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A Multi-color Splitting Method and Convergence Analysis for Local Grid Refinement *

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Abstract

This paper consists of two parts. In the first part, a multi-color splitting method is proposed for a multi-processor computer, which can be viewed as an algorithm combining successive subspace correction with parallel subspace correction. In the second part, error estimation and numerical test for a discrete Green's function are presented on local refinement grids.

1. Algorithm of Splitting by color

The features of a typical SIMD (Single Instruction Multiple Data) computer are:

- it contains many processors and hence it works highly in parallel;
- it works highly synchronously.

The method of splitting by color is specially designed for the SIMD architecture. We illustrate it as follows. Consider a second order elliptic equation

$$\begin{aligned} Lu \equiv -\sum_{i,j=1}^{d} \frac{\partial}{\partial x_i}(a_{ij}\frac{\partial}{\partial x_j}u) &= f, \quad \text{in } \Omega \subseteq \mathcal{R}^d, \\ u &= 0, \quad \text{on } \partial\Omega. \end{aligned} \tag{1.1}$$

The associated variational problem is to find $u \in H_0^1(\Omega)$ satisfying

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$$a(u,v) \equiv \int_\Omega \sum_{i,j=1}^{d} a_{ij} \frac{\partial u}{\partial x_i} \frac{\partial v}{\partial x_j} dx = \int_\Omega f v dx, \quad \forall v \in H_0^1(\Omega). \tag{1.2}$$

In order to solve (1.2) on a SIMD computer, subdivide Ω into nonoverlapping subregions: $\bar{\Omega} = \cup \bar{\Omega}_{ij}$, where Ω_{ij} denotes the j^{th} element belonging to the i^{th} color ($i = 1, 2, \cdots, N_c; j = 1, 2, \cdots, m$), and assume that for each color i, Ω_{ij} and Ω_{ik} are disjoint, i.e. $\bar{\Omega}_{ij} \cap \bar{\Omega}_{ik} = \emptyset$, $\forall j, k = 1, \cdots, m$ and $j \neq k$. Let H denote the grid size of the initial grid $\{\Omega_{ij}\}$, h denote that of the refined grid, and S_0^h denote the corresponding finite element space. If $\mathcal{N}(\Omega_{ij})$ denotes the nodal index set of $\bar{\Omega}_{ij}, V_{ij} = Span\{\varphi_k \in S_0^h; k \in \mathcal{N}(\Omega_{ij})\}$, and $V_i = \bigcup_{j=1}^m V_{ij}$, then

$$S_0^h = V_1 + V_2 + \cdots + V_{N_c}. \tag{1.3}$$

We are now looking for $u_h \in S_0^h$ satisfying

$$a(u^h, v) = (f, v), \quad \forall v \in S_0^h. \tag{1.4}$$

Define the projection operator $P_i : S_0^h \longrightarrow V_i$ satisfying

$$a(P_i u, v) = a(u, v), \qquad \forall v \in V_i$$

and $P_{ij} : S_0^h \longrightarrow V_{ij}$ satisfying

$$a(P_{ij} u, v) = a(u, v), \qquad \forall v \in V_{ij}.$$

Evidently, $P_i = \sum_{j=1}^m P_{ij}$ and $P_{ij} P_{ik} = 0$ $(j \neq k)$.

Algorithm 1 (Parallel Algorithm of Splitting by color)

Step 1. Choose an initial $u^0 \in S_0^h$ and relaxation factor $\omega \in (0, 2)$. Set $n := 0$, and $i := 0$.

Step 2. For $j = 1, 2, \cdots, m$, solve for $\delta_{ij} \in V_{ij}$ in parallel, according to

$$a(\delta_{ij}, v) = (f, v) - a(u^{n+i/N_c}, v), \quad \forall v \in V_{ij}.$$

Step 3. Set $u^{n+(i+1)/N_c} = u^{n+i/N_c} + \omega \sum_{j=1}^m \delta_{ij}$.

Step 4. If $i + 1 < N_c$, let $i := i + 1$ and goto Step 2;
If $i + 1 = N_c$, let $i := 0, n := n + 1$ and goto Step 2.

Denote the error by $e^{n+i/N_c} = u - u^{n+i/N_c}$, obviously

$$\begin{aligned} e^{n+1} &= (I - \omega P_{N_c})(I - \omega P_{N_c - 1}) \cdots (I - \omega P_1) e^n \\ &= \prod_{i=1}^{N_c} \prod_{j=1}^{m} (I - \omega P_{ij}) e^n. \end{aligned} \tag{1.5}$$

Hence Algorithm 1 mainly belongs to the framework of SSC(successive subspace correction), but Step 2 is a PSC(parallel subspace correction) algorithm [8].

In order to estimate the convergence rate of Algorithm 1, let $T = P_1 + \cdots + P_{N_c}$. Evidently,

$$\lambda_{max}(T) \leq \|T\| \leq N_c. \tag{1.6}$$

Proposition 1. *There exists a constant $C > 0$ independent of H and h such that*

$$\lambda_{min}(T) \geq C\frac{h^2}{H^2}. \tag{1.7}$$

Proof. Let $\hat{\Omega}_{ij} = \bigcup_{\varphi \in V_{ij}} Supp(\varphi)$, $\hat{\Omega}_{ij} \supset\supset \Omega_{ij}$, and $\{\hat{\Omega}_{ij}\}$ be an open covering of Ω. Construct the piecewise constant functions:

$$Q_i(x) = \begin{cases} 1, & x \in \bar{\Omega}_{ij},\ j = 1, 2, \cdots, m, \\ 0, & \text{elsewhere}, \end{cases}$$

and

$$\varphi_i(x) = \frac{Q_i(x)}{\sum_{i=1}^{N_c} Q_i(x)},$$

then $\sum_{i=1}^{N_c} \varphi_i(x) \equiv 1$. Notice that $diam(\hat{\Omega}_{ij}) = O(H)$, and consider

$$u^h = I^h u^h = \sum_{i=1}^{N_c} I^h(\varphi_i u^h) = \sum_{i=1}^{N_c} u_i^h, \tag{1.8}$$

where $u_i^h = I^h(\varphi_i u^h) \in V_i$, and I^h is the interpolating operator on S_0^h. By the theory of inverse estimation, we have

$$\begin{aligned} a_\Omega(u_i^h, u_i^h) &= \sum_{j=1}^{m} a_{\hat{\Omega}_{ij}}(u_i^h, u_i^h) \leq C_1 \sum_{j=1}^{m} \|u_i^h\|_{1,\hat{\Omega}_{ij}}^2 \\ &\leq C_2 \frac{H^2}{h^2} \sum_{j=1}^{m} \|u_i^h\|_{0,\hat{\Omega}_{ij}}^2. \end{aligned} \tag{1.9}$$

Again from

$$\sum_{j=1}^{m} \|u_i^h\|_{0,\hat{\Omega}_{ij}}^2 = \sum_{j=1}^{m} \int_{\hat{\Omega}_{ij}} (I^h(\varphi_i u^h))^2 dx \leq C_3 \sum_{j=1}^{m} \int_{\hat{\Omega}_{ij}} (u^h)^2 dx,$$

we have

$$\begin{aligned} \sum_{i=1}^{N_c} a(u_i^h, u_i^h) &\leq C_3 \sum_{i=1}^{N_c} \sum_{j=1}^{m} \int_{\hat{\Omega}_{ij}} (u^h)^2 dx \\ &\leq C_4 \frac{H^2}{h^2} \int_\Omega (u^h)^2 dx \leq C_5 \frac{H^2}{h^2} a(u^h, u^h). \end{aligned} \tag{1.10}$$

Finally, (1.7) follows from Lions' Lemma [5].

Corollary 1 (cf.[8]). *The norm of $E = (I-\omega P_{N_c})\cdots(I-\omega P_1)$ can be estimated by*

$$\|E\|^2 \leq 1 - \frac{Ch^2(2-\omega)}{H^2(1+N_c)^2}. \tag{1.11}$$

2. Estimation of the Convergence Rate of Local Grid Refinement

The error estimation of the composite grid method, in particular the error at the neighbourhood of singular points, is essential for engineering problems. Ewing, Lazarov and Vassilevski [2] discussed the error behavior of the finite difference scheme. For the finite element scheme, under the assumption that $u \in H^{1+\alpha}(\Omega)$ $(0 < \alpha < 1)$, Lin and Yan [4] proved that the superconvergence is of $O(H^{3/2} + h^\alpha)$ in the $H^1(\Omega)$-norm. From the engineering point of view, the accuracy of solving a problem with logarithmic singularity on a locally refined grid is very important. For example, consider the following elliptic equation

$$\begin{aligned} Lu &= \delta(x_1 - z_1)\delta(x_2 - z_2), \quad \text{in } \Omega \subset \mathcal{R}^2, \\ u &= 0, \quad \text{on } \partial\Omega, \end{aligned} \tag{2.1}$$

where $\delta(x)$ is the Dirac-function and $z = (z_1, z_2) \in \Omega$ is a singular point. The solution u is the Green function $G_z(x_1, x_2)$ which has a logarithmic singularity at z. In fact, $G_z \notin H_0^1(\Omega)$, but $G_z \in \overset{0}{W}{}_p^1(\Omega), 1 \leq p < 2$.

If $z \in \Omega_1 \subset\subset \Omega$, let H be the grid size of the original coarse grid of Ω, h be the grid size of the refined grid of Ω_1, $S_0^H(\Omega) \subset H_0^1(\Omega), S_0^h \subset H_0^1(\Omega_1)$ be the linear finite element spaces on the original coarse grid and the refined grid respectively, and $H_c = S_0^H(\Omega) + S_0^h(\Omega_1)$ be the finite element space on the composite grid.

Denote the finite element approximation of G_z by G_z^H which satisfies

$$a(G_z^H, v) = v(z), \qquad \forall v \in H_c. \tag{2.2}$$

Let Ω_0 be a small subregion of Ω_1, and $z \in \Omega_0 \subset\subset \Omega_1 \subset\subset \Omega$. We have the following error estimation of G_z^H on Ω_0.

Proposition 2. *If (1) the coefficients $a_{ij}, b_{ij} \in W_1^\infty(\Omega)$, and (2) $[a_{ij}]$ is uniformly positive definite, then there is an $L_p(\Omega)$ estimation*

$$\|G_z - G_z^H\|_{0,p,\Omega_0} \leq C(hH^{2/p-1} + H^2)|\ln H|^2, \quad 1 \leq p < 2, \tag{2.3}$$

where $\|\cdot\|_{k,p,\Omega_0}$ denotes the $W_p^k(\Omega_0)$ norm, $C > 0$ is a constant that is independent of h and H, but dependent on Ω_0 and Ω_1.

Proof. Construct a function $\omega(x) \in C_0^\infty(\Omega_1)$ satisfying $\omega(x) \equiv 1$, $\forall x \in \Omega_0$. Also for all $\varphi \in L_q(\Omega), (\frac{1}{p} + \frac{1}{q} = 1)$, construct an auxiliary function $w(x)$ satisfying:

$$\begin{aligned} Lw &= \varphi, \quad \text{in } \Omega, \\ w &= 0, \quad \text{on } \partial\Omega. \end{aligned} \tag{2.4}$$

Evidently, by the Sobolev Embedding Theorem,

$$c_0\|w\|_{1,\infty,\Omega} \leq \|w\|_{2,q,\Omega} \leq C\|\varphi\|_{0,q,\Omega}, \quad 2 < q < \infty. \tag{2.5}$$

Denote ωu by $\tilde{u}$. Since $G_z \in W_p^1(\Omega), 1 < p < 2$, from integration by parts, we have

$$\begin{aligned}(\widetilde{G_z - G_z^H}, \varphi) &= a(\widetilde{G_z - G_z^H}, w) + I = a_{\Omega_1}(G_z - G_z^H, \tilde{w}) + I \\ &\leq Ch\|G_z - G_z^H\|_{1,p,\Omega_1}\|w\|_{2,q,\Omega_1} + I \\ &\leq Ch\|G_z - G_z^H\|_{1,p,\Omega}\|\varphi\|_{0,q,\Omega} + I, \end{aligned} \tag{2.6}$$

where

$$\begin{aligned} I &= \int_\Omega \{\sum_{i,j=1}^{2} a_{ij}\frac{\partial \omega}{\partial x_i}\frac{\partial w}{\partial x_j} + \frac{\partial}{\partial x_i}(a_{ij}\frac{\partial \omega}{\partial x_j}w)\}(G_z - G_z^H)dx \\ &\leq C\|G_z - G_z^H\|_{0,1,\Omega}\|w\|_{1,\infty,\Omega} \\ &\leq C\|G_z - G_z^H\|_{0,1,\Omega}\|\varphi\|_{0,q,\Omega}. \end{aligned} \tag{2.7}$$

Substituting (2.7) into (2.6),

$$|(\widetilde{G_z - G_z^H}, \varphi)| \leq C(h\|G_z - G_z^H\|_{1,p,\Omega} + \|G_z - G_z^H\|_{0,1,\Omega})\|\varphi\|_{0,q,\Omega}. \tag{2.8}$$

Using the following two known results [9],

$$\|G_z - G_z^H\|_{1,p,\Omega} \leq CH^{2/p-1}|\ln H|,$$
$$\|G_z - G_z^H\|_{0,1,\Omega} \leq CH^2|\ln H|^2,$$

it follows that

$$\|G_z - G_z^H\|_{1,p,\Omega_0} \leq \|\widetilde{G_z - G_z^H}\|_{0,p,\Omega} \leq C(hH^{2/p-1} + H^2)|\ln H|^2.$$

With more detailed analysis [3], the following estimations can be obtained:

$$\|G_z - G_z^H\|_{0,p,\Omega} \leq \begin{cases} C(h^{2/p} + H^2), & 1 < p < +\infty, \\ C(h^2|\ln h|^2 + H^2), & p = 1, \end{cases}$$

and

$$\|G_z - G_z^H\|_{1,p,\Omega_0} \leq \begin{cases} C(h + H^2)h^{2/p-2}|\ln h|^{1/2}, & 1 < p < \infty, \\ C(h + H^2)|\ln h|, & p = 1. \end{cases}$$

Numerical Test. Consider $L = \Delta$, $\Omega = (0,2)\times(0,2)$ in equation (2.1) with $z = (1,1)$. Locally refine $(1/2, 3/2) \times (1/2, 3/2)$ by size $H/2$ and $(1/4, 5/4) \times (1/4, 5/4)$ by size $H/4$. Computed results using the Fast Adaptive Composite Grid Method (FAC)[6] for $G_z(599/600, 601/600)$ are as follows:

Uniform grid			Local refinement grid			
H	Error	CPU	H	Level No.	Error	CPU
1/600	1.30E-2	12.84		1	1.30E-2	12.84
1/1200	2.61E-3	1175.74	1/600	2	2.12E-3	243.97
				3	4.53E-4	452.96

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