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Domain Decomposition For Linear And Nonlinear Elliptic Problems Via Function Or Space Decomposition

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ABSTRACT. In this article, we use a function decomposition method and a space decomposition method of [5] to derive some parallel overlapping and nonoverlapping domain decomposition methods for self-adjoint linear and nonlinear elliptic problems. The function decomposition method and the space decomposition method use different starting points in doing the domain decomposition.

1. The function decomposition and space decomposition methods

In [5], function decomposition and space decomposition methods were proposed for a general convex programming problem. Here, we shall briefly show how we can use the methods for overlapping and nonoverlapping domain decomposition methods for linear and nonlinear elliptic problems.

It was shown, see [5], that by suitably decomposing the energy function for an elliptic problem, we can derive the classical Alternating Direction methods, see [1, 4]. By using different function decompositions, we can also derive some nonoverlapping domain decomposition methods for these problems. This shows that the Alternating Direction methods and the domain decomposition methods are just different ways of decomposing a problem, or in other words that we can use the splitting methods to get domain decomposition methods for some linear and nonlinear elliptic and parabolic problems.

The concept of space decomposition was first introduced in a review paper [8]. There many multigrid and domain decomposition methods are presented and analyzed. It is known that the overlapping domain decomposition methods,

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the substructuring methods, [2], and the multilevel methods, [6, 7], give some nice ways to decompose the finite element spaces. In the published papers, their convergence behaviour has been carefully analyzed for linear problems. By using the space decomposition approach of [5], we try to show that if these methods can be used for linear problems, they can also be used for some nonlinear problems.

First, let us recall the results of [5]. We consider minimization

$$\min_{v\in K} F(v)\ , \qquad K\subset V\ . \tag{1}$$

In case of function decomposition, we need to assume:

(F1). The space V is a Hilbert space and there exist Hilbert spaces $V_i,\ i=1,2,\cdots,m$, such that $V=\cap_{i=1}^m V_i$.

(F2). The function $F: V\mapsto R$ is convex, lower–semicontinuous in V and there exist convex, lower–semicontinuous functions $F_i: V_i\mapsto R$ in $V_i,\ i=1,2,\cdots,m$, such that $F(v)=\sum_{i=1}^m F_i(v),\quad \forall v\in V$.

(F3). The subset K is closed and convex in the norm of V. There exist convex subsets $K_i\subset V_i,\ i=1,2,\cdots,m$, such that $K=\cap_{i=1}^m K_i$.

(F4). There exists a Hilbert space H such that $V\subset V_i\subset H,\ i=1,2,\cdots,m$.

In case of space decomposition, we need to assume:

(S1). The space V is a reflexive Banach and there exist reflexive Banach spaces $V_i,\ i=1,2,\cdots,m$, such $V=V_1+V_2+\cdots+V_m$.

(S2). The subset K is closed and convex in the norm of V. There exist closed and convex subsets K_i of V_i such that $K=K_1+K_2+\cdots+K_m$.

(S3). The function $F(v)$ is convex, lower-semicontinuous in the norm of V and satisfies $\lim_{\|v\|_V\to+\infty}\frac{F(v)}{\|v\|_V}=+\infty$.

(S4). There exist constants $C_0,\ C_1$ such that $C_0\|\sum_{i=1}^m v_i\|_V^2\le\sum_{i=1}^m\|v_i\|_V^2,\ \forall v_i\in V_i,\ i=1,2,\cdots,m$, and

$$\begin{cases}\forall v\in V,\ \exists v_i\in V_i, i=1,2,\cdots,m, \text{ such that}\\ \displaystyle\sum_{i=1}^m v_i=v \text{ and } \sum_{i=1}^m\|v_i\|_V^2\le C_1\|v\|_V^2\ .\end{cases}$$

Under (F1)–(F4), we find that the minimization (1) is equivalent to

$$\min_{\substack{(v_1,v_2,\cdots v_m)\in\prod_{i=1}^m K_i\\ v_1=v_2=\cdots=v_m}}\sum_{i=1}^m F_i(v_i)\ . \tag{2}$$

This is a minimization of a separable structure under the extra constraint $v_1=v_2=\cdots=v_m$. In order to use a parallel method, we need to introduce a new variable v and realize the above constraint by enforcing $v_i=v, i=1,2,\cdots,m$. We will use augmented Lagrangian methods to deal with it. We define L_r on $H\times\prod_{i=1}^m V_i\times H^m$ by

$$L_r(v,v_i,\mu_i)=\sum_{i=1}^m F_i(v_i)+\frac{1}{m}\sum_{i=1}^m(\mu_i,v_i-v)_H+\frac{r}{2m}\sum_{i=1}^m\|v_i-v\|_H^2\ .$$

We will seek a saddle point for L_r over $H \times \prod_{i=1}^m K_i \times H^m$. We say (u, u_i, λ_i) is a saddle point if

$$L_r(u, u_i, \mu_i) \le L_r(u, u_i, \lambda_i) \le L_r(v, v_i, \lambda_i), \quad \forall v \in H, v_i \in K_i, \mu_i \in H .$$

It is easy to prove that, if (u, u_i, λ_i) is a saddle point for L_r, then u is a minimizer for (1). Under (F1)–(F4), we get the following parallel algorithm for (1):

ALGORITHM 1.

Step 1. *Choose initial values $u_i^0 \in K_i$ and $\lambda_i^0 \in H$ $(i = 1, 2, \cdots m)$, and positive numbers $r > 0$, and $\rho \in (0, \frac{1+\sqrt{5}}{2})r$.*

Step 2. *For $n \ge 1$, set*

$$u^n = \frac{1}{m}\sum_{i=1}^m u_i^{n-1} + \frac{1}{rm}\sum_{i=1}^m \lambda_i^{n-1} .$$

Step 3. *Find $u_i^n \in K_i$, $i = 1, 2, \cdots, m$ in parallel such that*

$$\begin{aligned}(3)\qquad & F_i(u_i^n) + \frac{1}{m}(\lambda_i^{n-1}, u_i^n)_H + \frac{r}{2m}\|u_i^n - u^n\|_H^2 \\ \le & F_i(v_i) + \frac{1}{m}(\lambda_i^{n-1}, v_i)_H + \frac{r}{2m}\|v_i - u^n\|_H^2 , \quad \forall v_i \in K_i .\end{aligned}$$

Step 4. *Update the multipliers and go to step 2: $\lambda_i^n = \lambda_i^{n-1} + \rho(u_i^n - u^n)$.*

The following theorem (see [**5**]) shows the convergence:

THEOREM 1. *Suppose L_r has a saddle point over $H \times \prod_{i=1}^m K_i \times H^m$. There exists a unique solution u_i^n such that (3) is satisfied. If conditions (F1)–(F4) are valid, and F_i is Gateaux differentiable with the inner product of H, then we have estimate:*

$$\sum_{i=1}^N \sum_{i=1}^m (F_i'(u_i^n) - F_i'(u), u_i^n - u)_H \le \frac{C_2 m}{2\rho} + \frac{C_3 r}{2m}, \quad \forall N > 0 .$$

The constants C_2 and C_3 depend only on the initial functions λ_i^0, u_i^0 and the solution u of (1).

Next, we discuss the space decomposition. Under conditions (S1)–(S2), we can see that, if $(u_1, u_2, \cdots, u_m)$ is a minimizer for

$$(4)\qquad \min_{(v_1, v_2, \cdots v_m) \in \prod_{i=1}^m K_i} F(v_1 + v_2 \cdots + v_m) ,$$

then $\sum_{i=1}^m u_i$ is a minimizer for (1). We use Jacobi method to find a solution for the minimization (4):

ALGORITHM 2.

Step 1. *Choose $u_i^0 \in K_i$ and relaxation parameters $\alpha_i > 0$ such that $\sum_{i=1}^m \alpha_i \le 1$.*

Step 2. *For $n \ge 1$, find $u_i^{n+\frac{1}{2}} \in K_i$ in parallel for $i = 1, 2, \cdots, m$ such that*

$$F(\sum_{k=1, k \ne i}^m u_k^n + u_i^{n+\frac{1}{2}}) \le F(\sum_{k=1, k \ne i}^m u_k^n + v_i) , \quad \forall v_i \in K_i .$$

Step 3. *Set u_i^{n+1} as: $u_i^{n+1} = u_i^n + \alpha_i(u_i^{n+\frac{1}{2}} - u_i^n)$, and go to step 2.*

For this algorithm, we have the following convergence result (see [**5**]):

THEOREM 2. *Under conditions (S1)–(S4), we assume each K_i is a bounded subset in V or $K_i = V_i$, function F is Gateaux differentiable and locally uniformly convex over bounded subsets in V and F' is uniformly continuous over bounded subsets in V, then we have for Algorithm 2 the convergence*

$$u^{n+1} = \sum_{i=1}^{m} u_i^{n+1} \rightarrow u \text{ strongly in } V \text{ as } n \rightarrow \infty .$$

As was observed in [**8**], the Jacobi method may not converge for general space decomposition problems. Here, by using a suitable under relaxation, we get the sufficient condition of convergence even for general minimization problems. In case that F' is Lipschitz continuous and coercive, an error estimate in a weak form was proved in [**5**], which shows the dependence of the convergence on constant C_1/C_0.

2. Applications to domain decomposition

Let us consider the model problem:

$$\min_{v \in W_0^{1,s}(\Omega)} \left(\int_\Omega \left(\frac{1}{s} |\nabla v|^s - fv \right) dx \right) . \tag{5}$$

We assume $s \geq 2$. If $s = 2$, it represents a typical self–adjoint linear elliptic equation; if $s \neq 2$, it is a nonlinear elliptic equation. We will restrict our consideration only to the discrete case. As in Glowinski and Marrocco [**3**], if we replace the Sobolev space $W_0^{1,s}$ by a finite element space and carry out the minimization of (5) over it, the finite element solution will converge to the minimizer of (5).

Assume Ω has been partitioned into finite elements $\mathcal{T}_h$ and the union of the finite elements form a discrete domain Ω_h. Let us define S_h as the nonconforming finite element space and V_h as the conforming finite element space of k^{th} order polynomials, i.e.

$$S_h = \{v_h | \quad v_h \in P_k(e), \forall e \in \mathcal{T}_h, v_h = 0 \text{ on } \partial\Omega_h\} ,$$
$$V_h = \{v_h | \quad v_h \in C^0(\Omega), v_h \in P_k(e), \forall e \in \mathcal{T}_h, v_h = 0 \text{ on } \partial\Omega_h\} .$$

We define the inner product of S_h and V_h as $(u,v)_{S_h} = \sum_{e \in \mathcal{T}_h} (u,v)_{H^1(e)}$. The discrete version of (5) is:

$$\min_{v_h \in V_h} \sum_{e \in \mathcal{T}_h} \int_e \left(\frac{1}{s} |\nabla v_h|^s - f v_h \right) dx . \tag{6}$$

We assume Ω_h has been partitioned into nonoverlapping subdomains Ω_1 and Ω_2, and each subdomain is the union of some elements of $\mathcal{T}_h$. This does not limit us to two parallel processors, because each Ω_i can again contain many disjoint subdomains. Here we will consider only the case that each Ω_i is a single connected subdomain. We will report elsewhere on the case that each Ω_i contains many disjoint subdomains. In order to use our algorithm, let us take $m = 2$, $V_1 = V_2 = V = S_h$, $H = S_h$, $K = V_h$, and

$$F_{h,i}(v_h) = \sum_{e \in \mathcal{T}_h \cap \Omega_i} \int_e \left(\frac{1}{s} |\nabla v_h|^s - f v_h \right) dx, \forall v_h \in S_h, i = 1, 2.$$

In order to satisfy (F3), we extend each subdomain Ω_i to a larger subdomain O_i, $i = 1, 2$, and each O_i is the union of some elements of $\mathcal{T}_h$. The subdomains Ω_i are nonoverlapping subdomains, while the subdomains O_i will overlap with each other. We define $K_i = \{ v_h | \quad v_h \in P_k, \forall e \in \mathcal{T}_h, v_h \in C^0(O_i) \}$. If O_1 and O_2 overlap suitably, we can have $K = K_1 \cap K_2$. Thus, the assumptions (F1)–(F4) are all satisfied and $F'_{h,i}$ satisfies [**3**]:

$$(F'_{h,i}(v_{h,1}) - F'_{h,i}(v_{h,2}), v_{h,1} - v_{h,2})_{S_h} \geq \alpha \int_{\Omega_i} |\nabla (v_{h,1} - v_{h,2})|^s dx, i = 1, 2.$$

Therefore, we get the following convergent algorithm for (6) from Algorithm 1.

ALGORITHM 3.

Step 1. *Choose initial values and constants r, ρ. For $n \geq 1$, set*

$$u_h^n = \frac{1}{2}(u_{h,1}^{n-1} + u_{h,2}^{n-1}) + \frac{1}{2r}(\lambda_{h,1}^{n-1} + \lambda_{h,2}^{n-1}) .$$

Step 2. *Solve $u_{h,1}^n \in H^1(O_1), u_{h,2}^n \in H^1(O_2)$ in parallel from:*

$$\begin{aligned} &(|\nabla u_{h,1}^n|^{s-2} \nabla u_{h,1}^n, \nabla v_h)_{L^2(\Omega_1)} + \frac{1}{2} \sum_{e \in \mathcal{T}_h \cap O_1} (\lambda_{h,1}^{n-1}, v_h)_{H^1(e)} \\ &+ \frac{r}{2} \sum_{e \in \mathcal{T}_h \cap O_1} (u_{h,1}^n - u_h^n, v_h)_{H^1(e)} = (f, v_h)_{L^2(\Omega_1)}, \quad \forall v_h \in V_h , \end{aligned} \tag{7}$$

$$\begin{aligned} &(|\nabla u_{h,2}^n|^{s-2} \nabla u_{h,2}^n, \nabla v_h)_{L^2(\Omega_2)} + \frac{1}{2} \sum_{e \in \mathcal{T}_h \cap O_2} (\lambda_{h,2}^{n-1}, v_h)_{H^1(e)} \\ &+ \frac{r}{2} \sum_{e \in \mathcal{T}_h \cap O_2} (u_{h,2}^n - u_h^n, v_h)_{H^1(e)} = (f, v_h)_{L^2(\Omega_2)}, \quad \forall v_h \in V_h . \end{aligned} \tag{8}$$

We obtain the value of $u_{h,1}^n$ in $\Omega \backslash O_1$, and the value of $u_{h,2}^n$ in $\Omega \backslash O_2$ through:

$$u_{h,1}^n = u_h^n - \frac{1}{r}\lambda_{h,1}^{n-1} \text{ in } \Omega \backslash O_1, \qquad u_{h,2}^n = u_h^n - \frac{1}{r}\lambda_{h,2}^{n-1} \text{ in } \Omega \backslash O_2 .$$

Homogeneous Dirichlet boundary conditions should be enforced on $\partial\Omega$ for (7) and (8).

Step 3. *Update the multipliers as: $\lambda_{h,i}^n = \lambda_{h,i}^{n-1} + \rho(u_{h,i}^n - u_h^n)$, in $\Omega, i = 1, 2$, and go to step 2.*

In [**5**] several other algorithm were also obtained for (5) and other linear self-adjoint elliptic problems.

Next, we use overlapping domain decomposition for (6). As before, we assume we have partitioned Ω into finite elements. We then decompose Ω_h into overlapping subdomains and each subdomain is the union of some elements of

$\mathcal{T}_h$. We assume that the subdomains can be marked by m colors, so that the subdomains with the same color do not intersect with each other. We denote the union of the subdomains with the ith color as Ω_i. Let us take

$$V_{h,i} = \{v_h| \quad v_h \in C^0(\Omega), v_h \in P_k, \forall e \in \mathcal{T}_h \cap \Omega_i, v_h = 0 \text{ in } \Omega_h \backslash \Omega_i \text{ and on } \partial\Omega_h\} .$$

If the subdomains overlaps suitably, we will have $V_h = \sum_{i=1}^m V_{h,i}$, and the constants C_0, C_1 can be explicitly estimated, see [**2, 9**]. For simplicity, we define for Algorithm 2:

$$u^n = \sum_{i=1}^m u_i^n, \quad w_i^{n+1} = \sum_{k=1,k\neq i}^m u_k^{n+1} + u_i^{n+\frac{1}{2}} = u^n - u_i^n + u_i^{n+\frac{1}{2}}, \forall i, n .$$

We get from Algorithm 2 the following overlapping algorithm for (6):

ALGORITHM 4.

Step 1. *Choose* $u_{h,i}^1 \in V_{h,i}$ *and constants* $\alpha_i > 0$ *such that* $\sum_{i=1}^m \alpha_i \leq 1$.

Step 2. *For* $n \geq 1$, *solve in parallel in each subdomain* Ω_i *the following problem:*

$$\begin{cases} (|\nabla w_{h,i}^{n+1}|^{s-2} \nabla w_{h,i}^{n+1}, \nabla v_h)_{L^2(\Omega_i)} = (f, v_h)_{L^2(\Omega_i)}, & \forall v_h \in V_{h,i} , \\ w_{h,i}^{n+1} = 0 \text{ on } \partial\Omega_i \cap \partial\Omega_h , \quad w_{h,i}^{n+1} = u_h^n = \sum_{k=1,k\neq i}^m u_{h,k}^n \text{ on } \partial\Omega_i \backslash \partial\Omega_h . \end{cases}$$

Step 3. *Set* $u_{h,i}^{n+1}$ *as:* $u_{h,i}^{n+1} = u_{h,i}^n + \alpha_i(w_{h,i}^{n+1} - u_h^n)$ *in* $\Omega_i, i = 1, 2, \cdots m$, *and go to the next iteration.*

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