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Stable Subspace Splittings for Sobolev Spaces and Domain Decomposition Algorithms

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ABSTRACT. The notion of a stable subspace splitting is basic for the theoretical understanding of some modern iterative methods for solving variational problems. For Sobolev spaces over polyhedral domains, examples of such splittings into finite element subspaces are given along with typical applications to multilevel and domain decomposition algorithms.

1. Introduction

Many modern iterative algorithms for solving elliptic p.d.e. discretizations can be interpreted as additive (Jacobi-like) or multiplicative (Gauss-Seidel-like) subspace correction methods, see [**27, 30, 10**]. The key to their analysis is the study of some metric properties of the underlying splitting of the discretization space V into a sum of subspaces V_j and of the variational problem on V into auxiliary problems on these subspaces. In Section 2, we start with a brief overview of the abstract theory for the symmetric positive definite case based on our joint paper with M. Griebel [**12**].

Investigation of such splittings for the solution of variational problems on Sobolev spaces benefits from already existing experience with decomposing elements of function spaces into simple building blocks. Approximation theory, Fourier analysis, and the theory of function spaces are helpful in this respect. Some examples of useful splittings of $H^s(\Omega)$ with respect to finite element subspaces over polyhedral domains in R^d are given in Section 3. We restrict ourselves to applications to second order elliptic boundary value problems (and some problems that are closely related), an analogous theory holds for fourth order problems, see [**17, 18, 32, 6**], for similar developments involving wavelets we refer to [**5, 7**] and the papers cited therein.

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In the final Section 4 we show how some domain decomposition algorithms can be derived along the lines of our approach.

2. Abstract Schwarz methods

Let V be some fixed Hilbert space, with the scalar product given by a continuous symmetric positive definite (s.p.d.) form $a(\cdot,\cdot) : V \times V \to R$. Note that at this stage V may be finite- or infinite-dimensional. Consider an arbitrary additive representation of V by the sum of a finite or infinite number of subspaces $V_j \subset V$:

$$V = \sum_j V_j \, , \tag{2.1}$$

this means that any $u \in V$ possesses at least one V-converging representation $u = \sum_j u_j$ where $u_j \in V_j$ for all j. Suppose that the V_j are equipped with auxiliary continuous s.p.d. forms $b_j(\cdot,\cdot) : V_j \times V_j \to R$. We call

$$\{V; a\} = \sum_j \{V_j; b_j\} \, , \tag{2.2}$$

stable splitting of $\{V; a\}$ if the quantity

$$|||u||| = \inf_{u_j \in V_j \,:\, u = \sum_j u_j} \sqrt{\sum_j b_j(u_j, u_j)} \tag{2.3}$$

defines an equivalent norm on V, i.e. if the bounds

$$\lambda_{\min} = \inf_{u \neq 0} \frac{a(u,u)}{|||u|||^2} \, , \quad \lambda_{\max} = \sup_{u \neq 0} \frac{a(u,u)}{|||u|||^2} \, , \tag{2.4}$$

are positive and finite. The quantity

$$\kappa \equiv \kappa(\{V; a\} = \sum_j \{V_j; b_j\}) = \frac{\lambda_{\max}}{\lambda_{\min}} \tag{2.5}$$

will be called *stability constant* or simply *condition number* of the splitting (2.2). Note that if the splitting (2.2) is into a finite number of subspaces then it is automatically stable and the difference is only in the size of κ. Also, stability and condition do not change if we change the ordering of the subspaces.

Introduce the operators $T_j : V \to V_j$ by the auxiliary variational problems

$$b_j(T_j u, v_j) = a(u, v_j) \qquad \forall \, v_j \in V_j \, . \tag{2.6}$$

If the splitting (2.2) is stable then it is easy to show that the associated additive Schwarz operator

$$P = \sum_j T_j \, : \, V \to V \tag{2.7}$$

is well-defined and s.p.d. on V, with exact lower and upper bounds for its spectrum given by the constants $\lambda_{\min}$ and $\lambda_{\max}$ from (2.4). Moreover, if for a given linear continuous functional Φ on V we define elements $\phi_j \in V_j$ and $\phi = \sum_j \phi_j \in V$ via

$$b_j(\phi_j, v_j) = \Phi(v_j) \qquad \forall\, v_j \in V_j\ , \tag{2.8}$$

then the given variational problem

$$\textit{find } u \in V \textit{ such that} \qquad a(u,v) = \Phi(v) \qquad \forall\, v \in V\ , \tag{2.9}$$

is equivalent to the operator equation

$$\textit{find } u \in V \textit{ such that} \qquad Pu = \phi \quad . \tag{2.10}$$

This is the so-called additive Schwarz formulation of (2.9) associated with the splitting (2.2), see [**27, 30, 10, 12**] for historical references.

In [**12**] we gave another reformulation of (2.9) as operator equation in the product Hilbert space $\tilde{V} = \times_j V_j$ which is useful in connection with the treatment of additive and multiplicative subspace correction methods. In order not to talk about computationally irrelevant situations, let from now on (2.2) be a finite splitting, i.e. let $j = 0, \dots, J$, and suppose in addition that V is finite-dimensional. The additive algorithm associated with the splitting (2.2) is typically defined as the Richardson method applied to (2.10):

$$u^{(l+1)} = u^{(l)} - \omega(Pu^{(l)} - \phi) = u^{(l)} - \omega \sum_{j=0}^{J} (T_j u^{(l)} - \phi_j)\ , \tag{2.11}$$

$l = 0, 1, \dots,$ with $u^{(0)} \in V$ any given initial approximation to the solution u of (2.9) resp. (2.10), and ω a relaxation parameter. Alternatively, one may apply the conjugate gradient method to the equation (2.10), relying on the same theoretical analysis.

In contrast to the parallel incorporation of the subspace corrections $r_j^{(l)} = T_j u^{(l)} - \phi_j$ into the iteration (2.11), the multiplicative algorithm uses them in a sequential way:

$$v^{(l+(j+1)/(J+1))} = v^{(l+j/(J+1))} - \omega(T_j v^{(l+j/(J+1))} - \phi_j)\ , \tag{2.12}$$

where $j = 0, \dots, J\ ,\ l = 0, 1, \dots$.

The simple observation which was made in [**12**] is that the analysis of the abstract methods (2.11), (2.12) can be carried out in almost the same spirit as in the traditional block-matrix situation if one switches from the operator P to the matrix-operator

$$\tilde{P} = \{T_i|_{V_j}\} \tag{2.13}$$

acting in the auxiliary Hilbert space $\tilde{V}$. Let $\tilde{P} = \tilde{L} + \tilde{D} + \tilde{U}$ be the decomposition of $\tilde{P}$ into lower triangular, diagonal, and upper triangular parts (note that $\tilde{U} =$

$\tilde{L}^T$ since $\tilde{P}$ is symmetric positive semi-definite in $\tilde{V}$). The following result which explains also the central role of the above stability concept is contained in [**12**], see [**14**] for statements of this type in the matrix case.

THEOREM 2.1. *Suppose that V is finite-dimensional, and that the splitting (2.2) is finite. Let the characteristic numbers $\lambda_{\max}, \lambda_{\min}$, and κ of the splitting be given by* (2.4), (2.5).
(a) *The additive method* (2.11) *converges for $0 < \omega < 2/\lambda_{\max}$, with the optimal asymptotic convergence rate*

$$\rho^*_{as} = 1 - \frac{2}{1+\kappa} \qquad (\, \omega = \frac{2}{\lambda_{\max} + \lambda_{\min}} \,) \,. \tag{2.14}$$

(b) *The multiplicative method* (2.12) *converges for $0 < \omega < 2/\|\tilde{D}\|$, with a bound for the optimal asymptotic convergence rate given by*

$$(\rho^*_{ms})^2 \leq 1 - \frac{2}{(\sqrt{q^2+1}+1)\kappa} \,, \qquad (\, \omega = \frac{2(\sqrt{q^2+1}-1)}{\lambda_{\max}} \,) \,, \tag{2.15}$$

where the quantity $q = 2\|\tilde{L}\|/\lambda_{\max}$ can be further estimated under additional assumptions, e.g. assuming the validity of strengthened Cauchy-Schwarz inequalities for the splitting (see [**27, 30, 10**] *). Without additonal assumptions, we have a guaranteed worst case estimate $q \leq [\log_2(4J)]$ which leads to*

$$\rho^*_{ms} \leq 1 - \frac{1}{\alpha_J \kappa} \quad , \qquad \alpha_J \approx \log_2 J \,, \; J \to \infty \,. \tag{2.16}$$

For full proofs and more details, see [**12**]. Note that several modifications of the above basic additive and multiplicative schemes, e.g. the analog of symmetric SOR, may be studied along the same lines. Though the estimate (2.16) is asymptotically (for $J \to \infty$) best possible in the general case [**21**], it is too rough to explain the better convergence rates of the multiplicative scheme observed in practical applications to special problem classes.

Concluding this section, we want to emphasize the crucial role played by the condition number of the splitting for both the additive and multiplicative algorithms. In our opinion, the derivation of a computationally suitable algorithm should include a thorough analysis of the behavior of the stability constants in order to make sure that the method is close to an optimal one. In the remaining sections we implement this strategy in a particular situation: we briefly present basic splittings for typical variational problems in Sobolev spaces, and apply them to derive some known domain decomposition algorithms.

3. Splittings for C^0 finite elements

Let $\Omega \subset R^d$ be a bounded polyhedral domain equipped with a nested sequence of partitions

$$\mathcal{T}_0 \prec \mathcal{T}_1 \prec \ldots \prec \mathcal{T}_j \prec \ldots \tag{3.1}$$

into d-dimensional simplices (or, if the domain is rectangular-like, into d-dimensional rectangles etc.) which are regular and quasi-uniform, with constants that are independent of j. Suppose that

$$h_j \equiv \max_{\Delta \in \mathcal{T}_j} \operatorname{diam}(\Delta) \approx 2^{-j}\,, \quad j = 0, 1, \ldots\,. \tag{3.2}$$

In practice, (3.1) is often produced by regular dyadic refinement from an initial partition, and the constants characterizing regularity and quasi-uniformity in (3.1), (3.2) depend only on $\mathcal{T}_0$.

Consider the sequence of linear C^0 Lagrange finite element subspaces

$$S_0 \subset S_1 \subset \ldots \subset S_j \subset \ldots \tag{3.3}$$

corresponding to (3.1). The usual nodal basis of S_j will be denoted by $\mathcal{N}_j = \{N_{j,P} : P \in \mathcal{V}_j\}$ where $\mathcal{V}_j$ is the set of all vertices (or nodal points) of $\mathcal{T}_j$. Let $S_{j,P}$ be the one-dimensional subspace of S_j spanned by the nodal basis function $N_{j,P}$ $(P \in \mathcal{V}_j\,, j = 0, 1, \ldots)$.

For the definition of the Sobolev spaces $H^s(\Omega)$ we refer to [**25, 26, 13**]). The following theorem is essentially contained in [**15**], see also [**22**] or our survey [**19**], other proofs have recently been given by Bramble, Pasciak [**2**], Xu [**27**], Zhang [**31**], Dahmen, Kunoth [**5**], Bornemann, Yserentant [**1**].

THEOREM 3.1. *Let $0 < s < 3/2$. Suppose that $a(\cdot,\cdot)$ is a symmetric H^s-elliptic bilinear form on $H^s(\Omega)$. Then, under the above assumptions on* (3.3), *the following splittings are stable:*

$$\{H^s(\Omega); a(\cdot,\cdot)\} = \sum_{j=0}^{\infty} \{S_j; 2^{2sj}(\cdot,\cdot)_{L_2(\Omega)}\} \tag{3.4}$$

$$\{H^s(\Omega); a(\cdot,\cdot)\} = \sum_{j=0}^{\infty} \sum_{P \in \mathcal{V}_j} \{S_{j,P}; 2^{2sj}(\cdot,\cdot)_{L_2(\Omega)}\} \tag{3.5}$$

$$\{H^s(\Omega); a(\cdot,\cdot)\} = \sum_{j=0}^{\infty} \sum_{P \in \mathcal{V}_j} \{S_{j,P}; a(\cdot,\cdot)\} \tag{3.6}$$

The characteristic constants $\lambda_{\min}$, $\lambda_{\max}$, and κ for these splittings depend only on the constants characterizing the regularity and quasi-uniformity of the partitions, on s, and on the ellipticity constants of the bilinear form.

We will call these splittings *basic* since many other results about computationally relevant splittings can be deduced from Theorem 3.1. Note that (3.5) and (3.6) are consequences of (3.4). Indeed, the L_2-stability of the nodal basis and a comparison of the L_2 and H^s norms for nodal basis functions gives the stability of the splittings

$$(3.7)\ \{S_j; 2^{2sj}(\cdot,\cdot)_{L_2(\Omega)}\} = \sum_{P\in\mathcal{V}_j}\{S_{j,P}; 2^{2sj}(\cdot,\cdot)_{L_2(\Omega)}\} = \sum_{P\in\mathcal{V}_j}\{S_{j,P}; a(\cdot,\cdot)\}\,,$$

with uniformly bounded condition numbers κ_j in both cases. Thus, it remains to substitute (3.7) into (3.4) to get the remaining splittings of Theorem 3.1.

We do not state the immediate corollaries of Theorem 3.1 concerning splittings for the trace spaces

$$H^s(\Omega)|_\Gamma = \{h \in L_2(\Gamma)\ :\ \exists f \in H^s(\Omega) : h = f|_\Gamma,\ \|h\|_{H^s|_\Gamma} = \inf_{h=f|_\Gamma}\|f\|_{H^s(\Omega)}\}$$

(this class essentially coincides with $H^{s-1/2}(\Gamma)$), and for the spaces

$$H^s_\Gamma(\Omega) = \{f \in H^s(\Omega)\ :\ f|_\Gamma = 0\}$$

which are necessary to handle Dirichlet boundary conditions for second order elliptic problems (in both cases it is assumed that $1/2 < s < 3/2$, and that Γ is the union of some $(d-1)$-dimensional faces of simplices from $\mathcal{T}_0$). Instead, we quote some computationally relevant finite splittings which fit the assumptions of Theorem 2.1 and lead to fast subspace correction methods for solving elliptic finite element discretizations.

THEOREM 3.2. *Let the assumptions of Theorem* 3.1 *be fulfilled. Then, for all* $0 \le j_0 < J < \infty$, *the following finite splittings possess uniform condition number estimates depending only on* s *and on the* κ *in Theorem* 3.1*:*

$$(3.8)\qquad \{S_J; a(\cdot,\cdot)\} = \{S_{j_0}; a(\cdot,\cdot)\} + \sum_{j=j_0+1}^{J}\sum_{P\in\mathcal{V}_j}\{S_{j,P}; 2^{2sj}(\cdot,\cdot)_{L_2(\Omega)}\}$$

$$(3.9)\qquad \{S_J; a(\cdot,\cdot)\} = \{S_{j_0}; a(\cdot,\cdot)\} + \sum_{j=j_0+1}^{J}\sum_{P\in\mathcal{V}_j}\{S_{j,P}; a(\cdot,\cdot)\}$$

$$(3.10)\qquad \{S_J; a(\cdot,\cdot)\} = \{S_{j_0}; a(\cdot,\cdot)\} + \sum_{P\in\mathcal{V}_J}\{S_{P,j_0,J}; a(\cdot,\cdot)\}$$

where $S_{P,j_0,J} = span\{N_{j,P}\ :\ \forall j_0 < j \le J\ :\ P \in \mathcal{V}_j\}$. *In all cases,* $a(\cdot,\cdot)$ *is the natural restriction of the bilinear form defined on* $H^s(\Omega)$ *onto the respective subspaces. The statement holds for* $j_0 = -1$ *if the first term in the splittings is dropped (*$S_{-1} = \{0\}$*). The extension of the results to the subspaces* $S_{J,\Gamma} = \{g \in S_J\ :\ g|_\Gamma = 0\}$ *of* $H^s_\Gamma(\Omega)$, $1/2 < s < 3/2$, *is immediate.*

The algorithm behind (3.8) was introduced by Bramble, Pasciak, Xu [**4**], see also [**29, 1**], (3.9) goes back to X.Zhang [**31**]. The third splitting (3.10) was introduced and analyzed (for $j_0 = -1$) by Griebel, see [**11**]. For details of the proof of Theorem 3.2, see [**19, 22**].

All splittings presented in Theorem 3.2 are overlapping. One may ask for uniformly stable nonoverlapping splittings for $\{S_J; a(\cdot,\cdot)\}$. Stable nonoverlapping splittings of a Hilbert space into one-dimensional subspaces are actually equivalent to unconditional Schauder bases (or Riesz bases) which leads to a classical problem that is interesting on its own. Orthonormal bases are particular cases but hard to construct, especially for Sobolev spaces on domains. It is still an open question to define and implement a prewavelet-like system of locally supported finite element functions corresponding to (3.3) such that it forms an unconditional basis in $L_2(\Omega)$ for arbitrary regular and quasi-uniform sequences of partitions (3.1) (see [**19**] for the consequences of the existence of such a system in the spirit of Theorem 3.2). For nice domains and sequences of shift-invariant partitions, such constructions are essentially known.

Another early attempt to use splittings into a direct sum of one-dimensional subspaces is the hierarchical basis method introduced by Yserentant [**28, 29**]. The condition number estimates for the hierarchical basis splitting can also be obtained as consequences of Theorem 3.1, see [**16**].

Let us briefly mention the case of higher degree Lagrange elements. There are two ways to deal with them: on the one hand, we can develop the whole machinery of infinite splittings of Sobolev spaces for these elements, see [**15**]. On the other hand, if we are mostly interested in algorithms, we can simply reduce the construction of iterative methods for the new element types to the case of linear elements on the same sequence of partitions by a procedure which corresponds to condensation of inner variables. This latter approach seems to be preferable for several reasons, applications to nonconforming schemes have been described in [**20**]. However, we do not know about a serious performance comparison of these two possibilities. For Hermite or serendipity elements where the monotonicity of the family of finite element subspaces (3.3) is violated, one is recommended to use the second strategy.

It is well-known that iterative methods based on subspace splittings of the above type can be carried over to an adaptive environment. What is not so easy (and, therefore, a drawback of our approach) is to overcome the restrictions on the geometry of the domain and on the construction of the partitions implicitely contained in (3.1), (3.2). Domains that do not allow for a simple initial partition into a few simplices of diameter ≈ 1 or cannot be reduced to this situation after a dilation tend to produce theoretically larger condition number estimates. Since the underlying triangulation of a finite-element discretization space may be produced by some grid-generation or -optimization method which does not care about having a sequence of partitions (3.1) but rather provides us with some $\mathcal{T}_J$ we may have some trouble. Also, non-polyhedral domains and the treatment of isoparametric elements which occur in engineering problems require additional ideas and arguments to get a smooth theory (this does not necessarily mean that the algorithms, if properly adopted, will not work for these new situations).

4. Applications to domain decomposition

The use of Theorem 3.1 and 3.2 for domain decomposition methods is quite obvious, similar ideas are contained in [**23, 24**]. Throughout this section, we assume that the conditions on the domain, on the sequence of partitions (3.1), (3.2), and on the sequence of linear finite element subspaces (3.3) are the same as in Section 2. Moreover, to simplify the notation, we will consider a symmetric H^1-elliptic variational problem $a(\cdot,\cdot)$ on $H^1(\Omega)$, the case of essential boundary conditions is completely analogous.

4.1. Nonoverlapping domain decomposition schemes. Suppose that Ω is decomposed into nonoverlapping subdomains. In order to keep the considerations simple and to use the basic splittings in a straightforward way, we assume that the subdomains are identical with the simplices of some of the partitions $\mathcal{T}_{j_0}$ (we may actually allow groups of less than a fixed number of such simplices to form the subdomains). We denote the subdomains by Ω_l, and introduce the notations $S_{j,l}$ for the set of finite element functions from S_j with support in Ω_l, $\mathcal{V}_{j,l}$ for the set of nodal points interior to Ω_l, and set $\mathcal{V}_{j,\gamma} = \mathcal{V}_j \setminus \cup_l \mathcal{V}_{j,l}$ for the part of $\mathcal{V}_j$ located on $\gamma = \cup_l(\partial\Omega_l \cap \Omega)$. Here, $j \geq j_0$, by our assumptions the subspaces $S_{j_0,l}$ are trivial, accordingly, $\mathcal{V}_{j_0,l}$ is empty.

For any $J > j_0$, consider the splitting (3.8) of Theorem 3 and group the one-dimensional subspaces as follows:

$$\begin{aligned}(4.1)\quad \{S_J; a(\cdot,\cdot)\} = \sum_l \left(\sum_{j=j_0+1}^{J} \sum_{P\in\mathcal{V}_{j,l}} \{S_{j,P}; 2^{2j}(\cdot,\cdot)_{L_2(\Omega_l)}\} \right) \\ + \left(\{S_{j_0}; a(\cdot,\cdot)\} + \sum_{j=j_0+1}^{J} \sum_{P\in\mathcal{V}_{j,\gamma}} \{S_{j,P}; 2^{2j}(\cdot,\cdot)_{L_2(\Omega)}\} \right) .\end{aligned}$$

The first sum contains groups of subspaces that form splittings of the $S_{J,l}$. Using a simple scaling argument, we can apply Theorem 3, (3.8), on the subdomains Ω_l:

$$\{S_{J,l}; a(\cdot,\cdot)\} = \sum_{j=j_0+1}^{J} \sum_{P\in\mathcal{V}_{j,l}} \{S_{j,P}; 2^{2j}(\cdot,\cdot)_{L_2(\Omega_l)}\}$$

are stable splittings, with a common bound for their condition numbers which is independent of l. Thus, we arrive at another stable splitting

$$(4.2)\qquad \{S_J; a(\cdot,\cdot)\} = \sum_l \{S_{J,l}; a(\cdot,\cdot)\}$$

$$+\left(\{S_{j_0}; a(\cdot,\cdot)\} + \sum_{j=j_0+1}^{J} \sum_{P\in\mathcal{V}_{j,\gamma}} \{S_{j,P}; 2^{2j}(\cdot,\cdot)_{L_2(\Omega)}\}\right).$$

Before going on with rewriting our basic splittings, let us introduce the Schur complement form $s_J(\cdot,\cdot)$ on $S_J|_\gamma$ induced by the form $a(\cdot,\cdot)$ and corresponding to the given subdomain structure, by setting

$$s_J(u_\gamma, u_\gamma) = \inf_{u\in S_J\,:\,u_\gamma=u|_\gamma} a(u,u) \quad \forall\, u_\gamma \in S_J|_\gamma\,. \tag{4.3}$$

This bilinear form corresponds to the Schur complement problem on γ which arises if in (2.9) the unknowns corresponding to interior nodal points (i.e. $P \in \mathcal{V}_{j,l}$ for some l) are eliminated.

THEOREM 4.1. *Under the above assumptions on γ and the bilinear form, the following splitting for the Schur complement problem on γ is stable, with a condition number that is independent of j_0 (coarse level) and J:*

$$\{S_J|_\gamma; s_J(\cdot,\cdot)\} = \{S_{j_0}|_\gamma; s_{j_0}(\cdot,\cdot)\} + \sum_{j=j_0+1}^{J} \sum_{P\in\mathcal{V}_{j,\gamma}} \{S_{j,P}|_\gamma; 2^{j}(\cdot,\cdot)_{L_2(\Omega)}\}\,. \tag{4.4}$$

The proof of the stability of (4.4) runs as follows. By (4.3) and (4.2)

$$\begin{aligned}
s_J(u_\gamma,u_\gamma) &= \inf_{u\in S_J\,:\,u_\gamma=u|_\gamma} a(u,u)\\
&\approx \inf_{u\in S_J\,:\,u_\gamma=u|_\gamma}\ \inf_{u=u_{j_0}+\sum_{j=j_0+1}^{J}\sum_{P\in\mathcal{V}_{j,\gamma}} u_{j,P}} \Big(a(u_{j_0},u_{j_0})\\
&\qquad + \sum_{j=j_0+1}^{J}\sum_{P\in\mathcal{V}_{j,\gamma}} 2^{2j}(u_{j,P},u_{j,P})_{L_2(\Omega)}\Big)\\
&\approx \inf_{u_\gamma=u_{j_0,\gamma}+\sum_{j=j_0+1}^{J}\sum_{P\in\mathcal{V}_{j,\gamma}} u_{j,P,\gamma}} \Big(a(u_{j_0},u_{j_0})\\
&\qquad + \sum_{j=j_0+1}^{J}\sum_{P\in\mathcal{V}_{j,\gamma}} 2^{2j}(u_{j,P},u_{j,P})_{L_2(\Omega)}\Big)\,.
\end{aligned}$$

Here we have already used that $u_{j,l}|_\gamma = 0$ for all subdomains. u_{j_0} resp. $u_{j,P}$ denote the unique extensions of $u_{j_0,\gamma} \in S_{j_0,\gamma}$ resp. $u_{j,P,\gamma} \in S_{j,P}|_\gamma$ to functions in S_{j_0} resp. $S_{j,P}$ (note that the latter are one-dimensional). Now it remains to express the bilinear forms in terms of $u_{j_0,\gamma}$ resp. $u_{j,P,\gamma}$ which leads to (4.4). To this end, use the definition (4.3) (with J replaced by j_0), and

$$\|N_{j,P}|_\gamma\|^2_{L_2(\gamma)} \approx 2^j \|N_{j,P}\|^2_{L_2(\Omega)}\,, \quad P \in \mathcal{V}_{j,\gamma}$$

where, once again, the regularity and quasi-uniformity of the partitions comes in.

One can prove analogs of (3.9) or (3.10) as well. Another possibility is to group the subspaces of (4.4) according to the geometrical structure of γ. E.g., in 2D applications one could associate groups with each edge of γ and with each vertex of $\mathcal{T}_{j_0}$. Then the computations of the subspace corrections for edges can be carried out independently, e.g. on different processors, the same is true for the vertex components if one allows for a certain redundancy in the computations. There were different proposals to neglect the vertex components to simplify the computations, but it is clear that this results in an increase of the condition number of the reduced splitting. In 3D, the corresponding substructures are faces and the wirebasket (composed of edges and vertices). One can introduce a lot of modifications (especially on the wirebasket), and also change the auxiliary problems on the substructures. For some of the many algorithms of this type which have been derived and analyzed by other researchers using different techniques, we refer to work by Dryja, Widlund et.al. (see [**8, 10**] and the references cited therein). Note that these authors deal also with more complicated situations which are not covered by our reasoning.

4.2. Domain decomposition methods with overlap. In the domain decomposition methods with overlap, the subregions Ω_l are enlarged to a certain extent giving domains on which local Dirichlet problems are solved (these are the subspace problems involved in such type of algorithms). More precisely, under the same assumptions on $\{\Omega_l\}$ as in the previous subsection we will compose the enlarged regions $\hat{\Omega}_{l,j_1}$ of all simplices (rectangles etc.) in $\mathcal{T}_{j_1}$ that intersect with the closure of Ω_l. The number j_1 is chosen between j_0 and J and characterizes the amount of overlap (if $j_1 = j_0$ the overlap is called generous, if $j_1 = J$ we have minimal overlap). Let $\hat{S}_{j;l,j_1}$ denote the subspace consisting of all finite element functions from S_j that vanish outside of $\hat{\Omega}_{l,j_1}$.

Consider first $j_1 = j_0$ or $j_1 = j_0 + 1$. In this case, any of the one-dimensional subspaces $S_{j,P}$ ($P \in \mathcal{V}_j$, $j = j_0+1, \ldots, J$) belongs to at least one and at most $d+1$ (if simplicial triangulations are used) of the $\hat{S}_l \equiv \hat{S}_{J;l,j_1}$. Since, generally,

$$\{V; a(\cdot,\cdot)\} = \sum_{j=1}^{m} \{V; a(\cdot,\cdot)\}$$

is a stable splitting with $\lambda_{\min} = \lambda_{\max} = m$ and condition number exactly 1, we can refine the splitting (3.8) from Theorem 3.2 by adding the necessary number of copies of one-dimensional subspaces without destroying the condition number too much, i.e. under the assumptions of Theorem 3 we get the uniform stability of the splittings

$$\{S_J; a(\cdot,\cdot)\} = \{S_{j_0}; a(\cdot,\cdot)\} + \sum_l \Big(\sum_{j=j_0+1}^{J} \sum_{supp\, N_{j,P} \in \hat{\Omega}_{l,j_1}} \{S_{j,P}; 2^{2j}(\cdot,\cdot)_{L_2(\Omega)}\} \Big).$$

Now it remains to apply once again a scaled version of Theorem 3.2, (3.8), to $\hat{\Omega}_{l,j_1}$. This leads to the particular cases $j_1 = j_0, j_0+1$ of the following

THEOREM 4.2. *Under the above assumptions, the condition numbers of the splitting*

$$\{S_J; a(\cdot,\cdot)\} = \{S_{j_0}; a(\cdot,\cdot)\} + \sum_l \{\hat{S}_l; a(\cdot,\cdot)\}, \tag{4.5}$$

behave like $\approx 2^{j_1-j_0}$, *with constants that are independent of* $j_0 \leq j_1 \leq J$. *The result extends to* $S_{J;\Gamma}$.

This result is known, see the recent papers [**8, 9**] where the effect of smaller up to minimal overlap is studied in a different way. Our approach to the case of general j_1 is first to switch from (3.8) to the subsplitting

$$\{S_J; a(\cdot,\cdot)\} = \{S_{j_0}; a(\cdot,\cdot)\} + \sum_{j=j_0+1}^{J} \sum_{P\in\mathcal{V}_j \,:\, \exists l \;\; supp\, N_{j,P}\in\hat{\Omega}_{l,j_1}} \{S_{j,P}; 2^{2j}(\cdot,\cdot)_{L_2(\Omega)}\},$$

where the condition number degenerates by the factor $O(2^{j_1-j_0})$, and then to apply the above arguments to the groups of subspaces corresponding to the subdomains $\hat{\Omega}_{l,j_1}$. Due to the lack of space, we omit the technical details.

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