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On Generalized Schwarz Coupling Applied to Advection-Dominated Problems

K.H. TAN AND M.J.A. BORSBOOM

ABSTRACT. Schwarz methods can be interpreted as domain decomposition methods in which a local mechanism at interfaces is used to restore iteratively the connection between artificially decoupled subproblems. The speed of convergence of these iterative processes can be improved significantly by the choice of a proper coupling mechanism. If known, properties of the discretized problem can be taken into account in such an optimization of the coupling.

In this paper a flexible local coupling mechanism is proposed that can easily be tailored to properties of the discrete problem. It consists of parameterized interface conditions, which are formulated with the use of virtual unknowns. Besides a Dirichlet and a Neumann part, tangential and mixed derivatives are involved in these interface conditions. The coupling technique is combined with the block-Jacobi iteration, resulting in a generalized additive Schwarz method. Its convergence properties are analyzed and optimized for a number of simple time-dependent advection-dominated problems, discretized by means of central finite differences.

1. Introduction

Schwarz methods can be introduced and investigated in both a continuous and discrete setting. Consider the continuous problem $Lu = g$ in a domain Ω that has been subdivided into two non-overlapping subdomains $\Omega = \Omega_1 \cup \Omega_2$ with common interface Γ, $\Omega_1 \cap \Omega_2 = \Gamma$. Then in a continuous setting this problem is typically reformulated as:

$$Lu_1 = g_1 \qquad \text{in } \Omega_1 \,, \tag{1}$$

$$\Phi_i(u_1) = \Phi_i(u_2) \qquad \text{on } \Gamma_i \subset \Gamma,\ i = 1, ..., I \,, \tag{2}$$

$$\Psi_j(u_2) = \Psi_j(u_1) \qquad \text{on } \Gamma_j \subset \Gamma,\ j = 1, ..., J \,, \tag{3}$$

$$Lu_2 = g_2 \qquad \text{in } \Omega_2 \,, \tag{4}$$

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complemented with appropriate boundary conditions on $\partial\Omega_1\backslash\Gamma$ and $\partial\Omega_2\backslash\Gamma$. If the coupling equations are chosen such that unique solutions u_1 and u_2 exist for which holds that $u_1 = u|_{\Omega_1}$ and $u_2 = u|_{\Omega_2}$, the Schwarz formulation is said to be equivalent with the original problem formulation. In general, many Φ_i and Ψ_j lead to equivalence. This freedom is exploited in optimizing the convergence speed of domain decomposition methods derived from (1)–(4). A suitable choice, resulting into fast convergence, should be made from the many coupling operators that guarantee equivalence. For some recent examples of operators proposed in the literature we refer to [**1**] and [**3**].

In the construction of coupling operators that maximize the convergence speed of domain decomposition methods for discretized problems, the discrete equivalent of (1)–(4), rather than the continuous problem itself, is to be considered. For this reason we describe in section 2 a discrete equivalent of (1)–(4). It is formulated with the use of so-called enhanced systems of equations, as introduced in [**3**]. In section 3, a simplified model problem is formulated. For this problem, a specific form of a parameterized enhancement is proposed. In section 4, the convergence behavior as a function of the parameters of the enhanced system will be examined and optimized. The resulting convergence rates are considerably better than the ones that can be obtained with the usual combinations of Neumann and Dirichlet conditions, as will be illustrated with some numerical results in section 5.

2. A discrete Schwarz formulation

Consider a system of linear equations $Au = f$ for which a unique solution exists. To introduce a discrete counterpart of (1)–(4), let this system be partitioned as:

$$\begin{pmatrix} A_{11} & A_{12}^l & A_{12}^r & A_{13} \\ \hat{A}_{21} & \hat{A}_{22}^l & \hat{A}_{22}^r & \hat{A}_{23} \\ \check{A}_{21} & \check{A}_{22}^l & \check{A}_{22}^r & \check{A}_{23} \\ A_{31} & A_{32}^l & A_{32}^r & A_{33} \end{pmatrix} \begin{pmatrix} u_1 \\ u_I^l \\ u_I^r \\ u_2 \end{pmatrix} = \begin{pmatrix} f_1 \\ f_I^l \\ f_I^r \\ f_2 \end{pmatrix}. \tag{5}$$

The dimensions of the subvectors are n_1, n_I^l, n_I^r, and n_2, respectively. Usually, the partitioning will be such that n_I^l, $n_I^r \ll n_1$, n_2.

Consider the following enhanced system of equations:

$$\begin{pmatrix} A_{11} & A_{12}^l & A_{12}^r & 0 & 0 & A_{13} \\ \hat{A}_{21} & \hat{A}_{22}^l & \hat{A}_{22}^r & 0 & 0 & \hat{A}_{23} \\ 0 & R^l & S^l & -R^l & -S^l & 0 \\ 0 & -R^r & -S^r & R^r & S^r & 0 \\ \check{A}_{21} & 0 & 0 & \check{A}_{22}^l & \check{A}_{22}^r & \check{A}_{23} \\ A_{31} & 0 & 0 & A_{32}^l & A_{32}^r & A_{33} \end{pmatrix} \begin{pmatrix} c_1 \\ c_I^l \\ \tilde{c}_I^r \\ \tilde{c}_I^l \\ c_I^r \\ c_2 \end{pmatrix} = \begin{pmatrix} f_1 \\ f_I^l \\ 0 \\ 0 \\ f_I^r \\ f_2 \end{pmatrix}, \tag{6}$$

where $\tilde{c}_I^l$ and $\tilde{c}_I^r$ are vectors of dimension n_I^l and n_I^r, respectively.

Let $\bar{c}_1 = (c_1, c_I^l, c_I^r, c_2)^T$, $\bar{c}_2 = (\tilde{c}_I^l, \tilde{c}_I^r)^T$, $g_1 = (f_1, f_I^l, f_I^r, f_2)^T$ and $g_2 = (0,0)^T$.

Equation (6) can be written as:

$$\begin{pmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{pmatrix} \begin{pmatrix} \bar{c}_1 \\ \bar{c}_2 \end{pmatrix} = \begin{pmatrix} g_1 \\ g_2 \end{pmatrix}, \tag{7}$$

where $B_{11} = \begin{pmatrix} A_{11} & A^l_{12} & 0 & A_{13} \\ \hat{A}_{21} & \hat{A}^l_{22} & 0 & \hat{A}_{23} \\ \check{A}_{21} & 0 & \check{A}^r_{22} & \check{A}_{23} \\ A_{31} & 0 & A^r_{32} & A_{33} \end{pmatrix}$, $B_{12} = \begin{pmatrix} 0 & A^r_{12} \\ 0 & \hat{A}^r_{22} \\ \check{A}^l_{22} & 0 \\ A^l_{32} & 0 \end{pmatrix}$,

$B_{21} = \begin{pmatrix} 0 & R^l & -S^l & 0 \\ 0 & -R^r & S^r & 0 \end{pmatrix}$, and $B_{22} = \begin{pmatrix} -R^l & S^l \\ R^r & -S^r \end{pmatrix}$. We have the following theorem from [**2**].

THEOREM 1. *Enhanced system of equations (6) has an unique solution c for which holds $c_k = u_k$ for $k = 1, 2$, $c^l_I = \bar{c}^l_I = u^l_I$ and $c^r_I = \bar{c}^r_I = u^r_I$, if and only if B_{22}^{-1} exists.*

Proof: If B_{22}^{-1} exists, the Schur complement system $(B_{11} - B_{12}B_{22}^{-1}B_{21})\bar{c}_1 = g_1$ is precisely the original system of equations $A\bar{c}_1 = f$. As A was assumed non-singular, it follows that $\bar{c}_1 = u$, and hence $\bar{c}^l_I = u^l_I$ and $\bar{c}^r_I = u^r_I$. Conversely, let c be the (unique) solution of (6) for which the equalities of theorem 1 hold. Then $\begin{pmatrix} -R^l & S^l \\ R^r & -S^r \end{pmatrix} \begin{pmatrix} c^l_I - \bar{c}^l_I \\ c^r_I - \bar{c}^r_I \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ has only the trivial solution. Hence B_{22} is non-singular. □

Enhanced system (6) is the discrete counterpart we were looking for. The two middle rows represent the (additional) coupling equations, the first two and last two rows represent the original discrete problem, while the existence of B_{22}^{-1} guarantees equivalence with the original problem $Au = f$.

To solve $Au = f$, we propose the domain decomposition method that consists of block-Jacobi iteration, applied to an enhanced system of the form (6). In this block-Jacobi iteration the blocks correspond with the substructures, indicated in (6).

For discretizations with local support, c^l_I and c^r_I can be chosen such that A_{13}, $\hat{A}_{23}$, $\check{A}_{21}$ and A_{31} vanish. The optimization of the convergence speed boils down to a proper choice of R^l, S^l, R^r and S^r. The only constraint on this choice is given by the condition of existence of B_{22}^{-1}, which is a condition independent of the discretization. For a cell-vertex scheme using three-times-three-point computational molecules on a two-dimensional structured grid, these submatrices already disappear if c^l_I and c^r_I are chosen as the vectors consisting of the unknowns defined on two adjacent grid lines, with the interface in between.

3. A discrete model problem

Consider the two-dimensional advection-diffusion problem, with uniform flow field:

$$\frac{\partial c}{\partial t} + u\frac{\partial c}{\partial x} + v\frac{\partial c}{\partial y} = D\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) \quad \text{in } \Omega \times \mathcal{T}, \tag{8}$$

with $\Omega = [0,2] \times [0,1]$, time interval $\mathcal{T} = [t_0, t_0 + T]$; constants $u, v, D > 0$, and $c : \Omega \times \mathcal{T} \to R$. We restrict ourselves to cases where advection is dominant over diffusion (Pe $\gg 1$). On Ω, a uniform cartesian grid with size ($\Delta x = \frac{2}{I_1+I_2}$, $\Delta y = \frac{1}{J}$) and $(I_1 + I_2 + 2) \times (J+2)$ nodes has been defined to discretize equation (8), using three-point central differences in space and Euler backward in time. Depending on the flow direction, inflow or outflow conditions are used at the physical boundaries. For domain decomposition purposes, the domain is subdivided into two vertical strips, with common interface Γ normal to the x-direction, having $I_1 \times J$ and $I_2 \times J$ internal grid points respectively. Courant and cell-Peclet numbers are defined as: $\mathrm{CFL}_x := \frac{u\Delta t}{\Delta x}$ and $\mathrm{Pe}_x^c := \frac{u\Delta x}{D}$. Similar expressions are defined for the y-direction.

Conforming to the theory of the previous section, $(I_k+2) \times (J+2)$ unknowns are defined for each subdomain k, $k = 1, 2$, thus effectively using extended grids with $2J+4$ additional unknowns. These unknowns are denoted by $c_k^{i,j}$, $i = 0, ..., I_k+1$, $j = 0, ..., J+1$, $k = 1, 2$. On the virtual overlap consisting of two adjacent vertical grid lines, the subvectors $c_I^l = (c_1^{I_1,0}, ..., c_1^{I_1,J+1})^T$ and $c_I^r = (c_2^{1,0}, ..., c_2^{1,J})^T$ are defined, together with the enhancement vectors $\tilde{c}_I^r = (c_1^{I_1+1,0}, ..., c_1^{I_1+1,J+1})^T$ and $\tilde{c}_I^l = (c_2^{0,0}, ..., c_2^{0,J+1})^T$ (cf. (6)). So to close the system of equations, $2J+4$ coupling equations have to be added. For our advection-dominated model problem (8) they have been chosen as:

$$
(9) \quad
\begin{aligned}
&\Big(\, \mu_x\big(\tfrac{1}{2} + \tfrac{\beta}{2\Delta y}\mu_y\delta_y\big) + \delta_x\big(\tfrac{\alpha}{\Delta x} + \tfrac{\beta}{2\Delta y}\mu_y\delta_y\big) \,\Big)\, c_1^{I_1+\frac{1}{2},j} = \\
&\Big(\, \mu_x\big(\tfrac{1}{2} + \tfrac{\beta}{2\Delta y}\mu_y\delta_y\big) + \delta_x\big(\tfrac{\alpha}{\Delta x} + \tfrac{\beta}{2\Delta y}\mu_y\delta_y\big) \,\Big)\, c_2^{\frac{1}{2},j} \qquad ,j = 1, ..., J\,, \\
&\Big(\, \mu_x\big(\tfrac{1}{2} + \tfrac{\delta}{2\Delta y}\mu_y\delta_y\big) + \delta_x\big(\tfrac{\gamma}{\Delta x} - \tfrac{\delta}{2\Delta y}\mu_y\delta_y\big) \,\Big)\, c_2^{\frac{1}{2},j} = \\
&\Big(\, \mu_x\big(\tfrac{1}{2} + \tfrac{\delta}{2\Delta y}\mu_y\delta_y\big) + \delta_x\big(\tfrac{\gamma}{\Delta x} - \tfrac{\delta}{2\Delta y}\mu_y\delta_y\big) \,\Big)\, c_1^{I_1+\frac{1}{2},j} \qquad ,j = 1, ..., J\,,
\end{aligned}
$$

where μ_x and δ_x are defined as: $\delta_x c^i \equiv c^{i+\frac{1}{2}} - c^{i-\frac{1}{2}}$, $\mu_x c^i \equiv c^{i+\frac{1}{2}} + c^{i-\frac{1}{2}}$. Operators μ_y and δ_y are defined analogously. Note that equations (9) are the discrete equivalent of (2) and (4) involving c, $\frac{\partial c}{\partial x}$ at the interface, and $\frac{\partial c}{\partial y}$ at the boundaries of the extended subdomains. The 4 remaining coupling equations (at the intersection of the interface with the physical boundaries) are defined in a similar way, except for the number of unknowns involved (8 instead of 12), see [**2**].

4. Convergence analysis

The optimal value of the coupling parameters α, β, γ and δ is estimated by means of a simple Fourier analysis. We assume that the convergence error e^m of the mth iterand, $e^m = c^m - c$, can be written as:

$$
(10) \qquad e_k^{i,j,m} = \sum_{\ell=0}^{J-1} \left[F_{k,\ell}^m (\lambda_\ell^l)^i + G_{k,\ell}^m (\lambda_\ell^r)^{i-I_k-1} \right] \exp(\imath\omega_\ell j)\,,
$$

with $\imath = \sqrt{-1}$ and $\omega_\ell = 2\pi\ell\Delta y$; λ_ℓ^l and λ_ℓ^r are determined by the discretization while the amplitudes $F_{k,\ell}^m$ and $G_{k,\ell}^m$ are determined by (9) and by the left and right boundary conditions. Straightforward calculation shows:

PROPOSITION 1. *For sufficiently small CFL-numbers, the rate of convergence per Fourier mode can be approximated by:*

$$\rho_\ell^2 \approx \frac{[\frac{1+\lambda_\ell^l}{2} + \frac{\alpha(\lambda_\ell^l - 1)}{\Delta x} + \frac{\beta\lambda_\ell^l \sin(\omega_\ell)\imath}{\Delta y}][\frac{\bar{\lambda}_\ell^r + 1}{2} + \frac{\gamma(1-\bar{\lambda}_\ell^r)}{\Delta x} + \frac{\delta\bar{\lambda}_\ell^r \sin(\omega_\ell)\imath}{\Delta y}]}{[\frac{\bar{\lambda}_\ell^r + 1}{2} + \frac{\alpha(1-\bar{\lambda}_\ell^r)}{\Delta x} + \frac{\beta \sin(\omega_\ell)\imath}{\Delta y}][\frac{1+\lambda_\ell^l}{2} + \frac{\gamma(\lambda_\ell^l - 1)}{\Delta x} + \frac{\delta \sin(\omega_\ell)\imath}{\Delta y}]}$$

Here $\bar{\lambda}_\ell^r$ denotes $\frac{1}{\lambda_\ell^r}$. In our numerical experiments, we approximately solved the minimax problem: $\min_{\alpha,\beta,\gamma,\delta} \max_\ell \rho_\ell$ to estimate the optimal coupling parameters for (9). (The coupling parameters in the 4 remaining coupling equations were determined experimentally.)

5. Numerical results

In this section we report on some numerical experiments, where we have used results of the convergence analysis. In all cases we used $I_1 = I_2 = J = 10$, and $\mathrm{Pe}_x^c = \mathrm{Pe}_y^c = 10^5$.

To assess the importance of coupling parameters and the benefits of the inclusion of parameterized tangential derivatives in the coupling conditions, we varied the coupling parameters for a skew velocity field $\mathrm{CFL}_x = 1$, $\mathrm{CFL}_y = 2$ and computed the rate of convergence by determining the spectral radius of the error amplification matrix of the block-Jacobi method. The results are shown in figure 1. In the first figure α and β were varied, keeping γ and δ fixed at their (experimentally determined) optimal values. In the second figure, the opposite case was considered. For this example the convergence analysis predicts approximate optimal coupling

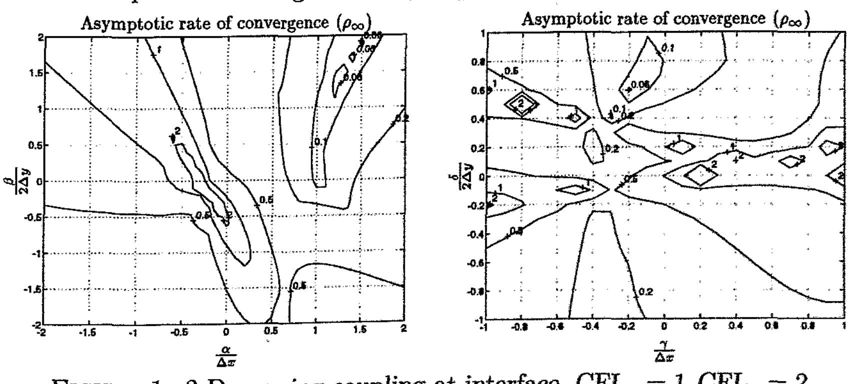

FIGURE 1. 2-D: varying coupling at interface, $\mathrm{CFL}_x = 1, \mathrm{CFL}_y = 2$

parameters: $\frac{\alpha}{\Delta x} \approx 1.27$, $\frac{\beta}{2\Delta y} \approx 1.44$, $\frac{\gamma}{\Delta x} \approx -0.19$, $\frac{\delta}{2\Delta y} \approx 0.56$. These are very close to the experimentally determined optimal values, both sets of (theoretically or experimentally determined) parameters resulting in convergence rates of approximately $\rho_\infty \approx 0.05$. The best convergence rate with the use of combinations of c and $\frac{\partial c}{\partial x}$ only, turns out to be $\rho_\infty \approx 0.3$, which is considerably worse.

We also compared several coupling mechanisms for a number of different flow directions. Velocities (u, v) were chosen such that $(\mathrm{CFL}_x)^2 + (\mathrm{CFL}_y)^2 = 2$. This time periodic boundary conditions in y-direction were used. In figure 2 the obtained convergence rates are shown. In the figure, DD denotes Dirichlet-Dirichlet coupling at the virtual grid points ($\alpha = \frac{\Delta x}{2}$, $\gamma = -\frac{\Delta x}{2}$, $\beta = \delta = 0$). ND1 is a

Neumann-Dirichlet coupling ($\alpha = \infty$ and $\beta = \gamma_2 = \delta_2 = 0$). ND2 is an extended Neumann-Dirichlet coupling where the 'Neumann' condition consists of a one-sided discretization of the advective part of (8) at the interface ($\frac{\alpha}{\Delta x} = \mathrm{CFL}_x$, $\beta = \mathrm{CFL}_y$, $\gamma = \delta = 0$). Finally, GND is our generalized Neumann-Dirichlet coupling, with parameters chosen according to the approximate optimization.

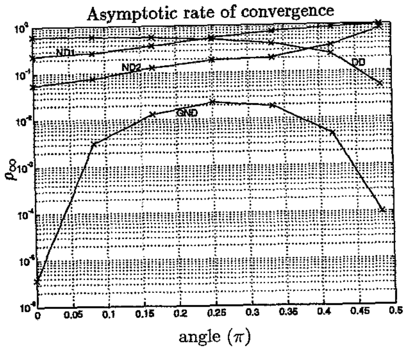

FIGURE 2. 2-D: 2 subdomains, varying angle of flow (w.r.t. x-axis)

The results indicate that for any flow angle, a suitable subdomain coupling is obtained with our approximate optimization of (9).

6. Concluding remarks

In this paper we have briefly described a flexible local coupling mechanism that allows an improvement of convergence speed of domain decomposition methods by tailoring the coupling to the discretized problem to solve. Full details can be found in [**2**]. Although in section 4 only a few coupling parameters were optimized, numerical results nevertheless show that in this way already some very good convergence rates can be obtained. The inclusion of a parameterized tangential component in the coupling algorithm appears to be essential. Recently obtained results with a slightly extended parameterized subdomain coupling by including several mixed derivatives show a further improvement of convergence rates.

The presented approach can be extended easily to variable-coefficient and non-linear problems when discretized on a curvilinear, structured grid, optimizing the coupling parameters locally in a problem-dependent way. This work is currently in progress and will be reported on in the near future.

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MATHEMATICAL INSTITUTE, UTRECHT UNIVERSITY, P.O. BOX 80010, 3508 TA UTRECHT, THE NETHERLANDS.

E-mail address: tan@math.ruu.nl

DELFT HYDRAULICS, P.O. BOX 152, 8300 AD EMMELOORD, THE NETHERLANDS.

E-mail address: mart.borsboom@wldelft.nl