Contemporary Mathematics
Volume 180, 1994

A Wire Basket Based Method for Spectral Elements in Three Dimensions

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ABSTRACT. A two-level iterative substructuring algorithm with a wire basket based coarse space is proposed and analyzed. The three dimensional model problem is scalar, elliptic, with discontinuous coefficients, and is discretized by conforming spectral elements. The condition number of the resulting iteration operator is bounded by $C(1+\log p)^2$, where the constant is independent of the number of elements, their diameters, their degree p, and the size of the jumps across element boundaries of the coefficients of the elliptic operator. The results of this paper have been obtained jointly with Olof B. Widlund.

1. Introduction

Iterative substructuring methods are two-level domain decomposition methods based on nonoverlapping subregions. For the h-version finite element method, extensive research has been conducted in the last decade and many algorithms have been proposed for three dimensional problems; see e.g. Bramble, Pasciak, and Schatz [2], Dryja [4], Dryja and Widlund [6], Smith [15], and Le Tallec, De Roeck and Vidrascu [7]. A recent paper by Dryja, Smith, and Widlund [5] summarizes the current knowledge of the h-version case. See also Chan and Mathew [3] for an overview of domain decomposition algorithms.

For p-version finite elements and spectral methods, the construction of iterative substructuring methods is more challenging, since the stiffness matrices can be much more ill-conditioned and different mathematical tools are needed. See Babuška, Craig, Mandel, and Pitkäranta [1] for two dimensional problems, Mandel [10] for three dimensional problems, Pavarino [12, 11] for overlapping

1991 *Mathematics Subject Classification.* 41A10, 65N30, 65N35, 65N55.

This work was supported by the U.S. Department of Energy under contract DE-FG-05-92ER25142 and by the State of Texas under contract 1059.

The final detailed version of this paper has been submitted for publication elsewhere.

0271-4132/94 $1.00 + $.25 per page

methods in two and three dimensions, Rønquist [**14**] and Maday and Patera [**8**] for spectral element methods.

In this paper, we propose a wire basket based algorithm (in the terminology of Dryja, Smith, and Widlund [**5**]) with condition number bounded by $C(1+\log p)^2$, where p is the degree of the spectral elements. This method is directly inspired by a method developed for the h-version by Smith [**15, 16**]. Complete proofs of the results can be found in Pavarino and Widlund [**13**].

2. The elliptic problem and its discretization

We consider a three dimensional domain $\Omega = \bigcup_{i=1}^{N} \Omega_i$, a union of elements which are cubes or smooth images of a reference cube. We consider a model, elliptic problem on Ω with Dirichlet boundary conditions on $\Gamma_D \subset \partial\Omega$ and Neumann boundary conditions on $\partial\Omega - \Gamma_D$: find $u \in V = H^1_{\Gamma_D}(\Omega)$ such that

$$a(u,v) = \sum_{i=1}^{N} \int_{\Omega_i} \rho_i \nabla u \cdot \nabla v \, dx = \int_{\Omega} f v \, dx \qquad \forall v \in V. \tag{1}$$

The values $\rho_i > 0$ can be very different in different subregions. (1) is discretized by a continuous, piecewise Q_p Galerkin method, using conforming spectral elements. The discrete space $V^p \subset V$ is given by:

$$V^p = \{v_p \in C^0(\Omega) : v_p|_{\Omega_i} \in Q_p, \; i = 1, \cdots, N\} \cap H^1_{\Gamma_D}(\Omega).$$

The finite element problem obtained is turned into a linear system of algebraic equations, $K\underline{u} = \underline{b}$, by choosing a basis in V^p. A basis particularly useful in the convergence analysis of the method will be given in the next sections, but more practical hierarchical bases can also be used. As usual in the literature for spectral and p-version finite elements, we distinguish between interior basis functions, with support in the interior of an element and interface basis functions, with support intersecting the interface $\Gamma = \bigcup_{i=1}^{N} \partial\Omega_i$ (these can be further divided into face, edge and vertex basis functions). The coefficients of the unknown functions are partitioned accordingly, $\underline{u} = (\underline{u}_I, \underline{u}_B)$. As in most iterative substructuring algorithms, the unknowns $\underline{u}_I$ associated to the interior basis functions are eliminated first. The reduced Schur complement obtained in this way, $S\underline{u}_B = \tilde{\underline{b}}$, is solved with a preconditioned conjugate gradient method (or a more general Krylov method for nonsymmetric or indefinite problems).

3. The new method

The Schur complement system corresponds to a discrete variational problem posed in the discrete harmonic subspace $\tilde{V}^p$ of V^p, with the inner product $s(u,v) = \underline{u}_B^T S \underline{v}_B$. The functions in $\tilde{V}^p$ are $a(\cdot,\cdot)$-orthogonal to the interior basis functions and they are completely specified by their interface values. The space $\tilde{V}^p$ is decomposed into the direct sum of the following subspaces:

$$\tilde{V}^p = V_0 + \textstyle\sum_{ij} V_{ij}.$$

Here V_{ij} is the discrete harmonic subspace of $V^p \cap H_0^1(\Omega_{ij})$, where $\Omega_{ij} = \Omega_i \cup F_{ij} \cup \Omega_j$, (one space for each face), and V_0 is a coarse space consisting of piecewise discrete harmonic functions defined solely by their values on the wire baskets. The values on the faces are given by an interpolation operator I^W defined by eq. (3), Section 5. On V_0, we use the bilinear form $\tilde{s}_0(\cdot,\cdot) = (1+\log p)\sum_i \rho_i \inf_{c_i} \|u - c_i\|^2_{L_2(\mathcal{W}_i)}$. We introduce $s(\cdot,\cdot)$-orthogonal projections P_{ij} onto V_{ij} by $s(P_{ij}u, v) = s(u, v)$, $\forall v \in V_{ij}$ and an approximate projection T_0 onto V_0 by $\tilde{s}_0(T_0 u, v) = s(u, v)$, $\forall v \in V_0$. Our iterative substructuring method consists in solving

$$Tu = (T_0 + \sum_{ij} P_{ij})u = g. \tag{2}$$

by a conjugate gradient method. This is equivalent to an additive Schwarz preconditioner for the original Schur complement system.

The following is the main result of the paper.

THEOREM 1. *For the iterative substructuring method just introduced,*

$$\kappa(T) \leq const.(1+\log p)^2.$$

Here the constant is independent of the number of elements, their diameters, the degree p, and the size of the jumps of the coefficient ρ_i across element boundaries.

4. Special sets of polynomials and a basis for $Q_p([-1,1]^3)$

We denote by P_0^p the space of degree p polynomials on $[-1,1]$ that vanish at the endpoints.

DEFINITION 1. *Let φ_0^+ be the degree p polynomial satisfying*

$$\min_{\varphi} \|\varphi\|_{L_2(-1,1)}, \ \varphi(-1) = 0, \ \varphi(1) = 1.$$

DEFINITION 2. *Let $\Phi_i \in P_0^p$ and $\lambda_i, i = 1, \cdots, p-1$, be the eigenfunctions (normalized with unit H^1-norm) and eigenvalues defined by*

$$\int_{-1}^{1} \frac{d\Phi_i(x)}{dx} \frac{dv(x)}{dx}\, dx = \lambda_i \int_{-1}^{1} \Phi_i(x) v(x)\, dx \qquad \forall v \in P_0^p.$$

DEFINITION 3. *Given the eigenvalues $\{\lambda_i\}_{i=1}^{p-1}$ of Definition 2, define a set $\{\varphi_i^+\}_{i=1}^{p-1}$ of degree p polynomials by $\varphi_i^+(-1) = 0$, $\varphi_i^+(1) = 1$ and*

$$\int_{-1}^{1} \frac{d\varphi_i^+(x)}{dx} \frac{dv(x)}{dx}\, dx + \frac{\lambda_i}{2} \int_{-1}^{1} \varphi_i^+(x) v(x)\, dx = 0 \qquad \forall v \in P_0^p.$$

DEFINITION 4. *Given the eigenvalues* $\{\lambda_i\}_{i=1}^{p-1}$ *of Definition 2, define a set* $\{\varphi_{ij}^+\}_{i,j=1}^{p-1}$ *of degree* p *polynomials by* $\varphi_{ij}^+(-1)=0$, $\varphi_{ij}^+(1)=1$ *and*

$$\int_{-1}^{1} \frac{d\varphi_{ij}^+(x)}{dx}\,\frac{dv(x)}{dx}\,dx + (\lambda_i+\lambda_j)\int_{-1}^{1}\varphi_{ij}^+(x)v(x)\,dx = 0 \qquad \forall v \in P_0^p.$$

Analogous families of polynomials satisfying opposite boundary conditions, obtained by changing x into $-x$, will be denoted by φ_0^-, $\{\varphi_i^-\}$ and $\{\varphi_{ij}^-\}$. By computing the Legendre expansion of φ_0, it is possible to prove that $\|\varphi_0^+\|^2_{L^2(-1,1)} = \frac{2}{p(p+2)}$ and $\|\varphi_0^+\|^2_{L^\infty(-1,1)} \le 1$. We can now define a basis for $Q_p([-1,1]^3)$.

- *Interior functions:* $\Phi_i(x)\Phi_j(y)\Phi_k(z)$, $i,j,k=1,\cdots,p-1$. They are $a(\cdot,\cdot)-$ and L_2-orthogonal.
- *Face functions:* $\varphi_{i,j}^+(x)\Phi_i(y)\Phi_j(z)$, $i,j=1,\cdots,p-1$, for the face defined by $x=1$.
- *Edge functions:* $e_i^{(1)}(x,y,z)=\varphi_i^+(x)\varphi_i^+(y)\Phi_i(z)$, $i=1,\cdots,p-1$, for the edge E_1 defined by $x=1, y=1$.
- *Vertex functions:* $v_i(x,y,z)=\varphi_0^\pm(x)\varphi_0^\pm(y)\varphi_0^\pm(z)$, for the vertices $V_i=(\pm1,\pm1,\pm1)$.

The following result follows from a direct computation.

LEMMA 1. *Face, edge and vertex basis functions are discrete harmonic.*

5. Extension from the wire basket

In order to carry out a local analysis of the algorithm (i.e. global bounds from local bounds on individual elements), it is crucial to include the constants in the coarse space V_0, see Lemma 2.2 in Dryja, Smith and Widlund [5] or Theorem 5.1 in Mandel [9]. V_0 can be seen as the range of an interpolation operator. We first introduce locally a preliminary interpolation operator $\tilde{I}^W : V^p \to V_0$ by $\tilde{I}^W u = u_V + u_E$. Here $u_V = \sum_{i=1}^{8} u(V_i)\varphi_0^\pm\varphi_0^\pm\varphi_0^\pm$ is the sum of the vertex components of u and $u_E = \sum_{i=1}^{12}\sum_{j=1}^{p-1}\alpha_j^{(i)}\underbrace{\Phi_j\varphi_j^\pm\varphi_j^\pm}_{permutations}$ is the sum of the edge components of u, with coefficients $\alpha_j^{(i)} = \lambda_j\int_{-1}^{1}(u-u_V)\Phi_j dx$. The range of this interpolation operator does not contain the constants. We therefore consider $\mathcal{F} = \tilde{I}^W 1$, the image of the function identically equal to 1 on the wire basket. $\mathcal{F}$ is not equal to 1 on the faces. In order to recover the constants, consider the function $\kappa = 1-\mathcal{F}$, which vanishes on the wire basket. It can be split into six discrete harmonic components, each with nonzero values only on one face: $\kappa = \sum_{i=1}^{6}\kappa_i$. The new interpolation operator is then defined by

$$I^W u = \tilde{I}^W u + \sum_{i=1}^{6}\overline{u}_{\partial F_i}\kappa_i, \tag{3}$$

where $\overline{u}_{\partial F_i} = \frac{1}{8}\int_{\partial F_i} u$. If $u \equiv 1$ on W, then $I^W u \equiv 1$ on $\partial\Omega$. This interpolation operator defines a change of basis in the wire basket space, by mapping edge and

vertex basis functions into:

$$(4)\qquad \begin{array}{rcccl} \tilde{e}_j^{(k)} & = & I^W e_j^{(k)} & = & e_j^{(k)} + \sum_i \bar{e}_{j,\partial F_i}^{(k)} \kappa_i, \\ \tilde{v}_j & = & I^W v_j & = & v_j + \sum_i \overline{v}_{j,\partial F_i} \kappa_i. \end{array}$$

6. Matrix form of the preconditioner

We are solving the Schur complement system $S\underline{u}_B = \tilde{\underline{b}}$ obtained by eliminating the interior unknowns. Let us order the face basis functions first and then those of the wire basket. The contribution to the Schur complement S attributable to the element Ω_i can be written as:

$$S^{(i)} = \begin{pmatrix} S_{FF}^{(i)} & S_{FW}^{(i)} \\ S_{FW}^{(i)^T} & S_{WW}^{(i)} \end{pmatrix}.$$

The preconditioner $\hat{S}$ is similarly obtained by subassembly of local contributions $\hat{S}^{(i)}$, constructed for individual substructures. We first change basis for the wire basket space by using the new edge and vertex basis functions defined by (4). The face basis functions remain the same. The transformation from the old to the new basis can be represented by $\begin{pmatrix} I & 0 \\ R^{(i)} & I \end{pmatrix}$. In the new basis $S^{(i)}$ is transformed into

$$\begin{pmatrix} I & 0 \\ R^{(i)} & I \end{pmatrix} \begin{pmatrix} S_{FF}^{(i)} & S_{FW}^{(i)} \\ S_{FW}^{(i)^T} & S_{WW}^{(i)} \end{pmatrix} \begin{pmatrix} I & R^{(i)^T} \\ 0 & I \end{pmatrix} = \begin{pmatrix} S_{FF}^{(i)} & \text{nonzero} \\ \text{nonzero} & \tilde{S}_{WW}^{(i)} \end{pmatrix}.$$

We construct the local preconditioner by replacing $S_{FF}^{(i)}$ with its block diagonal part $\hat{S}_{FF}^{(i)}$ with one block for each face and by dropping the coupling between face and wire basket spaces. $\tilde{S}_{WW}^{(i)}$ is replaced by a rank-one perturbation of a multiple of the identity. It turns out that, in our basis, $\hat{S}_{FF}^{(i)}$ is the diagonal of $S_{FF}^{(i)}$ because each diagonal block is diagonal. We then return to the old basis:

$$\hat{S}^{(i)} = \begin{pmatrix} I & 0 \\ -R^{(i)} & I \end{pmatrix} \begin{pmatrix} \hat{S}_{FF}^{(i)} & 0 \\ 0 & \hat{S}_{WW}^{(i)} \end{pmatrix} \begin{pmatrix} I & -R^{(i)^T} \\ 0 & I \end{pmatrix}.$$

The preconditioner is obtained by subassembly:

$$\hat{S} = \begin{pmatrix} I & 0 \\ -R & I \end{pmatrix} \begin{pmatrix} \hat{S}_{FF} & 0 \\ 0 & \hat{S}_{WW} \end{pmatrix} \begin{pmatrix} I & -R^T \\ 0 & I \end{pmatrix}.$$

Therefore

$$(5)\qquad \hat{S}^{-1}S = R_0 \hat{S}_{WW}^{-1} R_0^T S + \sum_i R_{F_i} \hat{S}_{F_i F_i}^{-1} R_{F_i}^T S,$$

where $R_0 = (R, I)$ (see Dryja, Smith, and Widlund [5]). Clearly this is an additive preconditioner with independent parts associated with the wire basket and each face. (5) is the matrix form of the operator T of (2).

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