Contemporary Mathematics
Volume 180, 1994

Three-dimensional Domain Decomposition Methods with Nonmatching Grids and Unstructured Coarse Solvers

P. LE TALLEC, T. SASSI, M. VIDRASCU

December 1993

ABSTRACT. This paper deals with finite element approximations which are defined independently on each subdomain and which do not match at the interfaces. Weak matching conditions then impose that the solution on two neighboring subdomains share the same L^2 projection on the mortar space which is defined on their common interface. For elliptic problems, we will prove that such discretization strategies lead to optimal approximation errors.

On the numerical side, the resulting discrete problem can be reduced to a problem set on the interface and associated to a generalized Schur complement matrix. This interface problem can then be solved by a preconditioned Conjugate Gradient method, using a Neumann-Neumann preconditioner with coarse grid correction. This technique, illustrated on several numerical examples, is proved to be optimal in such cases.

1. Introduction

Variational methods for decomposing and solving elliptic problems by domain decomposition techniques are well established. Most applications use discretization grids which are defined globally over the whole domain and then split into subdomains. In mechanics, this results into an overall conforming approximation of the velocity field. However, it might be more convenient and efficient to use approximations which are defined independently on each subdomain and which do not match at the interfaces. This allows the user to make local and adaptive change of designs, models, approximation strategies, or grids on one domain without modifying the other ones, provided that the user has found an adequate

1991 *Mathematics Subject Classification.* Primary 65F30; Secondary 65F10.
The paper is in its final version and will not be submitted for publication elsewhere.

way of imposing the weak continuity of both the fluxes and the velocities across such nonconforming interfaces.

In this paper, we will solve this problem by introducing a three-dimensional variant of the so-called mortar spaces. On the mathematical side, this technique imposes that the solution on two neighboring subdomains share the same L^2 projection on the mortar space which is defined on their common interface. For elliptic problems, we will prove that such discretization strategies lead to optimal approximation errors.

On the practical side, the resulting discrete problem can be reduced to a problem set on the interface and associated to a generalized Schur complement matrix. This interface problem can then be solved by a preconditioned Conjugate Gradient method, using for example a Neumann-Neumann preconditioner. This algorithm is very flexible and can be used both in a *conforming* and in a nonconforming framework. In both cases, we will add an unstructured coarse grid solver when using decompositions with a large number of subdomains, and prove the optimality of the resulting preconditioner.

2. Construction of the discrete problem

Nonoverlapping domain decomposition algorithms compute interface values which usually are the values of the unknowns on interface nodes shared by neighboring subdomains. But, it might be more efficient to consider interface nodes which do not match across the interfaces (Figure 1) [**13**], [**8**], [**7**], [**9**].

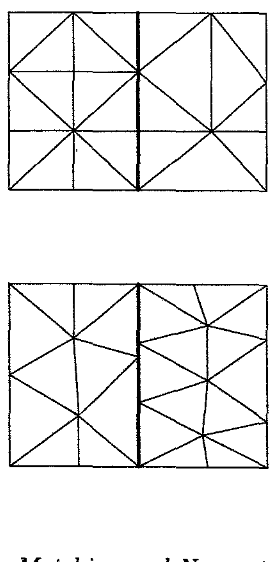

FIGURE 1. *Matching and Nonmatching Grids*

In such situations, the problem is then to impose global continuity of the unknowns. For this purpose, let us define the different subdomains Ω_i of Ω

either by direct juxtaposition of existing local meshes or by automatic partitioning of an existing global mesh. Such an automatic partitioning can be obtained for example by using spectral dissection techniques **[18]**, **[11]** or K-means techniques (dynamic clusters) as classically used in data analysis. On all resulting subdomains, let us then introduce independent finite element spaces $\ _h(\Omega_i)$. Functions of these local finite element spaces are supposed to belong to $\ ^1(\Omega_i)$ and to vanish on the local Dirichlet boundary $\partial\Omega_D \cap \partial\Omega_i$. To match all these spaces together, we finally introduce finite dimensional *mortar* spaces $\ _{ijh}$ defined on each interface F_{ij}, $i < j$, and weak traces Tr_{ijh} and Tr_{jih} defined by L^2 projection :

$$(2.1) \qquad \int_{F_{ij}} (Tr_{ijh}v_i - v_i)\mu_h = 0,\ \forall \mu_h \in \ _{ijh}, \forall v_i \in \ _h(\Omega_i), Tr_{ijh}v_i \in \ _{ijh},$$

$$(2.2) \qquad \int_{F_{ij}} (Tr_{jih}v_j - v_j)\mu_h = 0,\ \forall \mu_h \in \ _{ijh}, \forall v_j \in \ _h(\Omega_j), Tr_{jih}v_j \in \ _{ijh}.$$

Then, the space of finite element approximations of u over Ω can be defined by

$$(2.3) \qquad \ _h(\Omega) = \{(v_{ih})_i \in \prod_i \ _h(\Omega_i), Tr_{ijh}v_{ih} = Tr_{jih}v_{jh} \text{ on } F_{ij}, \forall i < j\}.$$

On this space, we wish to solve the second order elliptic variational problem

$$\text{Find } u_h \in \ _h(\Omega) \text{ solution of}$$

$$(2.4) \qquad a(u_h, v_h) := \sum_i \int_{\Omega_i} \sigma(\nabla u_h) : \nabla v_h = L(v_h), \forall v_h \in \ _h(\Omega).$$

Above, $\sigma(\nabla u_h)$ denotes a given symmetric elliptic linear function of ∇u_h. For example, the classical case of isotropic linear elasticity corresponds to the choice

$$\sigma(\nabla u_h) = \lambda Tr(\nabla u_h) Id + \mu(\nabla u_h + \nabla^T u_h)$$

which relates the deformation tensor $(\nabla u_h + \nabla^T u_h)/2$ to the stress tensor σ. Moreover, the right-hand side $L(v_h)$ is usually of the form

$$L(v_h) = \int_\Omega f \cdot v_h.$$

By introducing the Lagrange multipliers of the interface continuity constraints $Tr_{ijh}v_{ih} = Tr_{jih}v_{jh}$, this global problem takes the mixed form :

$$\text{Find } (u_{ih}) \in \prod_i \ _h(\Omega_i), \text{ and } (\lambda_{ij}) \in \prod_{i<j} \ _{ijh} \text{ solution of}$$

$$\int_{\Omega_i} \sigma(\nabla u_{ih}) : \nabla v_{ih} + \sum_{i<j} \int_{F_{ij}} \lambda_{ij} v_{ih} - \sum_{i>j} \int_{F_{ji}} \lambda_{ji} v_{ih} = L(v_{ih}),$$

$$\forall v_{ih} \in \ _h(\Omega_i), \forall i,$$

$$\int_{F_{ij}} (u_{ih} - u_{jh})\mu_h = 0, \forall \mu_h \in \ _{ijh}, \forall i < j.$$

In algebraic form, we thus recover the classical subdomain by subdomain writing

$$A_iU_i + \sum_{i<j} Tr_{ijh}^T \Lambda_{ij} - \sum_{i>j} Tr_{ijh}^T \Lambda_{ji} = F_i, \forall i,$$
$$Tr_{ijh}U_i = \bar{U}_{ij}, \forall j \neq i.$$

This sequence of local problems can now be solved by any classical substructuring algorithms. Indeed, after elimination of U_i and Λ_{ij}, and with respect to the interface unknowns $\bar{U} = (\bar{U}_{ij})_{i<j} = (Tr_{ijh}U)_{i<j}$, the above problem takes the standard form:

$$\sum_i \tilde{R}_i^T \begin{pmatrix} A_i & Tr_{ijh}^T \\ Tr_{ijh} & 0 \end{pmatrix}^{-1} \tilde{R}_i\bar{U} = F. \tag{2.5}$$

Compared to the case of matching grids, the nonmatching case finally leads to the same algorithms with three major changes :

- the pointwise traces are replaced on each face F_{ij} by local L^2 averaged traces Tr_{ijh};
- the pointwise interface restriction $\bar{R}_i$ is replaced by a global restriction operator $\tilde{R}_i$ which maps the global trace $\bar{U}$ into the local right-hand side

$$\begin{pmatrix} 0 \\ \bar{U}_{ij} \end{pmatrix}, \forall j \neq i;$$

- the space of global traces $\bar{U}$ is now defined face by face in the (smaller) product space $\prod_{i<j} \ _{ijh}$. Edges and vertices do not play any role in this definition. In fact, the definition of the trace on any geometric vertex or edge will no longer be unique and will depend of the particular face $\bar{F}_{ij}$ on which it is taken.

REMARK 2.1. *At the limit where the local spaces* $\ _h(\Omega_i)$ *become dense in* $H^1(\Omega_i)$, *a straightforward integration by parts shows that the solution of the proposed discrete problem satisfies the interface flux continuity requirement:*

For each interface F_{ij}, *there exists a traction force* $\lambda_{ij} \in \ _{ijh}$ *such that*

$$\int_{F_{ij}} v_i(\sigma(u_i).n - \lambda_{ij}) = 0, \forall v_i \in \ _h(\Omega_i),$$
$$\int_{F_{ij}} v_j(\sigma(u_j).n - \lambda_{ij}) = 0, \forall v_j \in \ _h(\Omega_j).$$

We thus observe that the multiplier unknown λ_{ij} *plays the role of a common traction force or generalized normal derivative.*

3. Error analysis

The above framework reduces and simplifies the interface algebraic problem, improves the flexibility of the numerical method, but changes the discrete problem. In particular, the proposed discrete solution is not pointwise continuous across the different interfaces. Nevertheless, we prove in this section that this new approximate solution still converges optimally in the sense that

$$\|u - u_h\|_H \leq Ch^k \|u\|_{H^{k+1}(\Omega)}.$$

Here k is the order of the finite elements which are used, and u denotes the continuous solution of the original elliptic problem set on the space

$$(\Omega) = \{v \in H_0^1(\Omega), v = 0 \, \mathrm{on} \, \partial\Omega_D\}.$$

Moreover, the norm $\|u - u_h\|_H$ is a sum of local H^1 subdomain norms :

$$\|u - u_h\|_H = \left(\sum_i \|u - u_h\|^2_{H^1(\Omega_i)} \right)^{1/2}.$$

To prove such an optimal convergence result, we need two assumptions on the mortar spaces $_{ijh}$.

ASSUMPTION 3.1. *The space* $_{ijh}$ *is a consistent approximation of the dual of* $H^{\frac{1}{2}}(F_{ij})$ *in the sense that*

$$\inf_{\lambda_{ij} \in \ _{ijh}} \|\mu - \lambda_{ij}\|_{(H^{1/2}(F_{ij}))'} \leq Ch^k \|\mu\|_{H^{k-1/2}(F_{ij})}.$$

In practice, it is sufficient that $_{ijh}$ be a good approximation of $H^{k+1}(F_{ij})$ in L^2. This means that functions of $_{ijh}$ may be discontinuous but must be allowed to take non zero values next to the boundary ∂F_{ij}.

ASSUMPTION 3.2. *On each interface* F_{ij}*, one neighboring subdomain* k *(*$k = i$ *or* $k = j$*) has a regular triangulation and satisfies*

$$\inf_{\lambda_{ij} \in \ _{ijh}} \ \sup_{v_h \in Tr \ _h(\Omega_k) \cap H_0^1(F_{ij})} \frac{\int_{F_{ij}} \lambda_{ij} v_h}{\|\lambda_{ij}\|_{L^2(F_{ij}} \|v_h\|_{L^2(F_{ij})}} \geq \beta.$$

This assumption means that $_{ijh}$ must be small compared to the space of traces of $_h(\Omega_k)$. We refer to Maday [**14**] for examples of finite element spaces satisfying this assumption. From the technical point of view, the above assumption is written in the L^2 norm which makes its verification much easier ([**13**]).

With these assumptions, we can now prove :

LEMMA 3.1. *Under Assumption* 3.1, *we have*

$$\sup_{w_h \in \ _h(\Omega)} \frac{|a(u,w_h) - L(w_h)|}{\|w_h\|_H} \leq Ch^k \left(\sum_{i<j} \|\sigma.n\|_{H^{k-1/2}(F_{ij})} \right),$$

and under Assumption 3.2 , *we have*

$$\inf_{v_h \in \ _h(\Omega)} \|u - v_h\|_H \leq Ch^k \|u\|_{H^{k+1}(\Omega)}.$$

Proof. By integration by parts, and since u is by construction the solution of the partial differential equation

$$\text{div } (\sigma) + f = 0 \,\text{on}\, \Omega,$$

we first get

$$\begin{aligned} R &= a(u,w_h) - L(w_h) \\ &= \sum_i - \int_{\Omega_i} \{div\sigma + f\}.w_{ih} + \sum_{i<j} \int_{F_{ij}} (\sigma.n).(w_{ih} - w_{jh}) \\ &= \sum_{i<j} \int_{F_{ij}} (\sigma.n).(w_{ih} - w_{jh}), \ \forall w_h \in \ _h(\Omega). \end{aligned}$$

Since w_h belongs to $\ _h(\Omega)$, we have $Tr_{ijh} w_{ih} = Tr_{jih} w_{jh}$, and thus we can rewrite the above inequality as

$$\begin{aligned} R &= \sum_{i<j} \int_{F_{ij}} (\sigma.n).((w_{ih} - w_{jh}) - (Tr_{ijh} w_{ih} - Tr_{jih} w_{jh})) \\ &= \sum_{i<j} \int_{F_{ij}} (\sigma.n - \mu_h) \cdot ((w_{ih} - w_{jh}) - (Tr_{ijh} w_{ih} - Tr_{jih} w_{jh})), \end{aligned}$$

for any function μ_h in $\ _{ijh}$. Hence, we deduce

$$\begin{aligned} |R| &\leq \sum_{i<j} \inf_{\mu_h \in \ _{ijh}} \|\sigma.n - \mu_h\|_{H^{-1/2}(F_{ij})} \|(w_{ih} - w_{jh})\|_{H^{1/2}(F_{ij})}, \\ &\leq Ch^k \left(\sum_{i<j} \|\sigma.n\|_{H^{k-1/2}(F_{ij})} \right) \|w_h\|_H. \end{aligned}$$

For the second estimate, we construct the function v_h by

$$(v_h)_{|\Omega_i} = \mathcal{I}_i u - \sum_{ij \in J(i)} Tr^{-i} \circ Tr_{ijh}^{-i} \circ (Tr_{ijh} \mathcal{I}_i u - Tr_{jih} \mathcal{I}_j u).$$

Above, $\mathcal{I}_i$ is the usual interpolation operator,

$$Tr_{ijh}^{-i} : \ _{ijh} \rightarrow Tr \ _h(\Omega_i) \cap H_0^1(F_{ij})$$

is a continuous inverse of Tr_{ijh}, which is well defined in L^2 from Assumption 3.2, and Tr^{-i} is a continuous inverse of the usual trace operator. Moreover, the set $J(i)$ on which the summation is carried is defined as the set of all interfaces F_{ij} for which $_h(\Omega_i)$ satisfies Assumption 3.2.

On each face $ij \in J(i)$, we first have by construction

$$\begin{aligned} Tr_{ijh}(v_h) &= Tr_{ijh}\mathcal{I}_i u - (Tr_{ijh}\mathcal{I}_i u - Tr_{jih}\mathcal{I}_j u) \\ &= Tr_{jih}\mathcal{I}_j u = Tr_{jih} v_h \end{aligned}$$

which guarantees that the above function v_h does belong to $_h(\Omega)$.

On the other hand, from the inverse Sobolev inequality, the contracting properties of the L^2 projection Tr_{ijh} and from standard results on interpolation, we get :

$$\begin{aligned} &\|u - v_h\|^2_{H^1(\Omega_i)} \le C\|u - \mathcal{I}_i u\|^2_{H^1(\Omega_i)} \\ &\quad + \sum_{ij \in J(i)} \|Tr^{-i}\|^2 \|Tr^{-i}_{ijh} \circ (Tr_{ijh}\mathcal{I}_i u - Tr_{jih}\mathcal{I}_j u)\|^2_{H^{1/2}(F_{ij})} \\ &\le C\|u - \mathcal{I}_i u\|^2_{H^1(\Omega_i)} + \sum_{ij \in J(i)} \frac{C}{h_i} \|Tr^{-i}_{ijh} \circ (Tr_{ijh}\mathcal{I}_i u - Tr_{jih}\mathcal{I}_j u)\|^2_{L^2(F_{ij})} \\ &\le C\|u - \mathcal{I}_i u\|^2_{H^1(\Omega_i)} + \sum_{ij \in J(i)} \frac{C}{h_i} \|(\mathcal{I}_i u - \mathcal{I}_j u)\|^2_{L^2(F_{ij})} \\ &\le C h_i^{2k} \|u\|^2_{H^{k+1}(\Omega_i)} + \sum_{ij \in J(i)} \frac{C}{h_i} (h_i^{2k+1} + h_j^{2k+1}) \|u\|^2_{H^{k+1/2}(F_{ij})} \\ &\le C h_i^{2k} \|u\|^2_{H^{k+1}(\Omega_i)} + \sum_{ij \in J(i)} \left(h_i^{2k} \|u\|^2_{H^{k+1}(\Omega_i)} + \frac{h_j^{2k+1}}{h_i} \|u\|^2_{H^{k+1}(\Omega_j)} \right) \end{aligned}$$

which yields the desired estimate.

We are now ready to prove our final convergence result.

THEOREM 3.1. *For any continuous uniformly elliptic second order operator $a(.,.)$, and under the assumptions* 3.1 *and* 3.2, *the error between the nonconforming discrete solution u_h and the exact solution u is bounded by*

$$\|u - u_h\|_H \le C h^k \left(\|u\|^2_{H^{k+1}(\Omega)} + (\sum_{i<j} \|\sigma.n\|^2_{H^{k-1/2}(F_{ij})})^{1/2} \right).$$

Proof. From a classical lemma of Strang and Fix [**17**], any nonconforming finite element approximation of the solution of a continuous uniformly elliptic second order problem satisfies

$$\sum_i \|u - u_h\|^2_{H^1(\Omega_i)} \le \sum_i \|u - v_h\|^2_{H^1(\Omega_i)} + \sup_{w_h \in V_{0h}} \frac{|a(u, w_h) - L(w_h)|}{\|w_h\|_H}.$$

The conclusion follows then by a direct application of the above lemma. Observe finally that for smooth coefficients, we have

$$\begin{aligned}\|\sigma.n\|_{H^{k-1/2}(F_{ij})} &= \|\sum_l a_l \frac{\partial u}{\partial n_L}\|_{H^{k-1/2}(F_{ij})} \\ &\leq C\|u\|^2_{H^{k+1/2}(F_{ij})} \leq C\|u\|_{H(\Omega_i)}.\end{aligned}$$

4. Neumann-Neumann algorithm

4.1. Basic algorithm. Using either matching or nonmatching grids yields the same discrete interface problem :

$$\sum_i \tilde{R}_i^T \begin{pmatrix} A_i & Tr_{ijh}^T \\ Tr_{ijh} & 0 \end{pmatrix}^{-1} \tilde{R}_i \bar{U} = F. \tag{4.1}$$

The presence of nonmatching grids simply replaces the pointwise operator Tr_{ij} by the global projection Tr_{ijh}.

In any case, this problem can be solved by the usual Neumann-Neumann preconditioned conjugate gradient algorithm which acts on any given weak trace $\bar{U} = (\bar{U}_{ij})_{i<j} = (Tr_{ijh}U)_{i<j}$ as follows :

- On each subdomain, solve in parallel the local mixed problem

$$\begin{pmatrix} A_i & Tr_{ijh}^T \\ Tr_{ijh} & 0 \end{pmatrix} \begin{pmatrix} U_i \\ \Lambda_{ij}^i \end{pmatrix} = \begin{pmatrix} 0 \\ \bar{U}_{ij} \end{pmatrix}. \tag{4.2}$$

- On each interface $F_{ij}, i < j$, compute the residual jump of the generalized local normal derivatives

$$\Lambda_{ij} = \Lambda_{ij}^i + \Lambda_{ij}^j = \left((S_i + S_j)\bar{U}\right)_{|F_{ij}} = \left(S\bar{U}\right)_{|F_{ij}}.$$

- Project this residual onto the original space by solving in parallel the local Neumann problems

$$A_i \Psi_i = -\frac{\rho_i}{\rho_i + \rho_j} Tr_{ijh}^T \Lambda_{ij}.$$

- Update $\bar{U}$ by the preconditioned residual

$$\mathcal{F}(\bar{U})_{|F_{ij}} = \frac{\rho_i}{\rho_i + \rho_j} Tr_{ijh} \Psi_i + \frac{\rho_j}{\rho_i + \rho_j} Tr_{jih} \Psi_j.$$

The above algorithm is very flexible and has good localization properties. The coefficient ρ_i is a local average value of the coefficients of the elliptic operator $\sigma(\nabla u)$ and cancels the effects of large jumps of coefficients across the different interfaces. The stiffness matrix A_i is the usual finite element matrix of the local space $_h(\Omega_i)$. For matching grids, the interface matrix Tr_{ijh} is a restriction matrix with a unique nonzero element per row. For nonmatching grids, the element kl of this matrix is given by the L^2 interface scalar product of the

k finite element shape function ϕ_k of (Ω_i) and of the l finite element shape function ψ_l of the mortar space $_{ij}$:

$$(Tr_{ijh})_{kl} = \int_{F_{ij}} \phi_k \psi_l.$$

4.2. Coarse Grid Solver. In the original algorithm, the Neumann subproblems are defined to within a rigid body motion and the condition number of the associated preconditioned operator grows with the inverse of the diameters H of the subdomains ($cond(M^{-1}S) = C/H^2$). J. Mandel **[15]** proposed computing these arbitrary rigid body motions in order to optimize the quality of the preconditioner. For this purpose, we first build a small space Z_i on each subdomain. This space must contain the kernel of A_i (the rigid body motions) and any other local function whose energy scales badly with the size of the subdomain. The extended (balanced) Neumann-Neumann preconditioner then adds to the original local preconditioner Ψ_i the elements z_i of Z_i minimizing

$$\|\frac{\rho_i}{\rho_i+\rho_j} Tr_{ijh}(\Psi_i + z_i) \frac{\rho_j}{\rho_i+\rho_j} Tr_{jih}(\Psi_j + z_j) - S^{-1}\Lambda\|_S$$

This optimisation problem in z_i is a coarse problem with very few unknowns per subdomain (6 for three-dimensional linear elasticity). It can be written for all type of partitions and operators. The resulting algorithm amounts to projecting the interface problem onto the orthogonal of the coarse space $\prod Z_i$, which cancels the bad influence of the elements of $\prod Z_i$ on the preconditioner. Its convergence now becomes independent of the number of subdomains or of the coefficient discontinuities :

THEOREM 4.1. *For either matching or nonmatching grids, the condition number of the above balanced Neumann-Neumann algorithm is bounded by*

$$cond(M^{-1}S) \leq C\left(1 + \log(H/h)\right)^2.$$

The constant C above is independent of the number of subdomains, independent of the averaged coefficients ρ_i, but does depend on the aspect ratio of the different subdomains. In case of nonmatching grids, we have to assume that the weak trace has a continuous inverse in the following sense :

$$\inf_{(\lambda_{ij})\in\prod_j \ _{ijh}} \sup_{v_h \in Tr \ _h(\Omega_i)} \frac{\sum_j \int_{F_{ij}} \lambda_{ij} v_h}{(\sum_j \|\lambda_{ij}\|_{L^2(F_{ij})})\|v_h\|_{L^2(\partial\Omega_i)}} \geq \beta.$$

Proof. For two subdomains, we have formally

$$M^{-1}S = (S_1^{-1} + S_2^{-1})(S_1 + S_2) = 2I + S_1^{-1}S_2 + S_2^{-1}S_1.$$

Similarly in the general case, we can prove (Mandel **[15]**, Le Tallec **[12]**)

$$cond(M^{-1}S) = C \sup_{U_j \in V_j/Z_j} \frac{< S_i \frac{\rho_j}{\rho_i+\rho_j} U_j, \frac{\rho_j}{\rho_i+\rho_j} U_j >}{< S_j U_j, U_j >}.$$

The term $\frac{\rho_j}{\rho_i+\rho_j} S_i U_j$ is bounded in two steps:
i) interface mirror. Viewed from Ω_i, the function $\frac{\rho_j}{\rho_i+\rho_j} U_j$ has a singularity at the corner and thus, it is only bounded by **[10]** :

$$\rho_i^{1/2} \| \frac{\rho_j}{\rho_i + \rho_j} U_j \|_{H^{1/2}(\partial\Omega_i)} \leq C(1 + \log(H/h)) \rho_j^{1/2} \|U_j\|_{H^{1/2}(\partial\Omega_j)};$$

ii) weak harmonic extension. Introducing as in Lemma 2 the bounded extension Tr_h^{-1} defined on Ω_j by

$$Tr_h^{-1} = Tr^{-i} \circ \left(P_i - Tr_h^{-i} (\prod_k (P_i - I)_{|F_{ik}}) \right),$$

we can easily derive the upper bound

$$< S_i U_i, U_i > \leq \rho_i \|Tr_h^{-1}\|^2 \|U_i\|^2_{H^{1/2}(\Gamma_j)}.$$

Here, P_i denotes the L^2 projection on the trace of $_h(\Omega_i)$ on $\partial\Omega_i$. The norm of Tr_h^{-1} is independent of the subdomain diameter H because we work on a quotient space V_j/Z_j **[14]**, **[12]**. On the other hand, it depends strongly on the aspect ratio of the subdomain Ω_i.

5. Numerical Results

5.1. Matching grids. The following tests aim at comparing the original Neumann-Neumann algorithm and the new *global version* with coarse grid solver. The first example considers a three-dimensional cantilever beam. The domain was partitioned successively in 4, 8, 32, and 128 identical subdomains. Both slices and boxes were treated in the 4 domain case. The table below displays the characteristics of the partition (local number of elements NE and of degrees of freedom NTDL, number of words used for matrix storage LMUA, size of the coarse grid problem LRIGI, aspect ratio ASP) and the number of iterations which were required to obtain a residual below 10^{-6}. Two numbers are given, one without coarse solver and one with coarse grid solver (given in parenthesis).

Nber of subdomains	NE	NTDL	LMUA	LRIGI	ASP	NITER
4 = 1*1*4 (slices)	512	8019	6 712 830	18	0.8	23(5)
4 = 2*2*1 (boxes)	512	8331	3 970 029	12	0.1	34(8)
8 = 2*2*2	256	4347	1 914 113	36	0.2	62(10)
32 = 2*2*8	64	1275	295 571	168	0.8	157(13)
128 = 4*4*8	16	423	39 665	672	0.4	791(30)

FIGURE 2. *Description of the different partitions*

The second example describes a three-dimensional complex elastic structure, made of aluminium, fixed on three lateral bolts, and twisted through an imposed rotation of its internal axis. The finite element mesh and final deformed shape is depicted on Figure 3. It contains 46, 133 first order $P1$ tetrahedral finite elements, 31, 143 degrees of freedom, among which 4, 248 lie on subdomain interfaces. This mesh was automatically partitioned into 24 subdomains, and the calculation was performed using 1 or 24 processors of a KSR-1 parallel computer.

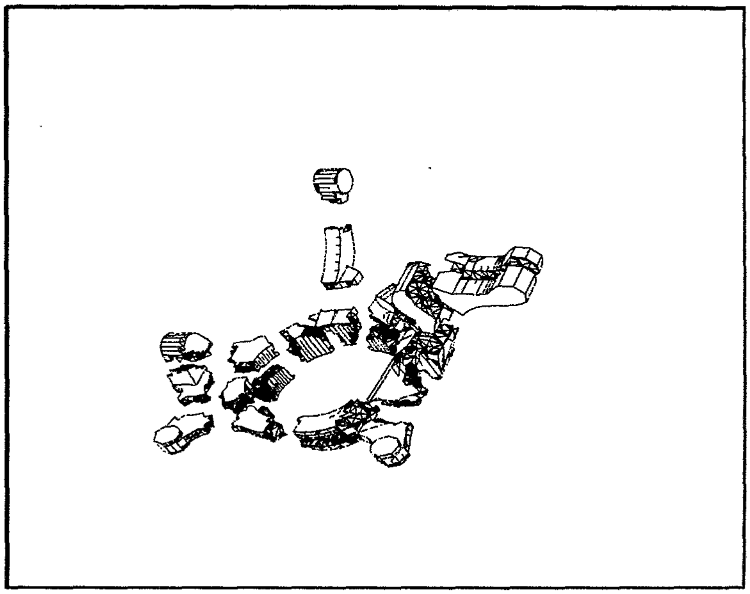

FIGURE 3. *Finite Element mesh of the structure*

The final solution was obtained after 116 iterations of the Neumann-Neumann algorithm without coarse grid solver or after 37 iterations of the Neumann-Neumann algorithm with coarse grid solver. On one processor, the calculation and assembly of the local stiffness matrices took 224 s, their factorisation 224 s, the construction of the interface data structure 3 s, the conjugate gradient initialisation 12.62 s, the local subdomain solves 1, 322 s, and the interface scalar products 47.15 s. After parallelisation of the subdomain solves on 24 processors, the timings for initialisation, local solves and interface scalar products were of 3.75 s, 58.24 s and 54.97 s, respectively. All these figures show the nice parallel properties of the Neumann-Neumann algorithm even for complex three-dimensional structures.

5.2. Nonmatching Grids without coarse grid solver. The domain considered is a beam of section 0.5m × 0.2m and length 1m or 2m. The beam is made of a quasi-incompressible material with $E = 10^{11}$MPa (Young modulus) and $\nu = 0.49$ (Poisson coefficient). As our main interest lies in the numerical solver, and not too much in the accuracy of the discretised problem, the beam is simply partitioned into first order tetrahedral finite elements. The beam has been sliced either along its leading dimension (nocross) or following a two-dimensional pattern, with edges and cross-points (cross)(see Figure 5). We show in the next table the effect of the number of subdomains p and of the mesh step h on the convergence rate of the Neumann preconditioned conjugate gradient algorithm (NPGC). The number of iterations does not appear to be very sensitive to the nonmatching character of the grid.

Number of subdomains and step size	iter	d.o.f in Ω	d.o.f in Γ
$p = 2$ $h = h_1$ (matching, nocross)	20	9180	270
$p = 4$ $h = h_1$ (matching, nocross)	39	9720	810
$p = 4$ $h = h_1$ (nonmatching, nocross)	50	8430	765
$p = 8$ $h = h_1$ (matching, nocross)	127	10800	2160
$p = 8$ $h = h_1$ (nonmatching, nocross)	107	9480	1785
$p = 4$ $h = h_2$ (matching, nocross)	67	2400	360
$p = 4$ $h = h_2$ (nonmatching, nocross)	76	1800	225
$p = 4$ $h = h_2/2$ (matching, nocross)	66	14580	1215
$p = 4$ $h = h_2/2$ (nonmatching, nocross)	88	10629	1080
$p = 4$ $h = h_2$ (nonmatching, cross)	41	2145	330
$p = 4$ $h = h_2/2$ (nonmatching, cross)	54	12852	972

FIGURE 4. *Test over the number of subdomains and the step size*

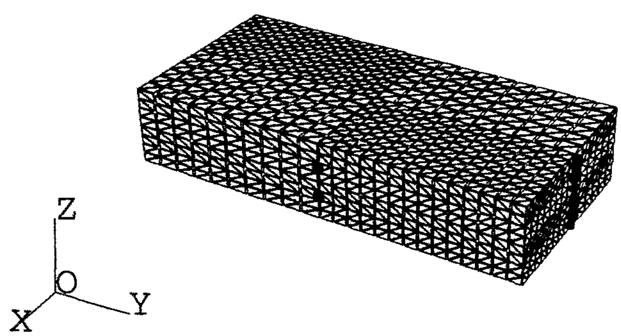

FIGURE 5. *Nonmatching finite element decomposition*

6. Conclusion

We have introduced and tested a theoretical and algorithmic framework which can handle nonmatching grids in three-dimensional situations. This approach leads to smaller interface problem because they are set on the product space $\prod_{ij}$, which has a better structure (the notion of corners and edges have disappeared), and an optimal order of approximation error.

We have also presented and tested a parallel implementation of the Neumann-Neumann preconditioner with coarse grid solver. This implementation handles three-dimensional elasticity and plates operators, matching or nonmatching grids, and any kind of unstructured partition of the mesh. We obtain such partitions by using automatic mesh partitioning strategies such as K-means techniques.

We would like now to extend these techniques to complex three-dimensional CFD problems. But two problems remain open in this direction :

- Which implicit solver to pick for a Navier-Stokes implementation? This choice does not affect the approximation strategy but has a direct consequence on the choice of the substructuring algorithm.

- What is a consistent nonmatching grid approximation of stabilized advection problems, especially in the limit of vanishing viscosity? Is it consistent with a nice multidomain approximation of the pure hyperbolic limit?

References

1. R. Glowinski, G. Golub, G. Meurant and J. Periaux (eds.), *Proceedings of the First International Symposium on Domain Decomposition Methods for Partial Differential Equations, Paris, France, January 7-9, 1987*, (SIAM, Philadelphia, 1988).
2. T. Chan, R. Glowinski, J. Periaux and O. Widlund (eds.), *Proceedings of the Second International Symposium on Domain Decomposition Methods for Partial Differential Equations, Los Angeles, California, January 14-16, 1988*, (SIAM, Philadelphia, 1989).
3. T. Chan, R. Glowinski (eds.), *Proceedings of the Third International Symposium on Domain Decomposition Methods for Partial Differential Equations, Houston, Texas, March 20-22, 1989*, (SIAM, Philadelphia, 1990).
4. R. Glowinski, Y. Kuznetsov, G. Meurant, J. Periaux and O. Widlund (eds.), *Proceedings of the fourth international symposium on Domain Decomposition Methods for Partial Differential Equations, Moscow, June 1990*, (SIAM, Philadelphia, 1991).
5. T. Chan, D. Keyes, G. Meurant, J. Scroggs and R. Voigt (eds.), *Proceedings of the fifth international symposium on Domain Decomposition Methods for Partial Differential Equations, Norfolk, May 1991*, (SIAM, Philadelphia, 1992).
6. A. Quarteroni (ed.), *Proceedings of the sixth international symposium on Domain Decomposition Methods for Partial Differential Equations, Como, June 1992*, (AMS, Providence, 1993).
7. Y. Achdou and O. Pironneau, *A fast solver for Navier-Stokes Equations in the laminar regime using mortar finite element and boundary element methods*, Technical Report 93-277 (Centre de Mathématiques Appliquées, Ecole Polytechnique, Paris, 1993)
8. C. Bernardi, Y. Maday and A. Patera, *A new nonconforming approach to domain decomposition: the mortar element method*, in H. Brezis and J.L. Lions (eds.) Nonlinear Partial Differential Equations and their Applications (Pitman, 1989).
9. J. Bramble, R. Ewing, R. Parashkevov and J. Pasciak, *Domain decomposition methods for problems with partial refinement*, SIAM J. Sci. Stat. Comp. **13** (1992) 397–410.

10. Y. H. De Roeck and P. Le Tallec, *Analysis and test of a local domain decomposition preconditioner*, in [4].
11. C. Farhat and M. Lesoinne, *Automatic Partitioning of Unstructured Meshes for the Parallel Solution of Problems in Computational Mechanics*, Int. J. Num. Meth. in Eng. **36** (1993) 745–764.
12. P. Le Tallec, *Domain Decomposition Methods in Computational Mechanics*, Computational Mechanics Advances (1994).
13. P. Le Tallec and T. Sassi, *Domain Decomposition with Nonmatching Grids: Schur Complement Approach*, Rapport CEREMADE 9323, Université de Paris Dauphine (1993).
14. Y. Maday, Proceedings of this Conference.
15. Jan Mandel, *Balancing Domain Decomposition*, Communications in Numerical Methods in Engineering, **9** (1993), 233–241.
16. Jan Mandel and M. Brezina, *Balancing Domain Decomposition: theory and performances in two and three dimensions*, Technical report (Computational Mathematics Group, University of Colorado at Denver, March 1993).
17. G. Strang and G. Fix, *An Analysis of the Finite Element Method*, Prenctice Hall (1973).
18. H. Simon, *Partitioning of unstructured problems for parallel processing*, Comp. Systems in Eng. **2** (1991) 135–148.

Université Paris Dauphine and INRIA - Rocquencourt, BP105, 78 153 Le Chesnay, France

E-mail address: letallec@menusin.inria.fr