Algorithms for the Mortar Element Method

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ABSTRACT

We consider the saddle point type linear systems obtained from the discretization of a second order symmetric elliptic equation by the mortar element method. An iterative method in a subspace for solving these systems is described. This algorithm is based on a special class of preconditioners. Several preconditioners are then proposed.

1 INTRODUCTION

The mortar element method introduced in [7], [6], [16] is a finite element method based on domain decomposition which permits to use meshes non necessarily matching at subdomain interfaces, or different finite element approximations in different subdomains. Conformity is impossible since continuity across the interfaces between subdomains cannot be achieved (the meshes do not match), and one has to impose only some kind of weak continuity: for each interface Γ_{kl}, one has to introduce a suitable space W_{kl} of finite element functions supported on Γ_{kl}, and the continuity constraint is that the $L^2(\Gamma_{kl})$ projection of the jump across Γ_{kl} on the space W_{kl} vanishes.

Such a method has many possible advantages:

- It is genuinely suited for parallel computing.

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- It provides flexibility for the construction of the finite element mesh. This flexibility may be exploited to avoid updating the finite element mesh (sliding meshes [5]) or on the contrary for adapting the mesh.

The mortar method has been used in [4] for designing a solver for the Navier Stokes equations.

The aim of this paper is to present a class of solvers for the linear systems obtained when applying the mortar element method to second order elliptic partial differential equations. Here we consider the saddle point formulation of the discrete problem where the weak continuity across interfaces is treated as a constraint and the related Lagrange multiplier is a discretization of the normal derivative at interfaces, and we choose to eliminate the degrees of freedom (d.o.f.) interior to subdomains. Other attractive algorithms can be designed: see [13] for a 3D algorithm avoiding the elimination of the d.o.f. interior to subdomains, [16],15] for a Neumann-Neumann algorithm, [3], [17] for substructuring algorithms based on a two level block diagonal preconditioners with suitably chosen coarse spaces.

The section 2 will be devoted to a brief review on the mortar element method. In §3, we discuss an iterative method for saddle point problems, namely the so called preconditioned conjugate gradient in a subspace of constraints introduced in [14]. In §4, we apply this method for designing an algorithm for the mortar method. In §5, we discuss possible preconditioners.

2 THE DISCRETE PROBLEM

We consider the symmetric elliptic equation

$$-div\alpha(x)gradu + \beta(x)u = f \quad \text{in } \Omega, \qquad u = 0 \quad \text{on } \partial\Omega, \tag{2.1}$$

where Ω is a domain of $\mathbb{R}^N$ $(N = 2, 3)$, and α (resp. β) is a positive (resp. nonnegative) function. For simplicity we suppose that Ω is polygonal and that $N = 2$.

2.1 The geometry.

Let $\{\Omega_k\}$ be a partition of Ω into K non-overlapping open polygonal subdomains:

$$\overline{\Omega} = \cup_{k=1}^{K} \overline{\Omega}_k \qquad \text{and} \quad \Omega_k \cap \Omega_l = \emptyset \quad \text{if } k \neq l. \tag{2.2}$$

For simplicity, we also suppose that the domain decomposition is geometrically conforming, which means that the intersection of the closures of two subdomains is either empty or a vertex or a whole edge. For any $1 \leq k, l \leq K$, let Γ_{kl} be the closed straight segment, possibly degenerate : $\Gamma_{kl} = \overline{\Omega}_k \cap \overline{\Omega}_l$.

2.2 The discretization.

With each $1 \leq k \leq K$, we associate a family of quasi uniform triangular finite element meshes $\mathcal{T}_{k,h}$ of Ω_k with the classical regularity assumption for F.E.M., and we denote X_{kh} the related space of P_1 finite element functions vanishing on $\partial\Omega$. Let h_k be the

maximal diameter of the elements of $\mathcal{T}_{k,h}$. Let X_h denote the product space :

$$X_h = \prod_{1 \le k \le K} X_{kh}. \tag{2.3}$$

Note that the meshes do not need to match at interfaces. Therefore, in order to build a finite element space approaching $H^1(\Omega)$ one has to write a weak continuity constraint at the subdomain interfaces. Let us define the space of the Lagrange multipliers for the continuity constraint: we denote by Tr_k the trace on $\partial\Omega_k$. If $|\Gamma_{kl}| \neq 0$, the space $\tilde{W}_{k,l,h} = \left\{ Tr_k v_{|\Gamma_{kl}}, \quad v \in X_{kh} \right\}$ has dimension $N_{kl} + 2$, ($N_{kl} + 2$ is the number of vertices of $\mathcal{T}_{k,h}$ lying on Γ_{kl}).
For each interface Γ_{kl}, one can build the Lagrange multiplier space either from $\tilde{W}_{k,l,h}$ or from $\tilde{W}_{l,k,h}$. One possibility is to choose the space corresponding to the finer mesh: assuming the Lagrange multiplier space is built from $\tilde{W}_{k,l,h}$, let us choose $W_{k,l,h}$ as the subspace of $\tilde{W}_{k,l,h}$ of codimension 2 of functions which are constant near the two ends of Γ_{kl}. Let us call W_h the Lagrange multiplier space:

$$W_h = \prod_{1 \le k < l \le K,\ |\Gamma_{kl}| \neq 0} W_{k,l,h}. \tag{2.4}$$

Calling b the bilinear form

$$\begin{aligned} b : \quad & X_h \times W_h \to \mathbb{R}, \\ & b(\mathbf{v}_h, \mu_h) = \sum_{k<l\ :\ |\Gamma_{kl}| \neq 0} \int_{\Gamma_{kl}} \mu_{klh}(v_{kh} - v_{lh}), \end{aligned} \tag{2.5}$$

We are now able to define the subspace Y_h of X_h :

$$Y_h \equiv \{ \mathbf{v}_h \in X_h : \quad \forall \mu_h \in W_h, \quad b(\mu_h, \mathbf{v}_h) = 0 \}. \tag{2.6}$$

Calling a the bilinear form :

$$\begin{aligned} a : \quad & X_h \times X_h \to \mathbb{R}, \\ & a(\mathbf{u}_h, \mathbf{v}_h) \equiv \sum_{k=1}^{K} \int_{\Omega_k} \alpha \nabla u_{kh} \cdot \nabla v_{kh} + \beta u_{kh} v_{kh} \quad , \end{aligned} \tag{2.7}$$

the discretization of (2.1) is to find $\mathbf{u}_h \in Y_h$ such that

$$\forall \mathbf{v}_h \in Y_h, \quad a(\mathbf{u}_h, \mathbf{v}_h) = \sum_{k=1}^{K} \int_{\Omega_k} f\, v_{kh}, \tag{2.8}$$

which is clearly a well posed problem. It is easily proved that (2.8) is equivalent to the following well posed saddle point problem: find $(\mathbf{u}_h, \lambda_h) \in X_h \times W_h$ such that

$$\begin{aligned} & \forall \mathbf{v}_h \in X_h, && a(\mathbf{u}_h, \mathbf{v}_h) + b(\mathbf{v}_h, \lambda_h) = (f, \mathbf{v}_h), \\ & \forall \mu_h \in W_h, && b(\mathbf{u}_h, \mu_h) = 0. \end{aligned} \tag{2.9}$$

For the numerical analysis of the method, we refer to [7], [6]. The choice of the space W_h was made in order to achieve the Babuska-Brezzi inf-sup condition, (see [6]).

Remark 1 In [1], a variant of the method has been studied, where the jump operator is modified by means of mass lumping: indeed, defining the bilinear forms $b^{kl} : X_{kh} \times W_{klh} \to \mathbb{R}$ and $b^{lk} : X_{lh} \times W_{klh} \to \mathbb{R}$ by

$$b(\mathbf{v}_h, \mu_h) = \sum_{|\Gamma_{kl}|>0} b^{kl}(v_{kh}, \mu_{klh}) + b^{lk}(v_{lh}, \mu_{klh}), \tag{2.10}$$

and assuming that the space W_{klh} is constructed with the mesh of Ω_k, the idea is to replace in (2.10) the bilinear form b^{kl} by $\tilde{b}^{kl}$ obtained by performing mass lumping on the matrix of b^{kl}, and by keeping b^{lk} unchanged. A new jump bilinear form $\tilde{b}$ is thus obtained by assembly, and this leads to a new approximation method. With this new approximation, the same error estimates as in the original method can be obtained provided the meshes on which the Lagrange multiplier space W_h is built are sufficiently close to being uniform. As explained later, this modified jump operator may permit to design easily preconditioners with optimal arithmetical complexity.

3 PRECONDITIONED ITERATIVE METHODS IN A SUBSPACE OF CONSTRAINTS

Consider the saddle point problem:

$$\mathcal{S}\begin{pmatrix} V \\ \Lambda \end{pmatrix} = \begin{pmatrix} S & B^T \\ B & 0 \end{pmatrix}\begin{pmatrix} V \\ \Lambda \end{pmatrix} = \begin{pmatrix} G \\ 0 \end{pmatrix} \tag{3.1}$$

where S is a block diagonal matrix with K blocks S^k. We assume that $\mathcal{S}$ and BB^T are non singular, and that the blocks S^k are symmetric and positive semi-definite. We define the preconditioner for matrix $\mathcal{S}$ as

$$\mathcal{R} \equiv \begin{pmatrix} R & B^T \\ B & 0 \end{pmatrix} \tag{3.2}$$

where R has the same block structure as S. We assume that $\mathcal{R}$ is non singular and that the blocks R^k are symmetric and positive semi-definite with $Ker(R^k) \subset Ker(S^k)$.

Following [14], [12], [2], we apply for solving system (3.1) the Preconditioned Conjugate Gradient method in the Subspace

$$V_B = (I - \mathcal{R}^{-1}\mathcal{S})^2 \hat{V}_B \subset \hat{V}_B, \tag{3.3}$$

with

$$\hat{V}_B = \left\{ \begin{pmatrix} V \\ \Lambda \end{pmatrix} \; : \; BV = 0 \right\}. \tag{3.4}$$

Is is possible to use the preconditioned conjugate gradient algorithm because from the assumptions above, the following results can be proved:

1. the matrix $\mathcal{R}^{-1}\mathcal{S}$ keeps the subspace V_B invariant, *i.e.* $\mathcal{R}^{-1}\mathcal{S}V_B \subset V_B$,
2. the matrix $\mathcal{S}$ defines a scalar product in V_B,
3. the matrix $\mathcal{R}^{-1}\mathcal{S}$ is symmetric and positive definite in V_B with respect to the energy scalar product generated by the matrix $\mathcal{S}$, *i.e.* $\mathcal{S}\mathcal{R}^{-1}\mathcal{S}$ is symmetric and positive definite in V_B.

Remark 2 Under the condition: $R - S$ positive (or negative) definite, it is possible to choose $V_B = (I - \mathcal{R}^{-1}\mathcal{S})\hat{V}_B \subset \hat{V}_B$ instead of (3.3), see [8], [11].

4 ELIMINATION OF THE D.O.F. INTERIOR TO SUBDOMAINS

We supply X_h and W_h with their natural basis of nodal functions. Then the matrix form of system (2.9) is

$$\begin{pmatrix} A & \mathbf{B}^T \\ \mathbf{B} & 0 \end{pmatrix} \begin{pmatrix} U \\ \Lambda \end{pmatrix} = \begin{pmatrix} F \\ 0 \end{pmatrix} \tag{4.1}$$

The matrix A is a block diagonal matrix (one block per subdomain), each block corresponds to a discrete Neumann problem in subdomain Ω_k, except if $\overline{\Omega_k} \cap \partial\Omega \neq \emptyset$. It is possible to eliminate the d.o.f. located in the interior of subdomains by solving discrete Dirichlet problems. This leads to the system

$$\mathcal{S}\begin{pmatrix} V \\ \Lambda \end{pmatrix} \equiv \begin{pmatrix} S & B^T \\ B & 0 \end{pmatrix} \begin{pmatrix} V \\ \Lambda \end{pmatrix} = \begin{pmatrix} G \\ 0 \end{pmatrix} \tag{4.2}$$

where B denotes the nonzero block of $\mathbf{B}$, and S is the block diagonal matrix whose k^{th} block S^k corresponds to a discretized version of the Steklov-Poincaré operator of subdomain Ω_k: $H^{\frac{1}{2}}(\Omega \cap \partial\Omega_k) \to H^{-\frac{1}{2}}(\Omega \cap \partial\Omega_k)$, $\quad v \to \frac{\partial u_k}{\partial n}$, where u_k is defined by :

$$-div\,\alpha\, grad\, u_k + \beta\, u_k = 0 \quad \text{in } \Omega_k, \quad u_k = v \quad \text{on } \Omega \cap \partial\Omega_k, \quad u_k = 0 \quad \text{on } \partial\Omega \cap \partial\Omega_k.$$

Clearly, the matrix S satisfies all the assumptions of §3 and it is thus possible to use the algorithm described above. We then have to choose properly the matrix R and therefore the preconditioner $\mathcal{R}$. This will be the topic of the next section. Note finally that the elimination procedure is not compulsory (see [13] for an algorithm avoiding elimination).

5 PRECONDITIONERS

We first introduce two inexpensive preconditioners, which are not optimal as regards the condition number estimates, but which lead to linear systems which can be solved at an arithmetical cost proportional to their number of unknowns. Then we discuss inner iterative methods for better preconditioners in terms of condition number estimates.

5.1 Two inexpensive preconditioners

The first preconditioner will be introduced for the following symmetric elliptic p.d.e. (2.1) where Ω is a domain of $\mathbb{R}^N$ $(N = 2, 3)$, and α (resp. β) are positive (resp. nonnegative) functions, for simplicity constant in each subdomain. The values of α and β in Ω_k are denoted α_k and β_k, and the jumps of α and β across the interfaces

can be arbitrarily large. Then following [9], we choose R as the block diagonal matrix whose k^{th} block is

$$R^k \equiv h_k^{N-2}\alpha_k(I^k - P^k) + d_k h_k^{N-1}\beta_k P^k. \tag{5.1}$$

Here d_k is the diameter of subdomain Ω_k (we assume that the aspect ratio of the subdomains is bounded by a constant), I^k is the identity, and P^k is the matrix of the operator which maps a function defined on $\partial\Omega_k$ to its mean value. We have the following result

Proposition 5.1 *The condition number $\kappa(\mathcal{R}^{-1}\mathcal{S})$ with respect to subspace V_B satisfies*

$$\kappa(\mathcal{R}^{-1}\mathcal{S}) \leq C \max_k \frac{d_k}{h_k}, \tag{5.2}$$

where the positive constant C does not depend on α, β, h_k and d_k.

Efficients algorithms can be designed for solving the preconditioning problem: indeed, it is easily observed that R is a low rank perturbation of a diagonal matrix D ($R = D + L$, $rank(L) = K$). Therefore we split the matrix $\mathcal{R}$ into

$$\mathcal{R} = \mathcal{D} + \mathcal{L} \equiv \begin{pmatrix} D & B^T \\ B & 0 \end{pmatrix} + \begin{pmatrix} L & 0 \\ 0 & 0 \end{pmatrix}. \tag{5.3}$$

Since the rank of $\mathcal{L}$ is exactly K, the preconditioning system will consist essentially of solving twice the linear systems with $\mathcal{D}$, and once a *coarse problem* of size K. In two or three dimensions, the system with $\mathcal{D}$ can be solved by eliminating first the unknown U. This leads to a linear system with matrix $B^T D^{-1} B$ which can be solved by a preconditioned iterative method where the preconditioner would be obtained by performing mass lumping on $B^T D^{-1} B$. For a desired precision, the total cost of solving the preconditioner problem is proportional to the number of unknowns.

Alternatively, in two dimensions a direct solver can also be proposed: we first reorder the unknowns into two groups: the second group is made of the d.o.f. of U located at the crosspoints of the domain decomposition and the first group contains the remaining d.o.f. of U and the d.o.f. of Λ. With this ordering, the matrix $\mathcal{D}$ becomes

$$\begin{pmatrix} D_e & B_e^T & 0 \\ B_e & 0 & B_c \\ 0 & B_c^T & D_c \end{pmatrix} \tag{5.4}$$

where e stands for *edges* and c for *crosspoints*. The idea is to eliminate first the unknowns of U_e and Λ, which yields a sparse system whose dimension is proportional to the number of crosspoints and which can be solved by means of a direct Choleski method. To eliminate the d.o.f. of U_e and Λ, we group together the unknowns of U_e and Λ corresponding to same interfaces, and the submatrix

$$\begin{pmatrix} D_e & B_e^T \\ B_e & 0 \end{pmatrix}$$

becomes a block diagonal matrix (one block per interface) and the block related to interface Γ_{kl} is denoted

$$\begin{pmatrix} D_e^{kl} & 0 & {B_e^{kl}}^T \\ 0 & D_e^{lk} & {B_e^{lk}}^T \\ B_e^{kl} & B_e^{lk} & 0 \end{pmatrix}. \tag{5.5}$$

For solving the systems with such a matrix, the unknowns of U_e^{kl}, U_e^{lk} are eliminated. This yields a system with the band matrix $B_e^{kl} {D^{kl}}^{-1} {B_e^{kl}}^T + B_e^{lk} {D^{lk}}^{-1} {B_e^{lk}}^T$, which can be solved in a direct manner.

The second inexpensive preconditioner is introduced for the Laplace operator ($\alpha_k = 1$, $\beta_k = 0$) in two dimensions : we choose R as the block diagonal matrix whose k^{th} block is

$$R^k \equiv \frac{h_k}{d_k}(I^k - P^k) + \Sigma^k, \tag{5.6}$$

where I^k and P^k have been introduced above and where Σ^k is the matrix corresponding to the Laplace-Beltrami operator $\Sigma_h^k \equiv -h_k \Delta_{\partial\Omega_k}$:

$$(\Sigma_h^k u_{kh}, v_{kh}) \equiv h_k \int_{\partial\Omega_k} \frac{d}{ds} u_{kh} \frac{d}{ds} v_{kh} ds, \qquad \forall u_{kh}, v_{kh} \in X_{kh}. \tag{5.7}$$

The choice of this preconditioner can be explained as follows : the term $\frac{h_k}{d_k}(I^k - P^k)$ approaches the Steklov-Poincaré operator for the lowest frequencies while the term Σ^k is used for the highest frequencies. The Steklov-Poincar'e operator is not so well approached in the intermediate frequencies. Quantitatively, we have the following result

Proposition 5.2 *The condition number $\kappa(\mathcal{R}^{-1}\mathcal{S})$ with respect to subspace V_B satisfies*

$$\kappa(\mathcal{R}^{-1}\mathcal{S}) \leq C \max_k \sqrt{\frac{d_k}{h_k}}. \tag{5.8}$$

Thus the condition number depends only on the maximal number of mesh points in one subdomain, and is much improved compared to the first preconditioner. Of course, this will be paid by more difficulty in solving the related linear systems.

The procedure for solving the preconditioning linear system is very close to the one discussed above for the first preconditioner: the matrix $\mathcal{R}$ is decomposed into $\mathcal{R} = \tilde{\mathcal{R}} + \mathcal{L}$, where $\mathcal{L}$ is a low rank matrix (rank K) and

$$\tilde{\mathcal{R}} \equiv \begin{pmatrix} \tilde{R} & B^T \\ B & 0 \end{pmatrix}, \tag{5.9}$$

and $\tilde{R}$ is the block diagonal matrix whose k^{th} block is

$$\tilde{R}^k \equiv \frac{h_k}{d_k} I^k + \Sigma^k. \tag{5.10}$$

Again the preconditioning system consists of solving twice a linear system with matrix $\tilde{\mathcal{R}}$ and once a coarse problem of dimension K. For the problems with $\tilde{\mathcal{R}}$, we eliminate first the unknowns non located at crosspoints and we are led to solving a small system

whose dimension is proportional to the number of crosspoints. The main difficulty is to solve the systems with the blocks

$$\begin{pmatrix} \tilde{R}_e^{kl} & 0 & B_e^{klT} \\ 0 & \tilde{R}_e^{lk} & B_e^{lkT} \\ B_e^{klT} & B_e^{lkT} & 0 \end{pmatrix}, \tag{5.11}$$

with self explanory notations. Here, eliminating first U_e^{kl} and U_e^{lk} would lead to a linear system on Λ^{kl} with a dense matrix. The cost of solving this system would be proportional to the square of the number of unknowns. Therefore, we prefer instead solving directly the system, after having reordered carefully the unknowns. The reordering procedure, fully described in [2], permits to solve the system at an arithmetical cost proportional to the number of unknowns. Thus, here again, the preconditioning system can be solved with the optimal arithmetical complexity. However, in this case the programming effort is important, because reordering the unknowns is needed. Alternatively, a very close preconditioner to the latter can be designed (see [1]) when the lumped jump operator $\tilde{B}$ described in remark 1 is used, with a much easier practical implementation.

5.2 Inner iterative procedure for better preconditioners

The preconditioner will be introduced and analysed again for the Laplace operator in 2 dimensions. Let us choose

$$R^k = (\Sigma^k)^{\frac{1}{2}}, \tag{5.12}$$

where Σ^k is either the matrix

$$\begin{pmatrix} 2 & -1 & & & & -1 \\ -1 & 2 & -1 & & & \\ & \cdot & \cdot & \cdot & & \\ & & \cdot & \cdot & \cdot & \\ & & & -1 & 2 & -1 \\ -1 & & & & -1 & 2 \end{pmatrix}$$

for interior subdomains or a diagonal block of it otherwise. It is well known that the matrices R^k and S^k are spectrally equivalent. Thus the corresponding preconditioner $\mathcal{R}$ is also spectrally equivalent to $\mathcal{S}$ in the subspace of constraints. However solving the systems with the above mentioned preconditioner is not easier than solving the original system. Therefore, following [10], we are going to replace the above matrix $\mathcal{R}$ with another matrix $\hat{\mathcal{R}}$, spectrally equivalent to R, but leading to much cheaper implementation costs.

Let us introduce the matrix

$$\mathcal{Q} = \begin{pmatrix} Q & B^T \\ B & 0 \end{pmatrix}, \tag{5.13}$$

where Q is the block diagonal matrix whose k^{th} block is $Q^k \equiv h_k(I^k - P^k)$ if Ω_k is an internal subdomain and $h_k I^k$ otherwise. Note that $\mathcal{Q}$ is exactly the first preconditioner introduced in (5.1), for the special case of the Laplace operator.

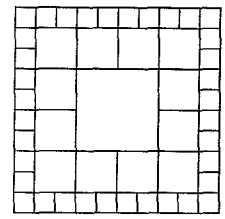

Figure 1 A smaller grid obtained by sparsening the initial one

Consider the eigenvalue problem : $\mu \mathcal{Q} W = \mathcal{R} W$. As in §5.1. it can easily be proved that the eigenvalues μ belong to segment $[c_1, c_2 \max_{1 \leq k \leq K} \frac{d_k}{h_k}]$ with positive constants c_1 and c_2 independant of d_k and h_k. Therefore, both matrices $\mathcal{Q}$ and $\mathcal{R}$ are symmetric and positive definite in subspace V_B and bounds on the spectrum of $\mathcal{Q}^{-1}\mathcal{R}$ can be computed by a Lanczos type method. Then it follows immediately from [19],[14] that the Preconditioned Chebyshev iterative method in the subspace V_B can be applied for solving system $\mathcal{R} X = Y$. The Preconditioned Chebyshev iterative method can be represented in a compact form by $X^s = [I - P_s(\mathcal{Q}^{-1}\mathcal{R})]\mathcal{R}^{-1} Y$ where $P_s(t)$ is the minimal polynomial of degree s related to a segment containing the eigenvalues of $\mathcal{Q}^{-1}\mathcal{R}$ (see [19]). Finally a matrix $\hat{\mathcal{R}}$ can be defined by

$$\hat{\mathcal{R}} = \mathcal{R} \left[I - P_m(\mathcal{Q}^{-1}\mathcal{R}) \right]^{-1} \tag{5.14}$$

with $m = O(\max_{1 \leq k \leq K} \sqrt{\frac{d_k}{h_k}})$. We have the following result :

Lemma 5.3 *Under the assumptions made, the matrix $\hat{\mathcal{R}}$ given by (5.17) is spectrally equivalent to matrix $\mathcal{S}$, in the subspace of constraints V_B.*

The crucial point in solving the preconditioning problem is the product by matrix R, which can be achieved thanks to fast Fourier transforms.

An alternative choice of R^k is proposed in [13]: let $\hat{\mathcal{T}}_{kh}$ be a finite difference grid such that the trace of $\hat{\mathcal{T}}_{kh}$ on $\partial\Omega_k$ coincides with that of $\mathcal{T}_{kh}$. We assume that $\hat{\mathcal{T}}_{kh}$ has much less nodes that $\mathcal{T}_{kh}$, namely $O(\frac{d_k}{h_k})$ nodes. When $\frac{d_k}{h_k} \approx 2^p$, an example of such a grid for a square is given on the figure 1. Let $\hat{A}^k$ be the corresponding stiffness matrix. It can be factorized with $O(\frac{d_k}{h_k})$ arithmetical operations. Calling $\hat{S}^k$ the related Schur complement, we have that $\hat{S}^k$ is spectrally equivalent to S^k (see [18]) and that the product of a vector by $\hat{S}^k$ costs $O(\frac{d_k}{h_k})$ operations. It is thus possible to choose $R^k = \hat{S}^k$ instead of (5.12).

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