

The Modified Vertex Space Domain Decomposition Method for Neumann Boundary Value Problems

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ABSTRACT. In this paper, we extend the vertex space domain decomposition method (VSDDM) [18] to solve singular systems arising from the discretization of partial differential equations with Neumann boundary conditions by finite differences or finite elements. We give a concrete discussion on how to deal with the null space in VSDDM so that it retains optimal condition number independent of sizes of coarse and fine grids. To reduce the complexity cost of VSDDM, we proposed several efficient variants VSDDM [6] based on Fourier approximation and a probing technique. Here, we further reduce the cost of the probing technique in VSDDM. Various numerical experiments have been conducted to test the efficiency of the modified VSDDM.

1. Introduction

The aim of this paper is to modify the vertex space domain decomposition method [18] so that this method can be applied to the symmetric positive semi-definite systems of linear algebraic equations arising from the discretization of elliptic systems with Neumann boundary condition by finite differences or finite elements. For these Neumann boundary value problems, the discrete stiffness matrix is singular and the solution is not unique. The VSDDM needs to be modified so that the results of preconditioning are orthogonal to the kernel space and the VSDDM retains an optimal convergence rate. Our motivation is to design

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an efficient parallel algorithm for solving the Navier-Stokes equations. When our scheme is based on the velocity and stream function, we have to solve a symmetric semi-positive system, resulting from Laplace's equation with "Neumann like" boundary conditions, in each time step. We also focus on the improvement of the probing technique in the construction of edge and vertex approximations. Based on the Fourier approximation [9, 14, 2, 3] and the probing technique [7, 15, 16, 5], several efficient variants of the VSDDM have been proposed and tested [4, 6, 17]. Here, we further reduce the cost of the probing technique in the VSDDM for two dimensional problems. Only four instead of six probing vectors [6] are used to multiply by the Schur complement and to form the approximate edge and vertex matrices. Various numerical tests have been conducted to show that the modified VSDDM has an optimal convergence rate for singular problems with smoothly varying coefficients, and highly jumping coefficients.

In section 2, we discretize a singular problem, reduce the problem on the whole domain to the interface and form the Schur complement system on the interface. We discuss the properties of the Schur complement matrix. In section 3, we apply the VSDDM to singular problems with known kernel spaces and state a theorem on convergence rate. In section 4, we further improve the probing technique in the VSDDM. Finally, in section 5 we conduct the numerical experiments on this modified VSDDM with various approximate edge and vertex matrices for solving singular problems with highly varying or jumping coefficients.

2. Neumann Boundary Value Problems

Let Ω in \mathbf{R}^d be a polygonal domain. In the Sobolev space $\mathbf{V} = (H^1(\Omega))^q$, we introduce a symmetric, bounded and semi-positive definite bilinear form $a(\cdot, \cdot) : \mathbf{V} \times \mathbf{V} \rightarrow \mathbf{R}$. Let (\cdot, \cdot) be the inner product in $(L^2(\Omega))^q$: $(\mathbf{f}, \mathbf{v}) = \int_{\Omega} \mathbf{f} \cdot \mathbf{v} dx$. The kernel space is defined by

$$KerA = \{\mathbf{u} | \mathbf{u} \in \mathbf{V}, \quad a(\mathbf{u}, \mathbf{v}) = 0, \quad \forall \mathbf{v} \in \mathbf{V}\},$$

which is known for most Neumann boundary value problems.

Consider a general variational problem with a natural boundary condition in the space \mathbf{V} : Find $\mathbf{u} \in \mathbf{V}$, and $\mathbf{u} \perp KerA$ such that

$$(2.1) \quad a(\mathbf{u}, \mathbf{v}) = (\mathbf{f}, \mathbf{v}), \quad \forall \mathbf{v} \in \mathbf{V},$$

where f satisfies the compatibility condition: $(\mathbf{f}, \mathbf{v}) = 0, \quad \forall \mathbf{v} \in KerA$.

As an example, we consider equation:

$$(2.2) \quad Lu = f \quad \text{in } \Omega, \quad \text{and} \quad \frac{\partial u}{\partial N} = 0 \quad \text{on } \partial\Omega,$$

where $Lv = -\sum_{i,j} \frac{\partial}{\partial x_i} (\alpha_{ij} \frac{\partial v}{\partial x_j})$ and $\frac{\partial v}{\partial N} = \sum_{i,j} \alpha_{ij}(x) \frac{\partial v}{\partial x_j} \cos(\vec{n}, \vec{e}_i)$ with α_{ij} uniformly bounded, and positive definite. Then $\mathbf{V} = H^1(\Omega)$ and $KerA = \text{span}\{\mathbf{1}\}$.

We partition the domain Ω as the the union of disjoint regions Ω_k of diameter H ,

$$\bar{\Omega} = \cup_k \bar{\Omega}_k \quad \text{and} \quad \Omega_i \cap \Omega_j = \emptyset \quad \text{if } i \neq j.$$

These subdomains form the elements of a coarse triangulation. We denote the union of these subdomain boundaries as $\Gamma = \cup_k \partial\Omega_k$. Each subdomain is further divided into the elements of diameter $O(h)$. So we have a fine triangulation on the domain Ω . Assume that these triangulations are shape regular in the sense common to finite element theory, cf. Ciarlet [8].

By using finite element or finite difference method, we obtain stiffness matrices A_h and A_H on the fine and coarse triangulations respectively. These matrices are symmetric, positive semi-definite. Denote the kernel spaces as $KerA_h = \{\mathbf{v}_h | A_h \mathbf{v}_h = 0\}$ and $KerA_H = \{\mathbf{v}_H | A_H \mathbf{v}_H = 0\}$. Then problem (2.1) has the following discrete forms:

$$(2.3) \quad \text{Find } \mathbf{u}_h \perp KerA_h \quad \text{such that} \quad A_h \mathbf{u}_h = \mathbf{f}_h.$$

on the fine grid and

$$(2.4) \quad \text{Find } \mathbf{u}_H \perp KerA_H \quad \text{such that} \quad A_H \mathbf{u}_H = \mathbf{f}_H,$$

on the coarse grid. Each of these problems has an unique solution if its right-hand side is orthogonal to the kernel space.

Now we restrict the problem on Ω to the interface Γ . By grouping the unknowns in the interior of the subdomains in the vector \mathbf{u}_I and those on the interface Γ in the vector \mathbf{u}_B , we can rewrite problem (2.3) as block form

$$\begin{pmatrix} A_{II} & A_{IB} \\ A_{BI} & A_{BB} \end{pmatrix} \begin{pmatrix} \mathbf{u}_I \\ \mathbf{u}_B \end{pmatrix} = \sum_k \begin{pmatrix} A_{II}^{(k)} & A_{IB}^{(k)} \\ A_{BI}^{(k)} & A_{BB}^{(k)} \end{pmatrix} \begin{pmatrix} \mathbf{u}_I^{(k)} \\ \mathbf{u}_B^{(k)} \end{pmatrix} = \begin{pmatrix} \mathbf{f}_I \\ \mathbf{f}_B \end{pmatrix}.$$

where $\mathbf{u}^{(k)} = ((\mathbf{u}_I^{(k)})^T, (\mathbf{u}_B^{(k)})^T)^T$ is a vector associated with $\bar{\Omega}_k$ and $\mathbf{u}_h = \sum_k \mathbf{u}^{(k)}$. Here $\mathbf{u}_I^{(k)}$ is a vector corresponding to the interior of subdomain Ω_k and $\mathbf{u}_B^{(k)}$ is associated with the nodal points on $\partial\Omega_k$. Note that each interior variable $\mathbf{u}_I^{(k)}$ is associated with only one of the substructures, it can be eliminated locally and simultaneously. The reduced global equation, called the Schur complement on Γ , can be written in assembled form,

$$(2.5) \quad S\mathbf{u}_B = (A_{BB} - A_{BI}A_{II}^{-1}A_{IB})\mathbf{u}_B = \sum_k S^{(k)}\mathbf{u}_B^{(k)} = \mathbf{g} = \sum_k \mathbf{g}^{(k)}$$

where $S^{(k)} = A_{BB}^{(i)} - A_{BI}^{(k)}(A_{II}^{(k)})^{-1}A_{IB}^{(k)}$ corresponds to the contribution from Ω_k to the boundary $\partial\Omega_k \subset \Gamma$, and $\mathbf{g}^{(k)} = \mathbf{f}_B^{(k)} - A_{BI}^{(k)}(A_{II}^{(k)})^{-1}\mathbf{f}_I^{(k)}$, comes from the value of \mathbf{f} on interior points Ω_k and boundary points $\partial\Omega_k$. The action of the inverse of $A_{II}^{(k)}$ is equivalent to solving a local problem on Ω_k with Dirichlet boundary condition. Note the reduced Schur complement is still singular i.e.

$$KerS = \{\mathbf{u}_B | S\mathbf{u}_B = 0\} \neq \{0\}.$$

Hence, problem (2.5) has an unique solution $\mathbf{u}_B \perp \text{Ker}S$ only if right-hand side \mathbf{g} satisfies the compatibility condition: $\mathbf{g} \perp \text{Ker}S$. For Schur complement system (2.5), we have the following properties, which can be easily proved [1, 17].

LEMMA 2.1. *If element $\mathbf{u}_h = (\mathbf{u}_I^T, \mathbf{u}_B^T)^T \in \text{Ker}A_h$, then $\mathbf{u}_B \in \text{Ker}S$. If $\mathbf{u}_B \in \text{Ker}S$, there exists \mathbf{u}_I such that $\mathbf{u}_h = (\mathbf{u}_I^T, \mathbf{u}_B^T)^T \in \text{Ker}A_h$.*

LEMMA 2.2. *If $\mathbf{f}_h \perp \text{Ker}A$, then $\mathbf{g} = \mathbf{f}_B - A_{BI}A_{II}^{-1}\mathbf{f}_I$ is orthogonal to $\text{Ker}S$.*

LEMMA 2.3. *If A_h is a symmetric semi-positive matrix, then for any \mathbf{v}_B*

$$\mathbf{v}_B^T S \mathbf{v}_B = \min_{\mathbf{v}_I} \mathbf{v}_h^T A_h \mathbf{v}_h$$

where $\mathbf{v}_h = (\mathbf{v}_I^T, \mathbf{v}_B^T)^T$. Hence, the capacitance matrix S is also symmetric positive semi-definite.

LEMMA 2.4. *For any \mathbf{v}_B , let $\mathbf{v}_h = (\mathbf{v}_I^T, \mathbf{v}_B^T)^T$ be discrete harmonic extension of \mathbf{v}_B , i.e. $(A_{II} \quad A_{IB})\mathbf{v}_h = 0$. Then $\mathbf{v}_B^T S \mathbf{v}_B = \mathbf{v}_h^T A_h \mathbf{v}_h$.*

From these lemmas, we can conclude that the direct restriction of $\text{Ker}A_h$ to the pseudo-boundary Γ equals $\text{Ker}S$. We can loosely state that $\text{Ker}S = \text{Ker}A_h$.

3. The Modified VSDD Algorithm

Let the interface Γ be partitioned as the union of faces $\Gamma_{F_{ij}}$, edges $\Gamma_{E_{ij}}$ and cross points x_i : $\Gamma = (\cup_{ij} F_{ij}) \cup (\cup_{ij} E_{ij}) \cup (\cup_i x_i)$, where the face F_{ij} is the interface of two neighbor substructures Ω_i and Ω_j , and the edge E_{ij} is the set with all the points on the substructure boundaries in the cylinder with radius $0(H)$ and the central line ending by adjacent cross points of x_i and x_j . Let the vertex space X_i be the region consisting of a vertex x_i and an overlap of order H onto adjacent faces and edges. We restrict the overlapping so that no portion of Γ is covered more than p times. Here p is a small finite integer. If Ω is in 2 dimension space, the interface Γ is partitioned into overlapping regions: edges and vertex spaces.

For each subregion $\hat{\Gamma}$, we introduce $R_{\hat{\Gamma}}$ as the pointwise restriction operator which returns only those unknowns that are associated with $\hat{\Gamma}$. Denote as $\mathbf{V}^h(\Gamma)$ the space of all the grid functions defined on Γ , and as \mathbf{V}^H the space of grid functions on the coarse mesh. Denote face, edge and vertex space submatrices by $S_{F_{ij}} = R_{F_{ij}}^T S R_{F_{ij}}$, $S_{E_{ij}} = R_{E_{ij}}^T S R_{E_{ij}}$, and $S_{X_i} = R_{X_i}^T S R_{X_i}$, respectively.

The criterion for choosing the restriction operator R_H is this: for any function $\mathbf{g} \in \mathbf{V}^h(\Gamma)$ with $\mathbf{g} \perp \text{Ker}S$, $R_H \mathbf{g}$ shall be orthogonal to the space $\text{Ker}A_H$. Let

R_H^T be a linear interpolation operator from V^H to $V^h(\Gamma)$. Then, R_H is the corresponding weighted restriction operator from the space $V^h(\Gamma)$ to the space V^H . For most Neumann boundary value problems, the corresponding kernel space $KerS$ consists of linear functions. Therefore, a straightforward computation gives that $R_H \mathbf{g}$ is orthogonal to the kernel space $KerA_H$ when \mathbf{g} is orthogonal to the kernel space $KerS$. Thus, the coarse problem $A_H \mathbf{u}_H = R_H \mathbf{g}$ is well defined and has only one solution $\mathbf{u}_H \perp KerA_H$.

We solve the singular Schur complement system (2.5) by using a preconditioned conjugate gradient iterative method with VSDD as a preconditioner M . The action of the inverse of the preconditioner M involves following block calculations.

Calculate $\mathbf{u}_B = M^{-1} \mathbf{g}$ where $\mathbf{g} \perp KerS$:

The VSDD Preconditioner

- (i) Solve subproblems on all faces $\Gamma^{F_{ij}}$, edges $\Gamma^{E_{ij}}$, and vertex spaces Γ^{X_i} :

$$S_{F_{ij}} \mathbf{u}_{F_{ij}} = R_{F_{ij}} \mathbf{g}; \quad S_{E_{ij}} \mathbf{u}_{E_{ij}} = R_{E_{ij}} \mathbf{g}; \quad S_{X_i} \mathbf{u}_{X_i} = R_{X_i} \mathbf{g};$$

- (ii) Find $\mathbf{u}_H \perp KerA_H$ so that: $A_H \mathbf{u}_H = R_H \mathbf{g}$;
- (iii) Calculate $\mathbf{w}_B = R_H^T \mathbf{u}_H + \sum_{ij} R_{F_{ij}}^T \mathbf{u}_{F_{ij}} + \sum_{ij} R_{E_{ij}}^T \mathbf{u}_{E_{ij}} + \sum_l R_{X_l}^T \mathbf{u}_{X_l}$;
- (iv) Find $\bar{\mathbf{w}}_B \in KerS$ such that $\bar{\mathbf{w}}_B + \mathbf{w}_B$ is orthogonal to $KerS$. Then

$$M^{-1} \mathbf{g} = \bar{\mathbf{w}}_B + R_H^T \mathbf{u}_H + \sum_{ij} R_{F_{ij}}^T \mathbf{u}_{F_{ij}} + \sum_{ij} R_{E_{ij}}^T \mathbf{u}_{E_{ij}} + \sum_l R_{X_l}^T \mathbf{u}_{X_l}.$$

Remark: All the subproblems in step (i) and (ii) can be solved simultaneously. The whole preconditioning procedure can be rewritten in a short form:

$$(3.1) \quad M^{-1} \mathbf{g} = R_H^T A_H^{-1} R_H \mathbf{g} + \bar{\mathbf{w}}_B + \sum_{ij} R_{F_{ij}}^T S_{F_{ij}}^{-1} R_{F_{ij}} \mathbf{g} + \sum_{ij} R_{E_{ij}}^T S_{E_{ij}}^{-1} R_{E_{ij}} \mathbf{g} + \sum_l R_{X_l}^T S_{X_l}^{-1} R_{X_l} \mathbf{g}$$

where $\bar{\mathbf{w}}_B \in KerS$ is determined by making $M^{-1} \mathbf{g}$ be orthogonal to the kernel space $KerS$. For problem (2.2), the vector $-\bar{\mathbf{w}}_B \in KerS$ in step (iv) is equal to the mean value of \mathbf{w}_B times a one-vector defined on the interface Γ .

After obtaining the approximate solution $\mathbf{u}_B \perp KerS$ on the interface Γ through using the PCG iterative method, we can calculate the approximate solution of problem (2.3) on the whole domain by solving concurrently all the subproblems defined on the substructures Ω_k with Dirichlet boundary value \mathbf{u}_B on $\partial\Omega_k \subset \Gamma$:

$$A_{II} \mathbf{u}_I = f_I - A_{IB} \mathbf{u}_B.$$

However, the extended solution $\mathbf{u}_h = (\mathbf{u}_I^T, \mathbf{u}_B^T)^T$, is not orthogonal to the kernel space $KerA_h$. Therefore, we have to find a function $\mathbf{w}_h \in KerA_h$ such that the approximate solution $\mathbf{u}_h + \mathbf{w}_h$ is orthogonal to $KerA_h$.

Using the general additive Schwarz framework [10, 11, 12, 13, 19], we can estimate the condition number κ of the modified VSDDM. The proof of the following theorem is similar to that in [11, 18, 19].

THEOREM 3.1. *Suppose the overlapping size is δ_i , then:*

$$\kappa(M^{-1}S) \leq \frac{\lambda_{\max}(M^{-1}S)}{\lambda_{\min}(M^{-1}S)} \leq C(1 + \max_i \frac{H}{\delta_i}),$$

where C is a constant independent of H and h .

4. The Variants of the Modified VSDDM

Assume that $\Omega \subset \mathbf{R}^2$. It is extremely expensive to form exact edge matrices $S_{E_{ij}}$ and vertex matrices S_{X_k} in the VSDDM. This expense in computation and storage can be significantly reduced when exact edge and vertex matrices are replaced by spectrally equivalent approximations [6] which are based on the Fourier approximation and the probing technique. In this section, we further reduce the cost of forming approximations by the probing technique. We use four instead of six probe vectors [6] in the modified VSDDM to form approximate edge matrices and vertex matrices. For simplicity, we first describe the procedure for constructing of these probing approximations on the common edge $\Gamma^{E_{ij}}$ of two adjacent rectangular substructures Ω_i and Ω_j . This technique can easily be extended to more general geometries. On the edge $\Gamma^{E_{ij}}$, we construct a symmetric tridiagonal matrix $\tilde{S}_{E_{ij}}$ to approximate the exact Schur complement $S_{E_{ij}}$ by using matrix vector products of $S_{E_{ij}}$ with two probing vectors. A heuristic motivation for using the tridiagonal approximations is that the entries of each $S_{E_{ij}}$ decay rapidly away from the diagonal : $(S_{E_{ij}})_{lm} = O(\frac{1}{(l-m)^2})$ for l, m away from the diagonal [14].

Let us introduce two probing vectors:

$$p_1 = [1, 0, 1, 0, \dots]^T \quad \text{and} \quad p_2 = [0, 1, 0, 1, \dots]^T.$$

From the fact

$$\tilde{S}_{E_{ij}} [p_1, p_2] = \begin{bmatrix} a_1 & b_1 & & & \\ b_1 & a_2 & b_2 & & \\ & b_2 & a_3 & b_3 & \\ & & b_3 & a_4 & \ddots \\ & & & \ddots & \ddots \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \\ 0 & 1 \\ \vdots & \vdots \end{bmatrix} = \begin{bmatrix} a_1 & b_1 \\ b_1 + b_2 & a_2 \\ a_3 & b_2 + b_3 \\ b_3 + b_4 & a_4 \\ \vdots & \vdots \end{bmatrix},$$

we can construct a symmetric tridiagonal approximate matrix $\tilde{S}_{E_{ij}}$ by letting

$$[\tilde{S}_{E_{ij}} p_1, \tilde{S}_{E_{ij}} p_2] = [S_{E_{ij}} p_1, S_{E_{ij}} p_2].$$

Computing the matrix vector product $S_{E_{ij}}p_k$ requires solving one problem on each sub-domain Ω_i and Ω_j . Hence, an approximate submatrix $\tilde{S}_{E_{ij}}$ can be obtained from the matrix vector products $S_{E_{ij}}p_k$ by using the algorithm [15, 16]:

Symmetric Probe Algorithm

$$\begin{aligned} &\text{For } l = 1, \dots, n_{ij} \\ &a_l = \begin{cases} (S_{E_{ij}}p_1)_l & \text{if } l \text{ is odd} \\ (S_{E_{ij}}p_2)_l & \text{if } l \text{ is even} \end{cases} \\ &b_1 = (S_{E_{ij}}p_2)_1 \\ &\text{For } l = 2, \dots, n_{ij} - 1 \\ &b_l = \begin{cases} (S_{E_{ij}}p_1)_l - b_{l-1} & \text{if } l \text{ is even} \\ (S_{E_{ij}}p_2)_l - b_{l-1} & \text{if } l \text{ is odd} \end{cases} \end{aligned}$$

THEOREM 4.1. Assume that $S_{E_{ij}}$ is a symmetric diagonally dominant $n_{ij} \times n_{ij}$ M-matrix and

$$|(S_{E_{ij}})_{l,l}| \geq |(S_{E_{ij}})_{l,l+1}| \geq \dots \geq |(S_{E_{ij}})_{l,n_{ij}}| \quad \text{for } l = 1, 2, \dots, n_{ij}.$$

Then the symmetric approximate matrix $\tilde{S}_{E_{ij}}$ is also an M-matrix.

PROOF. Without loss of generality, we assume $n_{ij} = 2k + 1$. From the assumption, we have

$$(4.1) \quad |(S_{E_{ij}})_{k,l}| \geq |(S_{E_{ij}})_{s,t}| \quad \text{if } s \leq k, l \leq t,$$

and for $1 \leq l \leq n_{ij}$:

$$a_l = \begin{cases} (S_{E_{ij}})_{l,1} + (S_{E_{ij}})_{l,3} + \dots + (S_{E_{ij}})_{l,2k+1} \geq 0, & \text{if } l \text{ is odd} \\ (S_{E_{ij}})_{l,2} + (S_{E_{ij}})_{l,4} + \dots + (S_{E_{ij}})_{l,2k} \geq 0, & \text{if } l \text{ is even} \end{cases}$$

$$b_1 = (S_{E_{ij}})_{1,2} + (S_{E_{ij}})_{1,4} + \dots + (S_{E_{ij}})_{1,2k} \leq 0$$

follows from the property of M-matrix $S_{E_{ij}}$. Then,

$$|a_1| - |b_1| = (S_{E_{ij}})_{1,1} + (S_{E_{ij}})_{1,2} + (S_{E_{ij}})_{1,3} + \dots + (S_{E_{ij}})_{1,n_{ij}} \geq 0.$$

The equation $b_1 + b_2 = (S_{E_{ij}})_{2,1} + (S_{E_{ij}})_{2,3} + \dots + (S_{E_{ij}})_{2,2k+1} \leq 0$, implies

$$\begin{aligned} b_2 &= (S_{E_{ij}})_{2,1} + (S_{E_{ij}})_{2,3} + \dots + (S_{E_{ij}})_{2,2k+1} \\ &\quad - (S_{E_{ij}})_{1,2} - (S_{E_{ij}})_{1,4} - \dots - (S_{E_{ij}})_{1,2k} \leq 0, \end{aligned}$$

by using inequality (4.1). Then,

$$\begin{aligned} |a_2| - |b_1| - |b_2| &= (S_{E_{ij}})_{2,2} + (S_{E_{ij}})_{2,4} + \dots + (S_{E_{ij}})_{2,2k} \\ &\quad + (S_{E_{ij}})_{1,2} + (S_{E_{ij}})_{1,4} + \dots + (S_{E_{ij}})_{1,2k} \\ &\quad + (S_{E_{ij}})_{2,1} + (S_{E_{ij}})_{2,3} + \dots + (S_{E_{ij}})_{2,2k+1} \\ &\quad - (S_{E_{ij}})_{1,2} - (S_{E_{ij}})_{1,4} - \dots - (S_{E_{ij}})_{1,2k} \\ &= (S_{E_{ij}})_{2,1} + (S_{E_{ij}})_{2,2} + (S_{E_{ij}})_{2,3} + \dots + (S_{E_{ij}})_{2,n_{ij}} \geq 0. \end{aligned}$$

By induction, for any $3 \leq l \leq n_{ij}$, we have

$$\begin{aligned}
 b_l &= (S_{E_{ij}})_{l,1} + (S_{E_{ij}})_{l,3} + \cdots + (S_{E_{ij}})_{l,2k+1} \\
 &\quad - (S_{E_{ij}})_{l-1,2} - (S_{E_{ij}})_{l-1,4} - \cdots - (S_{E_{ij}})_{l-1,2k} \\
 &\quad + b_{l-2} \quad \text{for } l \text{ even,}
 \end{aligned}$$

and

$$\begin{aligned}
 b_l &= (S_{E_{ij}})_{l,2} + (S_{E_{ij}})_{l,4} + \cdots + (S_{E_{ij}})_{l,2k} \\
 &\quad - (S_{E_{ij}})_{l-1,1} - (S_{E_{ij}})_{l-1,3} - \cdots - (S_{E_{ij}})_{l-1,2k+1} \\
 &\quad + b_{l-2} \quad \text{for } l \text{ odd.}
 \end{aligned}$$

So, we can obtain $b_l \leq 0$ through using $b_{l-1} \leq 0$ and $b_{l-2} \leq 0$, and inequality (4.1). Hence,

$$\begin{aligned}
 |a_l| - |b_{l-1}| - |b_l| &= a_l + b_{l-1} + b_l \\
 &= (S_{E_{ij}})_{l,1} + (S_{E_{ij}})_{l,2} + (S_{E_{ij}})_{l,3} + \cdots + (S_{E_{ij}})_{l,n_{ij}} \geq 0.
 \end{aligned}$$

Thus, the tridiagonal symmetric matrix $\tilde{S}_{E_{ij}}$ is positive definite. Note that above inequalities can be replaced by strict inequalities, if $S_{E_{ij}}$ is strictly diagonally dominant. \square

The assumption of this main theorem is always true according to our numerical tests. However, the theoretical proof of this assumption is still an open question.

To minimize the computational work and the approximate errors arising from boundary value on other edges in the constructing procedure of all approximate edge matrices, we will specify the same probe vectors p_k either on all horizontal edges simultaneously or on all vertical edges simultaneously. Let's define \mathbf{p}_k for $k = 1, 2$:

$$\mathbf{p}_k = \begin{cases} p_k & \text{on all horizontal edges} \\ 0 & \text{on all vertical edges} \end{cases}$$

and

$$\mathbf{p}_{k+2} = \begin{cases} 0 & \text{on all horizontal edges} \\ p_k & \text{on all vertical edges.} \end{cases}$$

which are as drawn in Fig.1.

Analogously, these approximations $\tilde{S}_{E_{ij}}$ resulting from the simultaneous probe vectors \mathbf{p}_k above preserve strict diagonally dominance and positive definiteness. Since the edge matrices $\tilde{S}_{E_{ij}}$ are tridiagonal, it is cheap and easy to calculate $\tilde{S}_{E_{ij}}^{-1} \mathbf{g}_{E_{ij}}$.

THEOREM 4.2. *If the Schur complement S on Γ is a strictly diagonally dominant M-matrix, then all the edge approximations $\tilde{S}_{E_{ij}}$ obtained from above are strictly diagonally dominant and positive definite.*

PROOF. From the assumption that M-matrix S is strictly diagonally dominant, $\tilde{S}_{E_{ij}}$ can be proved to be strictly diagonally dominant in the same way as in the proof of Theorem 4.1. \square

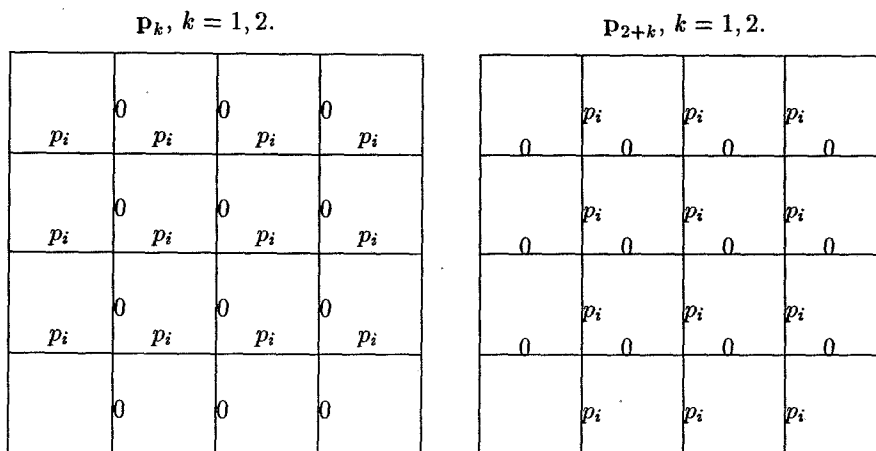


FIGURE 1. Probing Vectors in the Modified VSDDM

The probing approximate vertex matrices \tilde{S}_{X_i} can be easily constructed from the results of probing matrix-vector products $\{S\mathbf{p}_k\}_{k=1}^4$. Details of this technique can be found in [6]. The same argument as in [6] can be used to show that the vertex approximations \tilde{S}_{X_i} obtained from above probing procedure are diagonally dominant if Schur complement S is a diagonally dominant M-matrix.

5. Numerical Results

Now we present the numerical tests on the convergence rate of the modified VS methods with various edge and vertex approximations. The tests were performed for scalar elliptic problem (2.2) with Neumann boundary condition. The following three coefficient functions have been used in our tests:

- (i) $\alpha(x, y) = 1$, the Laplace operator;
- (ii) $\alpha(x, y) = e^{10xy}$, highly varying smooth coefficients;
- (iii) Highly discontinuous coefficients defined as Fig. 2.

The square domain $[0, 1]^2$ is first divided into $1/H^2$ square sub-domains with uniform size H . Then each square sub-domain was triangulated into finite element with uniform mesh size h on the square domain. These problems are discretized by standard finite element method with five stencil.

u is a randomly generated solution of the scalar elliptic problem normalized so that the mean value of u is zero. The integer K is defined to be the number of iterations required to reduce the A-norm of the error $e_n = u - u_n$ by a factor 10^{-5} . We list the iteration number K and the estimated condition number κ for these discrete problems with various coarse mesh size H and fine mesh size h in the following tables. We fix the size of vertex space matrices as 5×5 . So,

$\alpha = 300$	$\alpha = 10^{-4}$	$\alpha = 31400$	$\alpha = 5$
$\alpha = 0.05$	$\alpha = 6$	$\alpha = 0.07$	$\alpha = 2700$
$\alpha = 10^6$	$\alpha = 0.1$	$\alpha = 200$	$\alpha = 9$
$\alpha = 1$	$\alpha = 6000$	$\alpha = 4$	$\alpha = 140000$

FIGURE 2. Coefficient Function $\alpha(x, y)$

the overlapping size is $h/H = H^{-1}/h^{-1}$. FVS and PVS represent the modified VSDDM with the Fourier [6] and the probing approximations, respectively, on edges and vertex spaces. In our program, all sub-problems on coarse grid and on sub-domains are solved with high precision. The numerical results show that the modified VSDDM still has an optimal convergence rate for elliptic problems with Neumann boundary condition.

To compare six probing vectors with four probing vectors in probing VSDDM method, we list the results in Table 2 for problem (2.2) with harmonic Dirichlet boundary condition and various coefficient defined above. Ovlp is denoted as the overlapping size. It can be observed from these numerical results that our probing technique improves efficiency as well as retains the optimal convergence.

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