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Boundary Elements in Domain Decomposition Methods

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ABSTRACT. We give a brief survey of domain decomposition methods with boundary elements for the model problem of the potential equation, and we offer different parallel algorithms for solving these problems numerically.

1. Introduction

Let us consider the Dirichlet potential problem

$$(1.1)\qquad \begin{aligned} -\operatorname{div} a(x)\nabla u(x) &= 0 \qquad \text{for } x \in \Omega \subset I\!R^2\,, \\ u(x) &= g(x) \quad \text{for } x \in \Gamma := \partial\Omega, \end{aligned}$$

where Ω is a plane, bounded domain with a piecewise Lipschitz continuous boundary Γ. Suppose

$$(1.2)\qquad a(x) = a_i = \textit{constant} \quad \text{in } \Omega_i\,,$$

where the Ω_i are non–overlapping subdomains in a given domain decomposition

$$(1.3)\quad \overline{\Omega} = \bigcup_{i=1}^{p} \overline{\Omega}_i\,,\ \Gamma_i = \partial\Omega_i\,, \quad \Omega_i \cap \Omega_j = \emptyset \ \text{ and } \Gamma_{ij} = \overline{\Omega}_i \cap \overline{\Omega}_j \ \text{ for } i \neq j\,.$$

Now we introduce the skeleton Γ_S and the global coupling boundary Γ_C by

$$(1.4)\qquad \Gamma_S = \bigcup_{i=1}^{p} \Gamma_i\,, \quad \Gamma_C = \Gamma_S \backslash \Gamma\,,$$

and a q–dimensional set ω of all cross points.

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The variational formulation of the model problem (1.1) reads:
Find $u \in H^1(\Omega)$ *with* $u_{|\Gamma} = g(x)$ *such that*

$$\int_\Omega a(x)\,\nabla u(x)\,\nabla v(x)\,dx \;=\; 0 \tag{1.5}$$

for all test functions $v \in H^1_0(\Omega)$.

Before we apply the domain decomposition to (1.5) , we should introduce boundary integral equations corresponding to the local potential problems by use of the Calderon projector

$$\begin{pmatrix} u_{|\Gamma_i} \\ t_i \end{pmatrix} = \begin{pmatrix} \frac{1}{2}I - K_i & V_i \\ D_i & \frac{1}{2}I + K_i' \end{pmatrix} \begin{pmatrix} u_{|\Gamma_i} \\ t_i \end{pmatrix} = C_i \begin{pmatrix} u_{|\Gamma_i} \\ t_i \end{pmatrix}, \tag{1.6}$$

where V_i is the simple layer potential, K_i the double layer potential, K_i' its adjoint, and D_i the hypersingular operator. Further, $t_i = \frac{\partial u}{\partial n}_{|\Gamma_i}$ denotes the normal derivative of the potential u with respect to Γ_i. From (1.6) we get two representations of the local Steklov–Poincaré operator [**12**]:

$$\begin{aligned} t_i \;=\; S_i u_{|\Gamma_i} \;&=\; V_i^{-1}(\tfrac{1}{2}I + K_i)u_{|\Gamma_i} \\ &=\; \left[(\tfrac{1}{2}I + K_i')V_i^{-1}(\tfrac{1}{2}I + K_i) + D_i\right] u_{|\Gamma_i} \end{aligned} \tag{1.7}$$

Using the given domain decomposition (1.3) and Green's formula we may rewrite (1.5) into a variational formulation on the skeleton.
Find $u \in H^{1/2}(\Gamma_S)$ *with* $u_{|\Gamma} = g(x)$ *such that*

$$\sum_{i=1}^{p} \int_{\Gamma_i} S_i u_{|\Gamma_i}\, v_{|\Gamma_i}\, ds \;=\; 0 \tag{1.8}$$

for all test functions $v \in H^{1/2}_0(\Gamma_S)$.

2. Iterative methods with the "Symmetric Formulation"

One possibility for discretization of the Steklov–Poincaré operator in (1.8) is to replace the corresponding normal derivative t_i by the second equation of the Calderon projector. Furthermore, we need the first boundary integral equation to compute the local Cauchy datum [**3**].
Find $u \in H^{1/2}(\Gamma_S)$ *with* $u_{|\Gamma} = g$ *and* $t_i = \frac{\partial u}{\partial n}_{|\Gamma_i} \in H^{-1/2}(\Gamma_i)$ *such that*

$$\begin{aligned} \sum_{i=1}^{p} a_i \left\{ \langle D_i u_{|\Gamma_i}, v_{|\Gamma_i} \rangle_{\Gamma_i} + \tfrac{1}{2}\langle t_i, v_{|\Gamma_i} \rangle_{\Gamma_i} + \langle t_i, K_i v_{|\Gamma_i} \rangle_{|\Gamma_i} \right\} \;&=\; 0 \\ a_i \left(\langle \tau_i, V_i t_i \rangle_{\Gamma_i} - \tfrac{1}{2}\langle \tau_i, u_{|\Gamma_i} \rangle_{|\Gamma_i} - \langle \tau_i, K_i u_{|\Gamma_i} \rangle_{\Gamma_i} \right) \;&=\; 0 \end{aligned} \tag{2.1}$$

for all test functions $v \in H^{1/2}_0(\Gamma_S)$ *and* $\tau_i \in H^{1/2}(\Gamma_i)$ *and* $i = 1, \ldots, p$.

The equivalent discrete system can be written in the block form

$$\begin{pmatrix} V_h & -\frac{1}{2}M_h - K_h \\ \frac{1}{2}M_h^\top + K_h^\top & D_h \end{pmatrix} \begin{pmatrix} \underline{t} \\ \underline{u} \end{pmatrix} = \begin{pmatrix} \underline{f}_t \\ \underline{f}_u \end{pmatrix}, \tag{2.2}$$

where $\underline{t}$ denotes the vector of the local coefficients of the t_i and $\underline{u}$ the coupling values of the potential on the skeleton. The system matrix is obviously block skew–symmetric, but positive definite. Note that the first block equation in (2.2) consists of local equations only; this means that we can compute

$$\underline{t}_i = V_{h,i}^{-1}\left(\frac{1}{2}M_{h,i} + K_{h,i}\right)\underline{u}_{|\Gamma_i} + V_{h,i}^{-1}\underline{f}_{|\Gamma_i} \tag{2.3}$$

independently on the subdomains. Replacing the corresponding values in the second block equation, which describes the coupling across the boundaries, we get the BEM–Schur complement system

$$\left[\left(\frac{1}{2}M_h^\top + K_h^\top\right)V_h^{-1}\left(\frac{1}{2}M_h + K_h\right) + D_h\right]\underline{u} = \underline{f} \tag{2.4}$$

with a symmetric and positive definite stiffness matrix S_h . Therefore we can use a preconditioned conjugate gradient method to solve (2.4) in parallel. We note that the action of S_h includes the inverse of the simple layer potential V_h. However, in (2.3) we need only the action of the local operators $V_{h,i}^{-1}$, this can be done by local preconditoned conjugate gradient methods.

Alternatively, for solving system (2.2) without these additional inner iterations we can use a result of [**2**] . Suppose, that we have given a symmetric and positive definite matrix C_V which is spectrally equivalent to V_h satisfying the inequalities

$$\gamma_1(C_V\underline{t},\underline{t}) \leq (V_h\underline{t},\underline{t}) \leq \gamma_2(C_V\underline{t},\underline{t}) \tag{2.5}$$

for all vectors $\underline{t}$ of the structure according to (2.2). If $\gamma_1 > 1$, we can define a new inner product by

$$\left[\begin{pmatrix} \underline{t} \\ \underline{u} \end{pmatrix}, \begin{pmatrix} \underline{\tau} \\ \underline{v} \end{pmatrix}\right] = ((V_h - C_V)\underline{t},\underline{\tau}) + (\underline{u},\underline{v}) \tag{2.6}$$

and we have as a main result of [**2**] that the transformed matrix

$$A_h = \begin{pmatrix} C_V^{-1}V_h & -C_V^{-1}(\frac{1}{2}M_h + K_h) \\ (\frac{1}{2}M_h^\top + K_h^\top)C_V^{-1}(C_V - V_h) & D_h + (\frac{1}{2}M_h^\top + K_h^\top)C_V^{-1}(\frac{1}{2}M_h + K_h) \end{pmatrix} \tag{2.7}$$

now is self–adjoint and positive definite with respect to the newly defined product (2.6). Moreover, there hold spectral equivalence inequalities

$$\lambda_1\left[R\begin{pmatrix} \underline{t} \\ \underline{u} \end{pmatrix}, \begin{pmatrix} \underline{t} \\ \underline{u} \end{pmatrix}\right] \leq \left[A_h\begin{pmatrix} \underline{t} \\ \underline{u} \end{pmatrix}, \begin{pmatrix} \underline{t} \\ \underline{u} \end{pmatrix}\right] \leq \lambda_2\left[R\begin{pmatrix} \underline{t} \\ \underline{u} \end{pmatrix}, \begin{pmatrix} \underline{t} \\ \underline{u} \end{pmatrix}\right] \tag{2.8}$$

with

$$R = \begin{pmatrix} I & 0 \\ 0 & D_h + (\frac{1}{2}M_h^\top + K_h^\top)V_h^{-1}(\frac{1}{2}M_h + K_h) \end{pmatrix} . \tag{2.9}$$

Therefore we can use a parallel preconditioned conjugate gradient method with respect to the transformed matrix (2.7) in the special inner product (2.6) to solve the original system (2.2) [**10**]. As preconditioner for the discrete Schur complement in R we use the BPS preconditioner, as proposed in [**1**].

3. Dirichlet Schur complement CG with hierarchical bases

In the methods described above it is necessary to discretize the local hypersingular operators D_i, which may be complicated. If we use only the first integral equation, we will get from (1.8) a new variational formulation:
Find $u \in H^{1/2}(\Gamma_S)$ *and* $t_i = \frac{\partial u}{\partial n}_{|\Gamma_i} \in H^{-1/2}(\Gamma_i)$ *such that*

$$\begin{aligned} \sum_{i=1}^{p} a_i \langle t_i, v_{|\Gamma_i} \rangle_{\Gamma_i} &= 0 \\ \langle \tau_i, V_i t_i \rangle_{\Gamma_i} - \tfrac{1}{2}\langle \tau_i, u_{|\Gamma_i} \rangle_{\Gamma_i} - \langle \tau_i, K_i u_{|\Gamma_i} \rangle_{\Gamma_i} &= 0 \end{aligned} \tag{3.1}$$

for all test functions $v \in H_0^{1/2}(\Gamma_S)$ *and* $\tau_i \in H^{-1/2}(\Gamma_i)$ *and* $i = 1, \ldots, p$.

In analogy to (2.2) we can write the equivalent discrete system in the block form

$$\begin{pmatrix} M_h^\top & 0 \\ V_h & \frac{1}{2}M_h + K_h \end{pmatrix} \begin{pmatrix} \underline{t} \\ \underline{u} \end{pmatrix} = \begin{pmatrix} 0 \\ \underline{f}_u \end{pmatrix} . \tag{3.2}$$

Using (2.3) we can replace $\underline{t}$ in the first block equation to get the discrete Dirichlet representation S_D of the Steklov–Poincaré operator

$$M_h^\top V_h^{-1} \left(\frac{1}{2}M_h + K_h \right) \underline{u} = \underline{f} . \tag{3.3}$$

According to the properties of the symmetric representation of S_h in (2.4) we can symmetrize S_D. Therefore we can use a preconditioned conjugate gradient method to solve (3.3) in parallel.

If we use the BPS preconditioner, a transformation to a hierarchical basis is required to solve a coarse grid system to get the preconditioned values at the vertices. Here we use this idea already with respect to the ansatz functions. We introduce a coarse grid function

$$u_H(x) = \sum_{k=1}^{q} u_k \varphi_k^H(x) , \quad u_k = u(x_k) \text{ for all } x_k \in \omega , \tag{3.4}$$

to a given function $u(x)$. Then the resulting fine grid function $\tilde{u}(x) = u(x) - u_H(x)$ vanishes at all cross nodes $x_k \in \omega$. In general, this decomposition is not unique. Choosing discrete harmonic functions as basis functions in (3.4), the projection

of a function $u(x)$ onto the coarse grid is unique. Such basis functions can be described as follows:

(i) $\varphi_k^H(x_l) = \delta_{kl}$ for all $x_l \in \omega$ and $k = 1, \ldots, q$,

(ii) $\varphi_k^H(x)$ are piecewise linear on all edges Γ_{ij},

(iii) $\Delta\varphi_k^H(x) = 0$ in all subdomains Ω_l , $l = 1, \ldots, p$.

Using the hierarchical bases constructed above the Galerkin discretization of (3.1) leads to the block system

$$\begin{pmatrix} K_V & K_{VE} \\ K_{EV} & K_E \end{pmatrix} \begin{pmatrix} \underline{u}_V^H \\ \underline{u}_E^H \end{pmatrix} = \begin{pmatrix} \underline{f}_V \\ \underline{f}_E \end{pmatrix} \tag{3.5}$$

where every block matrix describes the action of the discrete Steklov–Poincaré operator (3.3) with respect to the corresponding basis functions. Finally we use a preconditioned conjugate gradient method to solve the Schur complement system to (3.5)

$$\left[K_E - K_{EV}K_V^{-1}K_{VE}\right] \underline{u}_E^H = \underline{f}, \tag{3.6}$$

which requires solving a coarse grid system in every iteration step.

4. Numerical results

As a simple example, we consider the problem (2.1) with a constant coefficient function $a(x) \equiv 1$ and a domain decomposition of the unit square into 16 subdomains. Method 1 is the symmetric Schur complement system (2.4). The conjugate gradient method with respect to the transformed matrix (2.7) is the method 2. Both cases result from the symmetric formulation including the full Calderon projector. In the other cases of the Dirichlet formulation it is necessary to solve local Dirichlet problems using the first boundary integral equation to realize the Steklov–Poincaré operator. Then we get method 3 with respect to the normal nodal bases and method 4 by using hierarchical bases.

All methods described above are iterative schemes which stop at a predefined bound with respect to a residual norm. For a relative error reduction of $\varepsilon = 10^{-6}$ we get the following results.

Table 1.

Nodes per	Symmetric Formulation		Dirichlet Formulation	
Subdomain	Method 1	Method 2	Method 3	Method 4
16	12	21	12	9
32	13	23	14	11
64	14	24	15	13
128	15	25	16	16
256	17	26	17	19

As preconditioners of the coupling nodes we use the BPS preconditioner. In all cases presented we need locally spectral equivalent matrices to the simple layer potentials $V_{h,i}$; here we use circulant matrices derived from the single layer potential with respect to a circle [**11**].

All the algorithms presented in this paper can be extended to more general problems, as e.g., to linear elasticity and, moreover, to three–dimensional problems.

All computations are executed on different parallel computer systems like the *Parsytec MC3*, the *Intel Paragon*, or at workstation clusters.

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