Contemporary Mathematics
Volume 180, 1994

AN OVERDETERMINED SCHWARZ ALTERNATING METHOD

VICTOR H. SUN AND WEI-PAI TANG

ABSTRACT. In this paper, a new type of coupling on the artificial boundary layer is proposed for the classical *SAM*. The convergence rate is demonstrated for this algorithm. Numerical testing has been carried out for a variety of problems.

1. INTRODUCTION

It is well known that the rate of convergence of the Schwarz alternating method increases with the amount of overlap, yet large overlap is not desirable. One interesting problem is how to improve convergence speed without increasing overlap. Tang proposed the *generalized Schwarz Alternating method* (*GSAM*) a few years ago [7], which applies the Robin boundary condition, $\omega u + (1-\omega)\frac{\partial u}{\partial n}$, on the artificial boundaries. A similar idea was presented by P. Lions [3]. Tang's work was generalized recently by Tan and Borsboom [6]. It was shown in [7] that rapid convergence can be achieved with the minimum overlap, if an optimal ω is chosen. The convergence rate is even faster than the classical *SAM* with a large overlap, but convergence can be a sensitive function of this parameter.

The *GSAM* applies the Robin condition only on the artificial boundaries. In this paper, we introduce an artificial boundary layer along the artificial boundary. The Robin condition is imposed on the entire layer. As we will demonstrate, this is beneficial to the convergence behavior of this approach.

In the next section, a new method, the *OSAM*, is introduced. The equivalent theorem and convergence behavior for the model problem is given in Section 3, and numerical results are shown in Section 4. Section 5 concludes the paper.

2. OVERDETERMINED *SAM* AND BOUNDARY LAYER RECONSTRUCTION

Consider the following boundary value problem:

$$\left\{\begin{array}{rcll} L(u) & = & f, & \Omega, \\ u & = & g, & \partial\Omega, \end{array}\right. \tag{1}$$

1991 *Mathematics Subject Classification.* 65F10, 65N10.
This research was supported by the Natural Sciences and Engineering Research Council of Canada.
This paper is in final form and no version of it will be submitted for publication elsewhere.

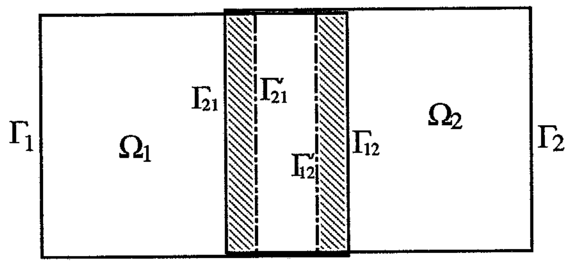

FIGURE 1. Artificial Boundary Layers

where L represents a general linear second order elliptic operator, and Ω is a bounded region in R^2. For simplicity, we consider only the two overlapping subdomain case (see Fig. 1). The generalization to an irregular solution domain or multi-subdomain problem is straightforward.

The rectangular solution region Ω is partitioned into two overlapping subdomains Ω_1 and Ω_2. Let

$$\Gamma_{21} = \partial\Omega_2 \cap \Omega_1,\ \Gamma_{12} = \partial\Omega_1 \cap \Omega_2,\ \Gamma_1 = \partial\Omega \cap \partial\Omega_1,\ \Gamma_2 = \partial\Omega \cap \partial\Omega_2.$$

We introduce two artificial boundary layers $\mathcal{L}_1$ and $\mathcal{L}_2$ (the shaded regions in Fig. 1), which are next to the corresponding artificial boundaries Γ_{12} and Γ_{21}. The thickness of these layers depends on the grid. When a uniform grid is employed, the corresponding thickness of the boundary layer will be the grid size h. For a general triangular mesh, the boundary layer will be the union of all triangles for which at least one of its edges or nodes is on the artificial boundary. The motivation for this choice is to allow the minimum overlap needed in the new algorithm. Let Γ_{12}' and Γ_{21}' denote the inner boundaries of the boundary layers $\mathcal{L}_1$ and $\mathcal{L}_2$, respectively.

The Robin condition is imposed on the entire boundary layer. Consequently, a new problem can be formulated as follows:

$$(2) \qquad \begin{cases} L(u_1) = f,\ \Omega_1, \\ u_1|_{\Gamma_1} = g, \\ b(u_1)|_{\mathcal{L}_1} = b(u_2)|_{\mathcal{L}_1}, \end{cases} , \qquad \begin{cases} L(u_2) = f,\ \Omega_2, \\ u_2|_{\Gamma_2} = g, \\ b(u_2)|_{\mathcal{L}_2} = b(u_1)|_{\mathcal{L}_2}, \end{cases}$$

where $b(u) = \omega u + (1-\omega)\frac{\partial u}{\partial n}$. There are many choices for the function ω. In this paper, we investigate only a very special case:

$$\omega = \begin{cases} 1, & (x,y) \in \textit{artificial boundary}, \\ 0, & \textit{otherwise}. \end{cases}$$

Problem (2) is overdetermined. In general, no solution may exist for an overdetermined problem, but in this case, the following result is trivially true:

Lemma 2.1. *If a solution of (1) exists and is unique, then a solution for problem (2) also exists and is unique.*

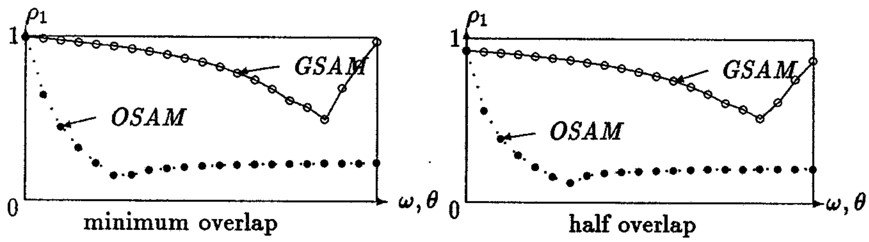

FIGURE 2. Maximum eigenvalues for 2-D seven subdomains

Actually, the solutions of (1) and (2) are identical. On the other hand, an iterative algorithm for solving problem (2) can easily be extended from the classical *SAM*.

To design a deterministic algorithm, reconstruction of the coupling on the boundary layer is necessary. When the problem is discretized, the constraint on the artificial boundary layer affects only the grid nodes on the boundary of the layer.

We observe that the solution on the inner side of the boundary layer satisfies both the difference operator and the derivative coupling between two solutions on the neighboring subdomain. Therefore, a natural approach is to apply a weighted combination of these two conditions. Namely, a linear combination of the difference operator and the Neumann condition is used to eliminate the extra constraint. In our numerical tests, the parameter $\theta(0 < \theta \leq 1)$ represents the weight for the difference operator part.

3. Convergence result

Applying matrix analysis as in [7], the *OSAM* can be formulated as the solution to an enhanced matrix problem $\tilde{A}'x' = \tilde{b}'$ of $Ax = b$, which is the discretized form of (1)[5]. The convergence rate is then determined by the spectral radius of the Jacobian iterative matrix. For the model problem, we have the following results.

Theorem 3.1. *For the model problem, matrix A is equivalent (in the sense of [7]) to its enhanced OSAM matrix $\tilde{A}'$, for $0 \leq \theta \leq 1$.*

Figure 2 shows the maximum eigenvalues of the Jacobi iterative matrix as a function of the parameters. Detailed analysis can be found in [5]. It can be seen that the *OSAM* shows better convergence behavior than *GSAM* or *SAM* (which is the special case of the *GSAM* with $\omega = 1$). The sensitivity of the convergence rate for the *OSAM* to its parameter is much reduced. Moreover, *OSAM* with a minimum overlap is still much better than *SAM* with a half overlap.

4. Numerical tests

Results for several test problems in the 2-D and 3-D cases are presented in this section. The differential equations are discretized by the standard central difference scheme. For each 2-D test problem, the solution domain is decomposed into a different number of subdomains. In all the cases, each subdomain contains a 20×20 grid and minimum overlap is considered. Domain decomposition is used as a preconditioner and Bi-CGSTAB is employed for the acceleration scheme. The convergence test is to require a residual reduction of 10^5.

In the tables of results, "SAM" represents the result for the traditional Schwarz alternating method. The "*" and "**" mean that the *OSAM* does not converge within 100 and 200 iterations, respectively, and the iteration is stopped. "Iter" and "SubD" represent the number of linear iterations needed to reach the precision and the total number of subdomains, respectively. We also define the improvement factor to be $\tau = \frac{SAM-OSAM}{SAM}$. The notation τ_b represents the best improvement factor of the results.

4.1. Stress in helical spring. The first problem tested is [2]

$$\Phi_{xx} + \Phi_{yy} + \frac{3}{5-y}\Phi_x = f, \ x \in \Omega,$$

where $\Omega = (-.5, .5) \times (-1, 1)$, and $\Phi = 0$ on the boundary. The problem has an exact solution [4]: $\Phi = (1-y^2)(1-4x^2)(5-y^3)(0.0004838y + 0.0010185)$.

The numerical results are shown in Table 1a. It can be seen that the *OSAM* performs better than the *SAM* with only one exception of $\theta = 0.1$. For the optimal case, $\theta = 0.3$, *OSAM* takes only half the number of iterations of *SAM*.

4.2. Discontinuous coefficient problem. In this test, the following equation was considered

$$(K_x u_x)_x + (K_y u_y)_y + u_x + u_y = \sin(\pi xy),$$

where $u = 0$ on the boundary of unit square, and

$$K_x = K_y = \begin{cases} 1, & [0,.5] \times [0,.5] \cup [.5,1] \times [.5,1], \\ 10^{+3}, & [.5,1] \times [0,.5], \\ 10^{-3}, & [0,.5] \times [.5,1]. \end{cases}$$

Stress in helical spring

SubD	8×12	6×8	3×4
θ	Iter	Iter	Iter
0.1	68	42	18
0.2	17	12	7
0.3	14	11	6
0.4	16	13	6
0.5	17	14	6
0.6	17	13	6
0.7	18	14	7
0.8	20	15	7
0.9	20	15	7
1.0	19	14	9
SAM	29	20	10
τ_b	0.517	0.450	0.400

a

Discontinuous coefficients

SubD	10×10	8×8	4×4
θ	Iter	Iter	Iter
0.1	*	*	*
0.2	87	63	*
0.3	16	15	10
0.4	16	12	8
0.5	13	11	7
0.6	16	11	7
0.7	17	12	7
0.8	19	11	9
0.9	16	11	9
1.0	17	12	10
SAM	31	22	13
τ_b	0.548	0.500	0.462

b

TABLE 1.

Results for indefinite problem

SubD	10×10		8×8		4×4	
θ	Iter		Iter		Iter	
	c=-20	c=-70	c=-20	c=-70	c=-20	c=-70
0.1	*	**	*	**	55	**
0.2	*	**	*	**	29	173
0.3	*	**	*	194	18	76
0.4	21	49	18	38	10	29
0.5	20	58	17	31	9	26
0.6	21	46	18	38	9	27
0.7	23	43	18	37	9	21
0.8	22	49	18	33	9	19
0.9	26	60	19	48	10	19
1.0	25	82	18	41	11	19
SAM	37	144	30	85	14	42
r_b	0.459	0.713	0.433	0.665	0.357	0.548

TABLE 2.

Harmonic weighting is used at points of discontinuity in K_x, K_y. The results are shown in Table 1b. It can be seen that the improvement factor in this test is larger than that in the previous case. From the result of this test, it appears that *OSAM* shows great potential for solving difficult problems. This will be further demonstrated by the following test.

4.3. Variable-coefficient, indefinite problem. The problem tested is

$$\begin{aligned} Lu \;=\; & -[(1+\frac{1}{2}\sin(50\pi x))u_x]_x - [(1+\frac{1}{2}\sin(50\pi x)\sin(50\pi y))u_y]_y \\ & + 20\sin(10\pi x)\cos(10\pi y)u_x - 20\cos(10\pi x)\sin(10\pi y)u_y + cu \end{aligned} \tag{3}$$

where $u = \exp(xy)\sin(\pi x)\sin(\pi y)$ is defined on a unit square [1], for $c < 0$. In this work, the cases for $c = -20, -70$ were tested. We will see that the second case requires much more work than the first one for the same decomposition form.

The numerical results are reported in Table 2. From these results, the difficulty of this problem is obvious as compared with the corresponding iteration count of previous tests.

For $c = -20$, the problem is more weakly indefinite than for $c = -70$ with the same grid size. The results show that the work for $c = -20$ is only a little more than the corresponding situations in the last two tests. The improvement factor varies from one-third to one-half. However, when $c = -70$, *OSAM* demonstrates its superiority over *SAM*. The improvement factors for all three decomposition cases are greater than one-half. This test further demonstrates that *OSAM* performs better for difficult problems.

4.4. Helmholtz equation. Here we present the results for the Helmholtz equation $-\Delta u + u = f$ in 3-D case. The true solution is $u = \exp(xy)\sin(\pi z)$ in the cube. The solution domain is decomposed into 8 strips in z-direction. The total number of unknowns is 73^3.

Results for Helmholtz equation

θ	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0	SAM
Iter	8	7	6	7	6	5	6	6	6	6	10

TABLE 3.

The numerical results are presented in Table 3. In the best case, *OSAM* took only one half as many iterations as did *SAM*.

5. CONCLUSION

In this paper, a new extension of the classical *SAM* – *OSAM* – is proposed. In this new approach, a stronger coupling is imposed on the artificial boundary layers. The superior convergence behavior is demonstrated for a variety of test problems. In particular, the weighted parameter θ does not have the sensitivity problem from which the *GSAM* suffers. So far, our testing is restricted to a single level approach. A multilevel preconditioner approach is the natural future extension of this work.

REFERENCES

1. X.-C. Cai, W. Gropp, and D. Keyes, *A comparison of some domain decomposition algorithm for nonsymmetric elliptic problems*, Fifth International Symposium on Domain Decomposition Methods for Partial Differential Equations (Philadelphia, PA) (T. Chan, D. Keyes, G. Meurant, J. Scroggs, and R. Voigt, eds.), SIAM, 1992.
2. L. Collatz, *The numerical treatment of differential equations*, 3rd ed., Spring-Verlag, New York, 1966.
3. P. L. Lions, *On the Schwarz alternating method III: A variant for nonoverlapping subdomains*, Third International Symposium on Domain Decomposition Methods for Partial Differential Equations (Philadelphia, PA) (T. Chan, R. Glowinski, J. Periaux, and O. Widlund, eds.), SIAM, 1989, pp. 202–216.
4. J. Rice and R. Boisvert, *Solving elliptic problems using ELLPACK*, Spring-Verlag, New York, 1985.
5. Huosheng Sun and Wei-Pai Tang, *An Overdetermined Schwarz alternating method*, Tech. Report CS-93-53, University of Waterloo, Dept. of Computer Science, 1993.
6. K.H. Tan and M.J.A. Borsboom, *Problem-dependent optimization of flexible couplings in domain decomposition methods, with an application to advection-dominated problems*, Seventh International Symposium on Domain Decomposition Methods for Partial Differential Equations (University Park, PA) (D. Keyes and J. Xu, eds.), AMS, 1993.
7. W.-P. Tang, *Generalized Schwarz splitting*, SIAM J. Sci. Stat. Comp. **13** (1992), 573–595.

DEPT. OF APPL. MATH., UNIVERSITY OF WATERLOO, WATERLOO, ON, CANADA, N2L 3G1
E-mail address, V.H. Sun: hsun@yoho.uwaterloo.ca

DEPT. OF COMP. SCI., UNIVERSITY OF WATERLOO, WATERLOO, ON, CANADA, N2L 3G1
E-mail address, W.-P. Tang: wptang@lady.uwaterloo.ca