Contemporary Mathematics
Volume 180, 1994

Error estimators based on stable splittings

U. RÜDE

ABSTRACT. Stable splittings have been used successfully to describe and analyze the performance of iterative solvers based on subspace corrections. The same theoretical foundation can be used to construct abstract upper and lower error estimates. This approach leads to a unified treatment of discretization and iteration errors and can therefore be used to guide iteration and mesh refinement strategies. The estimate does not require the exact solution of a fixed finite element problem, but can be used for any approximation that may be available. The estimate is based only on quantities that occur naturally in the solution process, so no extra work is required.

1. Introduction

Multilevel- and domain decomposition methods are generally based on a *splitting* of the solution space. Such subspaces, together with their Hilbert space structure, define elementary operations, that can in turn be used to construct iterative methods and preconditioners, the so-called *subspace correction methods*. This framework includes many iterative methods like classical relaxation schemes, domain decomposition algorithms, and multilevel preconditioners. The discussion of algorithms in this setup turns out to be useful, because the performance of solvers and preconditioners depends on a single abstract feature of the subspace system, the so-called *stability* of the splitting.

Additionally, the splitting of the space can be used to derive error estimates. If the subspace splitting is stable, we obtain uniformly bounded lower and upper error estimates. Depending on the interpretation of the spaces, the bounds apply to the (algebraic) iteration error or the (continuous) discretization error such that the error estimate combines in a natural way *discretization errors* and *algebraic errors*. This dual viewpoint distinguishes our approach from conventional error estimators. In contrast to more conventional estimates, this approach does not

1991 *Mathematics Subject Classification.* 65N22, 65N50, 65N55.
Key words and phrases. Fully adaptive multigrid, multilevel iteration, error estimate.
This paper is in final form and no version of it will be submitted for publication elsewhere.

require the exact solution of a finite element system, but can be applied to an approximate solution during the iterative solution process. It therefore provides useful criteria for the development of a strategy for switching from iteration to refinement, and vice versa.

Since iteration and discretization errors are estimated by a uniform approach, it becomes natural to consider algorithms that combine iteration, error estimation, and mesh refinement. An implementation of these ideas has been introduced as the the *virtual global grid* refinement technique and the *multilevel adaptive iteration* in Rüde [**7, 8**]. Our error estimate is based on subspace corrections, as they are computed in each iteration step such that the estimate involves no extra cost but only the correct interpretation of quantities occuring in the iterative solution process.

The integration of error estimates in a multilevel solution process has been discussed in several papers, see e.g. Bank, Smith, and Weiser [**2, 1**], Deuflhard, Leinen, and Yserentant [**3**], and Verfürth [**9**]. In this paper we will derive new error estimates based on the theory of stable splittings.

2. Stable splittings

We use an abstract setting, where V is a Hilbert space equipped with a scalar product $\langle \cdot, \cdot \rangle_V$ and a norm

$$\|u\|_V = \langle u, u \rangle_V^{1/2}.$$

Given a V-elliptic, symmetric, continuous bilinear form $a : V \times V \longrightarrow \mathbb{R}$ with constants $0 < c_1 \leq c_2 < \infty$, such that

$$c_1 \langle v, v \rangle_V \leq a(v, v) \leq c_2 \langle v, v \rangle_V \tag{2.1}$$

for all $v \in V$, we study the elliptic problem: Find $u \in V$ such that

$$a(u, v) = \Phi(v) \tag{2.2}$$

for all $v \in V$, where the functional $\Phi \in V^*$ is a continuous linear form.

To introduce a multilevel structure, we consider a finite or infinite collection $\{V_j\}_{j \in J}$ of subspaces of V, each with its own scalar product $(\cdot, \cdot)_{V_j}$ and the associated norm

$$\|u\|_{V_j} = (u, u)_{V_j}^{1/2}.$$

We further assume that the full space V can be represented as the sum of the subspaces V_j, $j \in J$,

$$V = \sum_{j \in J} V_j \tag{2.3}$$

and assume that the spaces are nested, that is $J \subset \mathbb{N}_0$, $V_i \subset V_j$, if $i \leq j$.

In typical applications, $\|\cdot\|_V$ is the H^1-Sobolev norm and the subspace norms $\|\cdot\|_{V_j}$ are properly scaled L_2-norms. Typically, $\|\cdot\|_{V_j} = 2^j \|\cdot\|_{L_2}$.

Any element of V can be represented as a sum of elements in V_j, $j \in J$. Generally, this representation is non-unique. The *additive Schwarz norm* $|||\cdot|||$ in V with respect to the collection of subspaces $\{V_j\}_{j\in J}$ is defined by

$$|||v||| \overset{\text{def}}{=} \inf\left\{ \left(\sum_{j\in J} \|v_j\|_{V_j}^2 \right)^{\frac{1}{2}} \,\middle|\, v_j \in V_j,\ \sum_{j\in J} v_j = v \right\}. \tag{2.4}$$

A collection of spaces $\{V_j\}_{j\in J}$ satisfying (2.3) is called a *stable splitting* of V, if $\|\cdot\|_V$ is equivalent to the additive Schwarz norm of V, that is, if there exist constants $0 < c_3 \le c_4 < \infty$ such that

$$c_3\|v\|_V^2 \le |||v|||^2 \le c_4\|v\|_V^2 \tag{2.5}$$

for all $v \in V$. The number

$$\kappa(V, \{V_j\}_{j\in J}) \overset{\text{def}}{=} \inf(c_4/c_3), \tag{2.6}$$

that is the infimum over all possible constants in (2.5), is called the *stability constant* of the splitting $\{V_j\}_{j\in J}$.

Next, we introduce auxiliary V_j-elliptic, symmetric, bilinear forms

$$b_j : V_j \times V_j \longrightarrow \mathbb{R}$$

in the spaces V_j, respectively. In classical multigrid terminology, the choice of b_j determines which kind of smoother we use. For the theory, we require that the b_j are uniformly equivalent to the respective inner product of the subspace, that is that there exist constants $0 < c_5 \le c_6 < \infty$ such that

$$c_5(v_j, v_j)_{V_j} \le b_j(v_j, v_j) \le c_6(v_j, v_j)_{V_j}, \tag{2.7}$$

for all $v_j \in V_j$, $j \in J$.

Multilevel algorithms are now described in terms of *subspace corrections* $P_{V_j} : V \longrightarrow V_j$, mapping the full space V into each of the subspaces V_j. P_{V_j} is defined by

$$b_j(P_{V_j}u, v_j) = a(u, v_j), \tag{2.8}$$

for all $v_j \in V_j$, $j \in J$. Analogously, we define $\phi_j \in V_j$ by

$$b_j(\phi_j, v_j) = \Phi(v_j), \tag{2.9}$$

for all $v_j \in V_j$, $j \in J$.

The *additive Schwarz operator* (also called BPX-operator) $P_V : V \longrightarrow V$ with respect to the multilevel structure on V is defined by

$$P_V = \sum_{j\in J} P_{V_j} \tag{2.10}$$

and

$$\phi = \sum_{j \in J} \phi_j.$$

With suitable bilinear forms b_j it is possible to evaluate P_V efficiently based on its definition as a sum. The explicit construction of P_V is not required.

The hierarchical structure in the subspace system seems to be essential for obtaining a stable splitting. Otherwise, the complexity of the original problem must be captured in the bilinear forms $b_j(\cdot,\cdot)$, and then the evaluation of the P_{V_j} is as expensive as the solution of the original problem.

The importance of the abstract multilevel structure for practical applications is indicated by the following theorem.

THEOREM 2.1. *Assume that the subspaces V_j, $j \in J$ of a Hilbert space V are a stable splitting. Assume further that P_{V_j} and P_V are defined as above with bilinear forms b_j satisfying (2.7). The variational problem (2.2) is equivalent to the operator equation*

$$P_V u = \phi, \tag{2.11}$$

and the spectrum of P_V can be estimated by

$$\frac{c_1}{c_4 c_6} \le \lambda_{\min}(P_V) \le \lambda_{\max}(P_V) \le \frac{c_2}{c_3 c_5}. \tag{2.12}$$

For a proof see Oswald [**4, 5**] or Rüde [**8**].

Remark. Results similar to Theorem 2.1 have been developed within the domain decomposition and multigrid literature. The interested reader is also referred to the survey articles of Xu [**10**], Yserentant [**11**], and the references given therein.

3. Multilevel Error Estimators

Besides providing the theoretical basis for the fast iterative solution of discretized PDEs, the multilevel splittings can also be used to provide error estimates.

The **scaled residuals** $\bar{r}_j \in V_j$ of u are defined by

$$b_j(\bar{r}_j, v_j) = a(u - u^*, v_j) = a(u, v_j) - \Phi(v_j) \tag{3.1}$$

for all $v_j \in V_j$, $j \in J$, where u^* is the solution of (2.2) in V. The following theorem is an abstract and more general version of a result given in [**6**], see also [**8**].

THEOREM 3.1. *Assume that the collection of spaces $\{V_j\}_{j \in J}$ is a stable splitting of V, and that u^* is the solution of (2.2) in V. Then there exist constants $0 < c_0 \le c_1 < \infty$ such that*

$$c_0 \sum_{j \in J} \|\bar{r}_j\|_{V_j}^2 \le \|u - u^*\|_V^2 \le c_1 \sum_{j \in J} \|\bar{r}_j\|_{V_j}^2. \tag{3.2}$$

Proof. With inequalities (2.1) and equation (2.7) it suffices to show that there exist constants $0 < \bar{c}_0 \leq \bar{c}_1 < \infty$ such that.

$$\bar{c}_0 \sum_{j \in J} b_j(\bar{r}_j, \bar{r}_j) \leq a(u-u^*, u-u^*) \leq \bar{c}_1 \sum_{j \in J} b_j(\bar{r}_j, \bar{r}_j). \tag{3.3}$$

From Theorem 2.1, we know that there exist such constants with

$$\bar{c}_0 a(P_V(u-u^*), u-u^*) \leq a(u-u^*, u-u^*) \leq \bar{c}_1 a(P_V(u-u^*), u-u^*).$$

Additionally,

$$\begin{aligned} a(P_V(u-u^*), u-u^*) &= \sum_{j \in J} a(u-u^*, P_{V_j}(u-u^*)) \\ &= \sum_{j \in J} b_j(\bar{r}_j, P_{V_j}(u-u^*)) \\ &= \sum_{j \in J} a(\bar{r}_j, (u-u^*)) \\ &= \sum_{j \in J} b_j(\bar{r}_j, \bar{r}_j), \end{aligned}$$

which concludes the proof. □

The value of Theorem 3.1 is that it estimates an unknown quantity, the error $u - u^*$, by known quantities, the *residuals* $P_{V_j} u - \phi_j$.

In contrast to the usual error estimators used for finite elements, (3.2) uses a sum of residuals from all levels. In general, this sum is infinite, so that, at a first glance, the practical usefulness seems to be limited. Estimate (3.2), however, gives valuable insight into the nature of adaptive processes because it relates the residuals from all levels of a hierarchical representation of the solution. Thus, iteration errors, which result in residuals on coarser levels, are included in the estimate. This information can now be used to guide the switching from iteration to refinement and vice versa. Suitable algorithms have been proposed in Rüde [**8**], where the idea is to treat refinement and iteration as essentially the same process.

Finite element nodes must be relaxed whenever the associated residuals are large relative to the overall error estimate. Using the virtual global grid data structure (see [**8**]), this strategy can be extended to unknowns that are not yet included in the finite element system. Of course, such unknowns must be generated by some refinement algorithm, before they can be relaxed.

4. Estimates based on residuals on one level

Based on (3.2) we will now develop an error estimator that uses the residuals of one level only. In view of Theorem 3.1 we need an additional assumption that can be used to bound the error contribution of an infinite sequence of levels by one of them alone. This can be accomplished by introducing the well known *saturation condition* (see, e.g., Bank and Smith [**1**])

$$a(u_{j+1}^* - u^*, u_{j+1}^* - u^*) \leq \gamma_j a(u_j^* - u^*, u_j^* - u^*), \tag{4.1}$$

for a constant $\gamma_j \leq \gamma < 1$, where u_j^* denotes the *exact* solution of the level j equations defined by

$$a(u_j^*, v_j) = \Phi(v_j), \tag{4.2}$$

for all $v_j \in V_j$.

The saturation condition (4.1) makes assumptions about the speed of convergence of u_j^* to u^* with respect to the multilevel system. Note that if V_{j+1} is a space with higher approximation order than V_j, then typically, $\gamma_j \to 0$, when $j \to \infty$ and if u^* is sufficiently smooth. In our paper we do not assume higher order approximation spaces, so that γ_j will tend to a constant like $1/4$ for linear finite elements and smooth solutions. Consequently, our error estimate cannot be asymptotically exact.

THEOREM 4.1. *Let $\bar{r}_{j+1}^*$ denote the residual of u_j on level $j+1$, defined by*

$$b_{j+1}(\bar{r}_{j+1}^*, v_{j+1}) = a(u_j^* - u^*, v_{j+1}) \tag{4.3}$$

for all $v_{j+1} \in V_{j+1}$. If the saturation condition (4.1) holds, then there exist constants $0 < c_1 \leq c_2 < \infty$ such that

$$c_1 b_{j+1}(\bar{r}_{j+1}^*, \bar{r}_{j+1}^*) \leq a(u_j^* - u^*, u_j^* - u^*) \leq c_2 b_{j+1}(\bar{r}_{j+1}^*, \bar{r}_{j+1}^*). \tag{4.4}$$

Proof. Using the theorem of Pythagoras

$$a(u_j^* - u^*, u_j^* - u^*) = a(u_{j+1}^* - u_j^*, u_{j+1}^* - u_j^*) + a(u_{j+1}^* - u^*, u_{j+1}^* - u^*)$$

and the saturation condition (4.1), we obtain

$$a(u_{j+1}^* - u_j^*, u_{j+1}^* - u_j^*) \leq a(u_j^* - u^*, u_j^* - u^*) \leq \frac{1}{1-\gamma_j} a(u_{j+1}^* - u_j^*, u_{j+1}^* - u_j^*).$$

Next, we must estimate the difference $u_{j+1}^* - u_j^*$ in terms of $\bar{r}_{j+1}^*$. First, note that in V_{j+1} equipped with the $\|\cdot\|_V$-norm, the finite system of spaces $V_0, V_1, \dots, V_{j+1}$, each with its own norm $\|\cdot\|_{V_0}, \|\cdot\|_{V_1}, \dots, \|\cdot\|_{V_{j+1}}$ is a stable splitting (see Rüde [**8**]). The finite additive Schwarz operator

$$\sum_{k=0}^{j+1} P_{V_k}$$

is spectrally equivalent to the identity in V_{j+1}, so that

$$\begin{aligned} a(u^*_{j+1}-u^*_j, u^*_{j+1}-u^*_j) &\le c_1 a\left(\sum_{k=0}^{j+1} P_{V_k}(u^*_{j+1}-u^*_j), u^*_{j+1}-u^*_j\right) \\ &= c_1 \sum_{k=0}^{j+1} b_{j+1}(P_{V_k}(u^*_{j+1}-u^*_j), P_{V_k}(u^*_{j+1}-u^*_j)). \end{aligned}$$

Noting that

$$P_{V_k}(u^*_j - u^*) = 0$$

for $k \le j$, and

$$P_{V_{j+1}}(u^*_{j+1} - u^*) = \bar{r}^*_{j+1},$$

we find the upper bound in (4.4). The proof of the lower bound is analogous. □

5. Conclusions

In this paper we have briefly outlined how multilevel error estimates can be derived from the theory of stable splittings, and how they are therefore linked to fast iterative solvers. We believe that this is of not only theoretical interest, because it can be used directly to construct and analyze efficient, adaptive multilevel solvers.

Future research must extend these ideas to more general problems, including non-selfadjoint and nonlinear equations. Additionally, the estimate must be tested for realistic problems and must be compared experimentally to alternative techniques.

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(U. Rüde) INSTITUT FÜR INFORMATIK, TECHNISCHE UNIVERSITÄT MÜNCHEN, D-80290 MÜNCHEN, GERMANY

Current address: Fachbereich Mathematik, Technische Universität Chemnitz-Zwickau, D-09009 Chemnitz, Germany

E-mail address: ruede@informatik.tu-muenchen.de