A Multi-Parameter Parallel Algorithm for Local Higher Accuracy Approximation

Aihui Zhou

1 Introduction

One may employ mesh refinement techniques if higher accuracy approximation is required. It is recognized that the global uniform refinement on the whole domain leads to simple and usually vectorizable algorithms but wastes time and memory, while local refinement on subdomains minimizes the size of the discrete problem and improves the accuracy locally but leads to a lower accuracy in the whole domain.

It has been shown recently that a combination of such discrete solutions related to refinement on subdomains can yield an approximation of higher accuracy and the procedure can be done in parallel using multi-processor computers if the exact solution is globally smooth (see [8, 9]). This technique is based on a so-called multi-parameter error resolution. The crucial point of this approach is to choose certain independent mesh parameters: According to its geometry, the domain is divided into some subdomains and covered with different meshes so that the number of independent mesh parameters, say p, is as large as possible, and an approximation of higher accuracy can be computed $(p+1)$ processors in parallel.

We shall prove here that the parallel algorithm based on the multi-parameter error resolution can produce an approximation of higher accuracy even if the exact solution is only locally smooth.

[1] Institute of Systems Science, Academia Sinica, Beijing 100080, China

Domain Decomposition Methods in Sciences and Engineering, edited by R. Glowinski *et al.*

2 A Parallel Algorithm

Consider the finite element solution of the following Dirichlet boundary value problem,

$$\begin{cases} -\Delta u &= f, \text{ in } \Omega, \\ u &= 0, \quad \text{on } \partial\Omega, \end{cases} \tag{2.1}$$

where $\Omega \subset R^2$ is a polygonal convex domain. Divide Ω into subdomains $T = \{\Omega_j : j = 1, 2, \cdots, m\}$ so that T is a quasi-uniform partition of Ω and each Ω_j is a triangle or a parallelogram. On each Ω_j a uniform mesh is imposed, and a quasi-uniform global partition of Ω is formed. Denote the mesh size(s) on Ω_j by $h_{j,1}$ if Ω_j is a trangle, and by $h_{j,1}$ and $h_{j,2}$, the mesh sizes along the directions of the edges of Ω_j, if Ω_j is a parallelogram ($j = 1, \cdots, m$). Among these mesh parameters, some are independent, say $h_1, \cdots, h_p$. It can be proved that the number of independent parameters can be equal to or greater than 2 even that $p >> 2$. Let $h = max\{h_j : j = 1, \cdots, p\}$, T^h be the partition on Ω and let S^h be the conforming finite element space on T^h consisting of functions which are linear or bilinear on each triangular element or each parallelogram element in T^h. The interpolated finite element approximation corresponding to $h_1, \cdots, h_p$ is denoted by $u(h_1, \cdots, h_p)$. Set $u_0 = u(h_1, \cdots, h_p)$ and $u_j = u(h_1, \cdots, h_{j-1}, h_j/2, h_{j+1}, \cdots, h_p)$. Then a parallel algorithm for approximations of higher accuracy is given by

Algorithm.

Step 1. Compute $u_j (0 \le j \le p)$ in parallel.

Step 2. Set $u^c = (4\sum_{j=1}^{p} u_j - (4p-3)u_0)/3$.

It is proved later that the composite numerical solution u^c is a higher accuracy approximation to the exact solution u of (2.1) in $D \subset\subset D^* \subset \Omega$ if u is smooth on D^*.

3 Multi-Parameter Error Resolution

In the following, $L^q(\Omega)(1 \le q < \infty), H^s(\Omega), H_0^s(\Omega)$ and $W^{s,q}(\Omega)(s = 1, 2, \cdots)$ are the usual Lebesgue and Sobolev spaces respectively. Set $S_0^h = H_0^1(\Omega) \cap S^h$.

The Galerkin projection $R_h u \in S_0^h$ of the solution u of (2.1) is determined by

$$\int_\Omega \nabla R_h u \nabla v = \int_\Omega fv, \quad \forall v \in S_0^h. \tag{3.1}$$

To discuss the multi-parameter error resolution, we shall compare $R_h u$ with the Lagrange's interpolation $i_h u$ of u. It is known that

$$\int_\Omega \nabla(R_h u - i_h u) \nabla v = \int_\Omega \nabla(u - i_h u) \nabla v, \quad \forall v \in S_0^h. \tag{3.2}$$

By the Euler-MacLaurin formula and the integral identity ([3, 5]), the following result is obtained.

$1, \cdots, p, i = 1, 2)$ *of 3rd, 2nd and 1st order, respectively, such that*

$$\int_\Omega \nabla(u - i_h u) \nabla v = \sum_{j=1}^{p} h_j^2 (\int_\Omega L_j u N_{j,1} v + \int_\Gamma M_j u N_{j,2} v)$$

$$+ \begin{cases} O(h^3)|v|_{1,1}, if u \in H_0^1(\Omega) \cap C^4(\Omega), \\ O(h^2)\|u\|_{3,s,\Omega}|v|_{1,t,\Omega}, if u \in H_0^1(\Omega) \cap W^{3,s}(\Omega)(s \geq 1), \end{cases} \tag{3.3}$$

where $t = s/(s-1)$ *and* $\Gamma = \cup_{1 \leq i,j \leq m}(\partial\Omega_i \cap \partial\Omega_j)$.

For any fixed $z \in \Omega$, let G_z, $G_z^h \in S_0^h$ be the Green function and discrete Green function, respectively. Then from Lemma A 1 and A 2 in [1], one can obtain the following result

Proposition 2. *If* T^h *is a quasi-uniform partition, then*

(i) for any $z \in \Omega, 1 \leq t < 2$,

$$\|G_z - G_z^h\|_{1,t,\Omega} \leq ch^{1-2/s} \mid \ln h \mid^{1/2} . \tag{3.4}$$

(ii) for any $z \in D \subset\subset D^* \subset \Omega$,

$$h^{-1}\|G_z - G_z^h\|_{1,\infty,\Omega\backslash D^*} + \|G_z\|_{1,\infty,\Omega\backslash D^*} + \|G_z^h\|_{1,\infty,\Omega\backslash D^*} \leq c. \tag{3.5}$$

Thus, we obtain the following relation between the Galerkin projection and the interpolation.

Theorem 1. *If* $u \in H_0^1(\Omega) \cap W^{3,s}(\Omega)(s > 2)$, *then there exists* $\{w_1(u), \cdots, w_p(u)\} \subset H_0^1(\Omega) \cap C(\Omega)$ *such that the following multi-parameter error resolution formula*

$$R_h u = i_h u + \sum_{j=1}^{p} w_j(u) h_j^2 + o(h^2) \tag{3.6}$$

holds on $C(\Omega_0)$, *where* $\Omega_0 \subset\subset \Omega \setminus \Gamma_{00}$ *with* $\Gamma_{00} \equiv$ *the set of all interior macro-vertices of* T^h.

Proof. From propositions 1 and 2,

$$\int_\Omega \nabla(u - i_h u) \nabla G_z^h = \sum_{j=1}^{p} h_j^2 (\int_\Omega L_j u N_{j,1} G_z + \int_\Gamma M_j u N_{j,2} G_z)$$

$$+ \sum_{j=1}^{p} h_j^2 (\int_\Omega L_j u N_{j,1}(G_z^h - G_z) + \int_\Gamma M_j u N_{j,2}(G_z^h - G_z)) + O(h^2)|G_z^h|_{1,t,\Omega}$$

$$= \sum_{j=1}^{p} w_j(u) h_j^2 + O(h^2)\|u\|_{3,s,\Omega}, \tag{3.7}$$

where $w_j(u) = \int_\Omega L_j u N_{j,1} G_z + \int_\Gamma M_j u N_{j,2} G_z$.

Combining (3.2) and (3.7), we obtain

$$R_h u = i_h u + \sum_{j=1}^{p} w_j(u) h_j^2 + O(h^2)\|u\|_{3,s,\Omega}. \tag{3.8}$$

Similarly, we also obtain the following result (cf [8, 9])

$$R_h u = i_h u + \sum_{j=1}^{p} w_j(u) h_j^2 + O(h^4 |\ln h|) \tag{3.9}$$

on $C(\Omega_0)$ with $\Omega_0 \subset\subset \Omega \setminus \Gamma_{00}$ if the exact soultion is smooth enough, e.g., $u \in H_0^1(\Omega) \cap C^4(\Omega)$.

It is easy to see that a linear functional F_h defined by

$$F_h(u) = h^{-2}(R_h u - i_h u - \sum_{j=1}^{p} w_j(u) h_j^2) \tag{3.10}$$

satisfies:

$$| F_h(u) | \leq c|u|_{3,s,\Omega}, \quad \forall u \in H_0^1(\Omega) \cap W^{3,s}(\Omega). \tag{3.11}$$

Thus, combining (3.9) and (3.11), we obtain the theorem by a functional argument.

Theorem 2. *If $f \in W^{1,r}(\Omega)(r > 1)$ and $u \in H_0^1(\Omega) \cap W^{3,s}(D^*)(s > 2)$, then there exists $\{w_1(u), \cdots, w_p(u)\} \subset H_0^1(\Omega) \cap C(\Omega)$ such that the following multi-parameter error resoltuion formula*

$$R_h u = i_h u + \sum_{j=1}^{p} w_j(u) h_j^2 + o(h^2) \tag{3.12}$$

holds on $C(D)$, where $D \subset\subset D^* \setminus \Gamma_{00} \subset \Omega$.

Proof. First of all, we have $u \in H_0^1(\Omega) \cap W^{3,r^*}(\Omega)$ for some $r^* > 1$(see Grisvard [2]). Let $\omega \in C_0^\infty(\Omega)$ satisfty $\omega \equiv 1$ on D_1 and $supp\omega \subset\subset D^*$, where $D \subset\subset D_1 \subset\subset D^*$. Define $u_1 = \omega u$ and $u_2 = u - u_1$. Then $u_1 \in H_0^1(\Omega) \cap W^{3,s}(\Omega)$ and $u_2 \in H_0^1(\Omega) \cap W^{3,r^*}(\Omega)$.

Let $z \in D$, and let l_h be a linear functional defined by $l_h(u) = F_h(u_2)$. Then from proposition 2

$$\begin{aligned} | l_h(u) | \ &\leq \ ch^{-2} \sum_{j=1}^{p} h_j^2 (\int_\Omega L_j u_2 N_{j,1}(G_z^h - G_z) + \int_\Gamma M_j u_2 N_{j,2}(G_z^h - G_z)) \\ &\quad + c|u_2|_{3,1,\Omega} \|G_z^h\|_{1,\infty,\Omega\setminus D_1} \\ &\leq \ c|u|_{3,1,\Omega}. \end{aligned} \tag{3.13}$$

Thus, theorem 1 and a functional argument yield $l_h(u) \longrightarrow 0$ as $h \longrightarrow 0$, i.e.

$$h^{-2}(R_h u_2 - i_h u_2 - \sum_{j=1}^{p} w_j(u_2) h_j^2) = o(1). \tag{3.14}$$

On the other hand, by theorem 1, we have

$$h^{-2}(R_h u_1 - i_h u_1 - \sum_{j=1}^{p} w_j(u_1)h_j^2) = o(1). \tag{3.15}$$

Combining (3.14) and (3.15), we complete the proof.

Remark. Using the argument in [3, 5], there exists an interpolation operator I_h so that

$$I_h R_h u = i_h u + \sum_{j=1}^{p} w_j h_j^2 + o(h^2) \tag{3.16}$$

holds on $C(D)$. $I_h R_h u$ is determined by the parameters $h_1, \cdots, h_p$ and is denoted by $u(h_1, \cdots, h_p)$. Thus, a so-called multi-parameter splitting extrapolation technique leads to

$$u^c \equiv (4\sum_{j=1}^{p} u_j - (4p-3)u_0)/3 = u + o(h^2), \tag{3.17}$$

which holds on $C(D)$, where $D \subset\subset D^* \setminus \Gamma_{00} \subset \Omega$.

4 Other Partitions with Multi-Parameter Resolution.

It is pointed out that similar results can be expected for other kinds of partitions (cf. [8]): Divide Ω into several convex quadrilateral $T = \{\Omega_1, \cdots, \Omega_m\}$ such that T is quasi-uniform. Let Φ_i:

$$x_1(\xi,\eta) = a_{i,1}(1-\xi)(1-\eta) + a_{i,2}\xi(1-\eta) + a_{i,3}\xi\eta + a_{i,4}(1-\xi)\eta,$$

$$x_2(\xi,\eta) = b_{i,1}(1-\xi)(1-\eta) + b_{i,2}\xi(1-\eta) + b_{i,3}\xi\eta + b_{i,4}(1-\xi)\eta$$

be the bilinear coordinate transformations from the unit square $[0,1]^2$ to $\Omega_i (i = 1, \cdots, m)$, where $(a_{i,j}, b_{i,j})(j = 1,2,3,4)$ are the 4 vertices of the Ω_i. Under the mapping, a line parallel to $\xi-$ or $\eta-$ axis in $[0,1]^2$ is transformed into the line linking the two equipartition points of a two opposite edges in Ω_i, and globally a quasi-uniform partition T^h on Ω is formed. For a function v defined on Ω_i, let $\hat{v}$ be the function defined on $[0,1]^2$ by

$$\hat{v} = v \circ \Phi_i. \tag{4.1}$$

Conversely, a function $\hat{v}$ defined on $[0,1]^2$ determines a function v on Ω_i satisfying (4.1). Let $[0,1]^2$ be covered by a uniform mesh. Define

$$S_0^h(\Omega) = \{v \in H_0^1(\Omega) : v \circ \Phi_i \text{ is piecewise bilinear on } [0,1]^2, i = 1,2,\cdots\}, \tag{4.2}$$

$$u = \hat{u} \circ \Phi_i^{-1}, \quad \text{on } \Omega_i,$$

$$i_h u = \widehat{i_h} \hat{u} \circ \Phi_i^{-1}, \quad \text{on } \Omega_i,$$

where $\widehat{i_h}\hat{u}$ is the piecewise bilinear interpolant of $\hat{u}$ on $[0,1]^2$. $i_h u(x) = u(x)$ holds for the nodal points x of Ω and $S_0^h(\Omega)$ is determined by some parameters, say $h_1, \cdots, h_p$.

By induction, it can be proved that for any polygonal domain and with a proper choice of $\{\Omega_1, \cdots, \Omega_m\}$, p satisfies $p \geq 2$ even $p >> 2$. Given

$$\int_{\Omega_i} \partial_1(u - i_h u)\partial_1 v = \int_{[0,1]^2} p_{12}(\partial_\xi(\hat{u} - \hat{i_h u})\partial_\eta \hat{v} + \partial_\eta(\hat{u} - \hat{i_h u})\partial_\xi \hat{v}) + \int_{[0,1]^2} p_{11}\partial_\xi(\hat{u} - \hat{i_h u})\partial_\xi \hat{v} + p_{22}\partial_\eta(\hat{u} - \hat{i_h u})\partial_\eta \hat{v},$$

where

$$p_{11} = (\partial_\eta x_2)^2/J, \quad p_{22} = (\partial_\eta x_2)^2/J,$$
$$p_{12} = -\partial_\eta x_2 \partial_\xi x_2/J, \quad J = |\partial_\xi x_1 \partial_\eta x_2 - \partial_\eta x_1 \partial_\xi x_2|,$$

if $R_h u$ satisfies (3.1) for $S_0^h(\Omega)$ defined by (4.2). Then there exists an interpolation operator I_h (cf. [3, 5]) and functions w_i such that the following formula

$$I_h R_h u = u + \sum_{i=1}^{p} w_i h_i^2 + o(h^2) \tag{4.3}$$

holds on $C(D)$, where $D \subset\subset D^* \setminus \Gamma_{00} \subset \Omega$.

REFERENCES

[1] Blum H., Lin Q. and Rannacher R. (1986) Asymptotic error expansion and Richardson extrapolation for linear finite elements. *Numer. Math.* 49:11-37.

[2] Grisvard P. (1976) Behavior of the solutions of an elliptic boundery value problem in a polygonal or polyhedral domain, in *Numerical Solution of Partial Differential Equations-III (B. Hubbard, ed.)*, Academic Press, New-York, 207-274.

[3] Lin Q. (1990) *An integral identity and interpolated postprocess in superconvergence*, Research Report 90-07, Inst. of Sys. Sci., Academia Sinica.

[4] Lin Q. and Lü T. (1983) The splitting extrapolation method for multidimensional problems. *J. Comp. Math.* 1:45-51.

[5] Lin Q., Yan N. and Zhou A. (1991) A rectangle test for interpolated finite elements. in *Proc. of Sys. Sci. & Sys. Engrg.*, Great Wall Culture Publish Co., Hong Kong, 217-229.

[6] Lin Q. and Zhu Q. D. (1986) Local asymptotic expansion and extrapolation for finite elements. *J. Comp. Math.*, 3: 263-265.

[7] Rannacher R. (1987) Extrapolation techniques in the finite element method (A Survey). in *Proc. of the Summer School in Numer. Analysis at Helsinki.*

[8] Zhou A., Liem C. B. and Shih T. M. (1994) A parallel algorithm based on multi-parameter asymptotic error expansion. in *Proc. of Conference on Scientific Comput.*, Hong Kong, March 17-19.

[9] Zhou A., Liem C. B., Shih T. M. and Lü T. (1994) A multi-parameter splitting extrapolation and a parallel algorithm. Research Report IMS-61, Inst. of Math. Sci., Academia Sinica.