagonal matrix T of the order $m-1$:

$$\begin{bmatrix} \frac{1}{(1-2\alpha)h_o^{2\alpha}} + \frac{1}{(2h_o)^{2\alpha}} & -\frac{1}{(2h_o)^{2\alpha}} & & \\ -\frac{1}{(2h_o)^{2\alpha}} & \frac{1}{(2h_o)^{2\alpha}} + \frac{1}{(3h_o)^{2\alpha}} & & \\ \ddots & \ddots & \ddots & \\ & & & -\frac{1}{((m-1)h_o)^{2\alpha}} \\ & & -\frac{1}{((m-1)h_o)^{2\alpha}} & \frac{1}{((m-1)h_o)^{2\alpha}} + \frac{1}{(mh_o)^{2\alpha}} \end{bmatrix}$$

The following Lemma is valid.

LEMMA 3.2. *There exist positive constants c_8, c_9, independent of α and h, such that*

$$c_8 \|V^h\|^2_{H^1_\alpha(D)} \le (BV, V) \le c_9 \|V^h\|^2_{H^1_\alpha(D)}, \quad \forall V^h \in F$$

where the components of the vector V are equal to the values of the function V^h in the corresponding nodes.

Remark 3.1. Using the spectral decomposition of the matrix A

$$A = Q\Lambda Q^T,$$

we can invert the matrix B :

$$B^{-1} = Q \otimes I_{m-1}(\Lambda^{-1} \otimes J^{-1} + I_{n-1} \otimes T^{-1})Q^T \otimes I_{m-1}$$

Then the multiplication of vectors by B^{-1} can be performed in $O(h^{-2} \log h^{-1})$ arithmetic operations using the Fast Fourier Transform. To define the easily invertible norm in the space V, using [**9**], we need to define operators R and T:

$$R : F \to V, \quad T : V \to F.$$

Let us define the operator R which puts into the correspondence to each function $V^h(z_{ij}) \in F$ a function $u^h \in V$ in the following way. Let z_L be a node of the triangulation D^h and let $z_e \in Q_{ij}$. Set

$$u^h(z_l) = V^h(z_{ij}).$$

Note that by the assumption on h_0 only one node z_l of the triangulation D^h belonging to the cell Q_{ij} can exist. Then the operator T is defined as follows. If the cell Q_{ij} contains a node z_l of the triangulation D^h, we set

$$V^h(z_{ij}) = u^h(z_l).$$

At other nodes of the mesh Q^h the function $V^h(z_{ij})$ can be defined in a sufficiently arbitrary way, for instance, as follows. Let the node z_{ij} belong to the triangle $\mathcal{T}_l$ of the triangulation D^h with the vertices z_{l_1}, z_{l_2} and z_{l_3}. Set

$$V^h(z_{ij}) = \frac{1}{3}(u^h(z_{l_1}) + u^h(z_{l_2}) + u^h(z_{l_3})),$$

It is easy to see that the above-defined operators R and T satisfy the hypothesis of Lemma 4.3 while the constants c_{10} and c_{11} are independent of α and h:

$$\begin{aligned} \|RV^h\|^2_{H^1_\alpha(D)} &\leq c_{10}(BV, V), \quad \forall V^h \in F, \\ (BTu^h, Tu^h) &\leq c_{11}\|u^h\|_{H^1_\alpha(D)}, \quad \forall u^h \in V. \end{aligned}$$

Then, the following theorem is valid:

THEOREM 3.1. *There exist positive constants c_{12} and c_{13}, independent of α and h , such that*

$$c_{12}\|u^h\|^2_{H^1_\alpha(D)} \leq (C^{-1}u, u) \leq c_{13}\|u^h\|^2_{H^1_\alpha(D)}, \quad \forall u^h \in V,$$

$$C = RB^{-1}R^T,$$

where the components of the vector u are equal to the values of the functions u^h in the corresponding nodes.

REFERENCES

1. P.L. Lions, *On the Schwarz alternating method,* I. First International Symposium on Domain Decomposition Methods for Partial Differential Equations, (R.Glowinski, G.H. Golub, G. Meurant and J. Périaux, eds.), SIAM, Philadelphia,1988
2. G.I. Marchuk,*Methods of Numerical Mathematics,* Springer, New York, 1982
3. A.M. Matsokin and S.V. Nepomnyaschikh, *Schwarz alternating method in subspaces,* Soviet Mathematics, 29 (1985), pp. 78–84.
4. A.M. Matsokin and S.V. Nepomnyaschikh, *Norms in the space of traces of mesh functions,* Sov. J. Numer. Anal. Math. Modeling, 3 (1988), 199–216.
5. A.M. Matsokin and S.V. Nepomnyaschikh, *Method of fictitious space and explicit extension operators,* Zh. Vychisl. Mat. Mat. Fiz , 33(1993), 52–68.
6. S.V. Nepomnyaschikh, *Domain decomposition and Schwarz methods in a subspace for the approximate solution of elliptic boundary value problems,* Thesis, Computing Center of the Siberian Branch of the USSR Academy of Sciences, Novosibirsk, USSR, 1986.
7. S.V. Nepomnyaschikh, *Domain decomposition method for elliptic problems with discontinuous coefficients,* Proc. 4th Conference on Domain Decomposition methods for Partial Differential Equations, Philadelphia, PA, SIAM, 1991, 242–251.
8. S.V. Nepomnyaschikh, *Method of splitting into subspaces for solving elliptic boundary value problems in complex-form domains,* Sov. J. Numer. Anal. Math. Modelling, 6(1991), 151–168.
9. S.V. Nepomnyaschikh, *Mesh theorems on traces, normalization of function traces and their inversion,* Sov. J. Numer. Anal. Math. Modelling, 6(1991), 223–242.
10. S.V. Nepomnyaschikh, *Decomposition and fictitious domain methods for elliptic boundary value problems,* 5th Conference on Domain Decomposition Methods for Partial Differential Equations, Philadelphia, PA , SIAM, 1992.
11. S.M. Nikol'ski, *Approximation of functions of many variables and embedding theorems,* Nauka, Moscow, 1977 (Russian).
12. G.N. Yakovlev, *On the traces of functions from spaces W_p^l on piecewise-smooth surfaces,* Mat. Sbornik 74(1967), 526–543, (Russian).

RUSSIAN ACADEMY OF SCIENCES, COMPUTING CENTER, LAVRENTIEV AV. 6, NOVOSIBIRSK, 630090, RUSSIA.

E-mail address: svnep@comcen.nsk.su