

Domain decomposition for the incompressible Navier-Stokes equations: solving subdomain problems accurately and inaccurately¹

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Abstract:

For the solution of practical flow problems in arbitrarily shaped domains, simple Schwarz domain decomposition methods with minimal overlap are quite efficient, provided Krylov subspace methods, such as the GMRES method, are used to accelerate convergence. With accurate subdomain solution, the amount of time spent in solving these problems may be quite large. To reduce computing time, inaccurate solution of subdomain problems is considered, which requires a different, GCR based, acceleration technique. Much emphasis is put on the multiplicative domain decomposition algorithm since we also want an algorithm which is fast on a single processor.

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1 Introduction

For the solution of the incompressible Navier-Stokes equations in domains of arbitrary shape, we use a finite volume method on structured boundary fitted grids. References [MWSK91, CP92, WSvK⁺92, SWV⁺92, ZSW95a, ZSW95b] describe the discretization in detail and [OWSB93, ZSW95a] discuss the capability of the method to accurately solve a number of laminar and turbulent flows. A Schwarz type domain decomposition iteration [Sch69, BW94, BSK95] in combination with GMRES [SS86] acceleration is used. In [BW94], it was shown that significant reductions in computing time can be obtained using the GMRES acceleration procedure.

The method described in [BW94] requires accurate solution of subdomain problems, As a result of this, the computing time can be much larger than with single-block solution for the same number of unknowns. Also, it is not known beforehand how accurate the subdomain problems must be solved. The required subdomain solution accuracy may be quite high, especially when grid cells are very much stretched near block interfaces. A possible solution to both problems is to abandon the assumption of exact subdomain solution and to allow (very) inaccurate subdomain solution. Since the preconditioner may now vary in each iteration, GMRES acceleration may no longer be applied. Instead, a method based on GCR [EES83] is used.

Considerable reductions in computing time can be obtained in this way, see [BVW95b, BVW95a]. The present paper is an abstract from [BVW95a]. Theoretical results and numerical experiments are presented to illustrate the effect of inaccurate solution of subdomain problems for the incompressible Navier-Stokes equations.

Parallel computing is of increasing importance. Therefore it is important to compare the parallel (additive) domain decomposition algorithms with the best multiplicative algorithms, which are known to be faster than additive algorithms. Therefore, we pay much attention to multiplicative algorithms.

2 Discretization

For the spatial discretization, we use a finite volume method employing a staggered grid and central discretization, see [MWSK91, CP92, WSvK⁺92, SWV⁺92, ZSW95a, ZSW95b]. For the time discretization, the implicit Euler method is used. With V^n and P^n representing the algebraic vectors of velocity and pressure unknowns at time t^n , we get

$$\frac{V^{n+1} - V^n}{\Delta t} = M(V^n, P^n)V^{n+1} - GP^{n+1}, \quad (1)$$

$$DV^{n+1} = 0, \quad (2)$$

where (1) represents the momentum equations and (2) represents the incompressibility condition $\text{div } u = 0$. The matrix M represents the linearized spatial discretization of the Navier-Stokes equations around time level n , G is the discretized gradient operator and D is the discretized divergence operator on a staggered grid.

The pressure correction method [HW65, Cho68, Van86] is used to solve (1) and (2). It consists of three steps. In the first step, an estimate V^* of V^{n+1} is computed by

solving (1) with the pressure fixed at the old time level:

$$\frac{V^* - V^n}{\Delta t} = M(V^n, P^n)V^* - GP^n. \quad (3)$$

In the second step, the pressure correction ΔP is solved from

$$DG\Delta P = \frac{DV^*}{\Delta t}. \quad (4)$$

Dirichlet boundary conditions are prescribed for ΔP on boundaries where the normal velocity is given, and in all other cases Neumann boundary conditions are used. The last step consists of correcting the pressure: $P^{n+1} = P^n + \Delta P$ and computing V^{n+1} satisfying the incompressibility condition (2)

$$V^{n+1} = V^* - \Delta t G \Delta P. \quad (5)$$

3 Domain decomposition

Domain decomposition amounts to the solution of (3) and (4) using an alternating Schwarz method with minimal overlap, see [BW94, BSK95]. It is written as a block iteration to solve a system $Au = f$, with the blocks defined by the subdomains

$$u^{m+1} = (I - N^{-1}A)u^m + N^{-1}f, \quad (6)$$

with N^{-1} an approximation to the inverse of the block diagonal or block lower-triangular matrix of A . The subdomain problems are solved using GMRES [SS86, Vui93].

With accurate subdomain solution, u^{m+1} in (6) only depends on the components of u^m corresponding to unknowns on or near the block interfaces, see [BW94, BSK95]. Collecting these unknowns in a vector v and defining the trivial injection operator $u = Qv$, which extends v by zeroes to the full vector length, we get

$$u^{m+1} = (I - N^{-1}A)Qv^m + N^{-1}f, \quad (7)$$

By premultiplying (7) with Q^T and taking the stationary solution of the iteration process we get

$$Q^T N^{-1} A Q v = Q^T N^{-1} f. \quad (8)$$

The system (8) is a system concerning only unknowns on or near the interfaces (similar to Schur's complement). The GMRES acceleration then solves (8); details are in [BW94, BSK95].

Inaccurate subdomain solution means that we replace (6) by

$$u^{m+1} = u^m + \tilde{N}^{-1}(f - Au^m), \quad (9)$$

where \tilde{N} represents inaccurate subdomain solution. Because GMRES is used for subdomain solution, \tilde{N} varies in each iteration step, and we may no longer use GMRES acceleration.

The GCR [EES83] method for solving $Ax = f$ can be easily adapted to cope with variable preconditioners. The GCR method seeks to minimize the residual $r_k = f - Ax_k$ over a search space $S_k = \langle s_1, s_2, \dots, s_k \rangle$. For this purpose, a subspace $V_k = \langle v_1, v_2, \dots, v_k \rangle$ with $As_i = v_i$ is stored. Gram-Schmidt orthogonalization of V_k is used to project f onto V_k . The search directions are updated during orthogonalization such that the property $As_i = v_i$ is preserved. This enables a simple construction of the optimal solution x_k . By extending the search space with appropriate search directions s_{k+1}, s_{k+2}, \dots , GCR reduces the residual further. With the choice $s_{k+1} = r_k$, the method is equivalent to GMRES [SS86].

GCR acceleration of (9) uses $s_{k+1} = \tilde{N}^{-1}r_k$, which corresponds to a single domain decomposition iteration. For the case of a single subdomain, the method simplifies to GMRESR [vdVV94].

Another Krylov method that enables variable preconditioners is FGMRES [Saa93]. In [Bör89], for instance, this algorithm was used together with inaccurate subdomain solution. The emphasis in [Bör89] was not on reduction of computing time but on restrictions on the subdomain solution accuracy to retain the h -independent convergence of Neumann-Dirichlet methods.

4 Theoretical motivation

Inaccurate solution of subproblems reduces the amount of work in each domain decomposition iteration at the cost of some additional iterations of the outer domain decomposition iteration. Therefore, a reduction of computing time is only possible if the number of additional iterations is not too large. A simple analysis of the condition number of the postconditioned matrix $A\tilde{N}^{-1}$ confirms this statement.

Each iteration involves solving $Nu = g$ with N the matrix from (6). With inaccurate solution of subdomains, we solve a problem $\tilde{N}\tilde{u} = g$ with \tilde{N} as in (9). All subproblems are solved using a relative accuracy.

Condition 1 *Each subproblem $A_{ii}u_i = g_i$ is solved using initial guess 0 and with a relative accuracy of ϵ so that $\|g_i - A_{ii}\tilde{u}_i\| \leq \epsilon\|g_i\|$ in the Euclidean norm.*

Theorem 1 relates N and \tilde{N} .

Theorem 1 *If condition 1 holds for all subdomains and all possible right-hand sides g_i , then*

$$(a) \|I - N_{gs}\tilde{N}_{gs}^{-1}\| \leq C\epsilon, \text{ for some constant } C > 0.$$

$$(b) \|I - N_{jac}\tilde{N}_{jac}^{-1}\| \leq \epsilon.$$

Proof:

Proof of (a): Combination of Condition 1 with $\tilde{A}_{ii}\tilde{u}_i = g_i$ (inaccurate subdomain solution) gives $\|g_i - A_{ii}\tilde{A}_{ii}^{-1}g_i\| = \|(I - A_{ii}\tilde{A}_{ii}^{-1})g_i\| \leq \epsilon\|g_i\|$ for all g_i . From the definition of a matrix norm it follows that $\|I - A_{ii}\tilde{A}_{ii}^{-1}\| \leq \epsilon$.

Without loss of generality we take two subdomains. We get

$$I - N\tilde{N}^{-1} = \begin{bmatrix} I - A_{11}\tilde{A}_{11}^{-1} & \emptyset \\ -(I - A_{22}\tilde{A}_{22}^{-1})A_{21}\tilde{A}_{11}^{-1} & I - A_{22}\tilde{A}_{22}^{-1} \end{bmatrix} \quad (10)$$

Partition $x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ and note that for the Euclidean norm

$$\|x\| \leq \left\| \begin{bmatrix} x_1 \\ 0 \end{bmatrix} \right\| + \left\| \begin{bmatrix} 0 \\ x_2 \end{bmatrix} \right\| = \|x_1\| + \|x_2\|, \text{ then we have}$$

$$\|I - N\tilde{N}^{-1}\| = \sup_{\|x\| \leq 1} \|(I - N\tilde{N}^{-1})x\| \leq \sup_{\|x\| \leq 1} (\|(I - A_{11}\tilde{A}_{11}^{-1})x_1\| + \|(I - A_{22}\tilde{A}_{22}^{-1})A_{21}\tilde{A}_{11}^{-1}x_1\| + \|(I - A_{22}\tilde{A}_{22}^{-1})x_2\|).$$

Furthermore, for the Euclidean norm $\|x\| \leq 1$ implies $\|x_1\| \leq 1$ and $\|x_2\| \leq 1$ so that finally (a) follows with $C = 2 + \|A_{21}\tilde{A}_{11}^{-1}\|$.

Proof of (b): For any block diagonal matrix $B = \text{diag}(D_1, D_2, \dots, D_n)$,

$$\text{we have: } \|B\| = \sqrt{\rho(B^T B)} = \sqrt{\rho(\text{diag}(D_1^T D_1, \dots, D_n^T D_n))} = \max\{\sqrt{\rho(D_1^T D_1)}, \dots,$$

$$\sqrt{\rho(D_n^T D_n)}\} = \max\{\|D_1\|, \dots, \|D_n\|\}. \text{ If we use the additive post-}$$

conditioner, then $I - N\tilde{N}^{-1}$ is a block diagonal matrix with blocks $D_i = I - A_{ii}\tilde{A}_{ii}^{-1}$, so that $\|I - N\tilde{N}^{-1}\| = \max_i \|I - A_{ii}\tilde{A}_{ii}^{-1}\| \leq \epsilon$.

Therefore, (b) holds.

□

Theorem 1 enables us to give a relation between the condition numbers of $A\tilde{N}^{-1}$ and AN^{-1} .

Theorem 2 *Under the conditions of Theorem 1 and $C\epsilon < 1$, the condition number of $A\tilde{N}^{-1}$ satisfies*

$$\kappa(A\tilde{N}^{-1}) \leq \frac{1 + C\epsilon}{1 - C\epsilon} \cdot \kappa(AN^{-1}). \quad (11)$$

Proof:

Application of Theorem 1, and noting that $\|\star\|$ is a least upper bound norm, gives $\|N\tilde{N}^{-1}\| = \|N\tilde{N}^{-1} - I + I\| \leq 1 + C\epsilon$ and $\|(N\tilde{N}^{-1})^{-1}\| = \|(N\tilde{N}^{-1})^{-1}(I - N\tilde{N}^{-1}) + I\| \leq 1 + \|(N\tilde{N}^{-1})^{-1}\|C\epsilon$.

Since $C\epsilon < 1$, $\kappa(N\tilde{N}^{-1}) = \|N\tilde{N}^{-1}\| \cdot \|(N\tilde{N}^{-1})^{-1}\| \leq \frac{1+C\epsilon}{1-C\epsilon}$.

Inequality (11) follows from $\kappa(A\tilde{N}^{-1}) = \kappa(AN^{-1}N\tilde{N}^{-1}) \leq \kappa(AN^{-1}) \cdot \kappa(N\tilde{N}^{-1})$.

□

Theorem 2 shows that the subdomain solution accuracy has only a small effect on the condition number of the postconditioned matrix. This means that, at least for symmetric problems, the number of outer iterations will not increase (significantly) when the subdomain accuracy is lowered. The sensitivity of outer loop convergence to ϵ is given by the constant C in Theorem 1, which can be chosen 1 for the additive algorithm, independent of the number of subdomains. For multiplicative algorithms

this sensitivity constant C will probably also be small and independent of the number of subdomains, however, sharper bounds may require a much more detailed analysis.

The theorems only hold for constant \tilde{N} , but the results in Section 6 show that the conclusions also hold in case \tilde{N} varies in each iteration.

5 The model problem

We consider flow around a cylinder in a wall-bounded shear flow. Figure 1 shows the geometry and decomposition of the domain, coarse versions of the multi-block and single-block grids, and a description of the boundary conditions. The multi-block and single-block grids consist of 12240 and 10800 grid cells respectively.

The cylinder has diameter $a = 2$. The Reynolds number is defined as $\text{Re} = \frac{1}{2} \left(\frac{au^*}{\nu} \right)^2$ with $u^* = \sqrt{\frac{\tau_0}{\rho}}$, where $\tau_0 = \mu \partial u / \partial y$ is the shear stress associated with the linear inlet velocity profile. Typical Reynolds numbers for this problem are $\text{Re} = 1 - 5$. Our results are given for $\text{Re} = 2$. In the computation we have used $L = H = 10$. For more details on this computation, see [BW94].

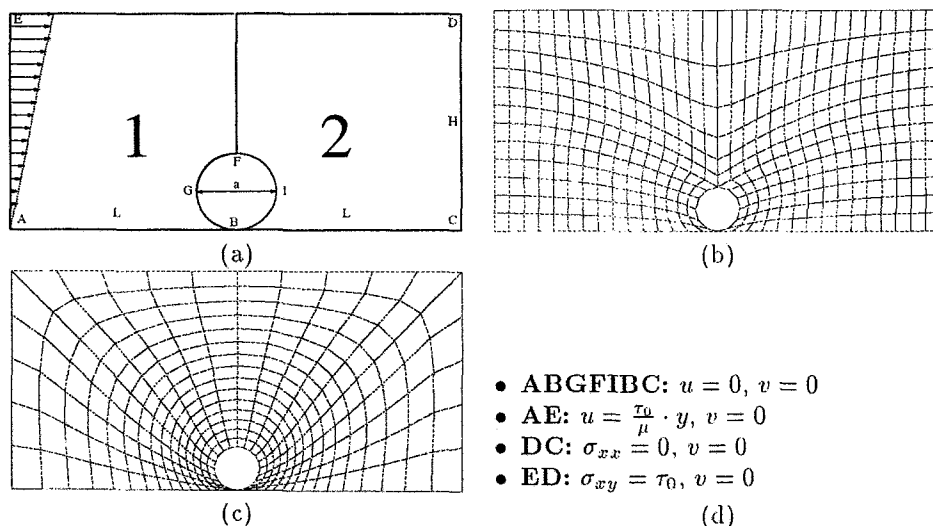


Figure 1 (a) Geometry and decomposition of the domain, (b) multi-block grid, (c) single-block grid, (d) boundary conditions

6 Results and conclusions

The time measurements in this section are given for only the first 10 time steps to avoid excessive computing times.

Table 1 lists the computing times (no. of iterations in brackets) for different

subdomain solution accuracies; the top row is for the algorithm which assumes accurate subdomain solution. As the subdomain solution accuracy is lowered from 10^{-4} to 10^{-1} , the number of outer GCR iterations shows only a small increase, which, because of the reduced work in solving subproblems, results in a reduction of total computing time. This is in accordance with Theorem 2. In these experiments, the most significant reductions are obtained for the pressure equations.

ϵ	<i>Tot.</i>	<i>Mom.</i>	<i>Pres.</i>
10^{-4*}	449.1	78.6(38)	315.2(154)
10^{-4}	465.2	58.4(37)	351.6(155)
10^{-2}	292.1	43.5(38)	193.4(168)
10^{-1}	230.3	47.5(48)	127.6(210)

Table 1 Results with varying accuracy of subdomain solution for the cylinder problem, multiplicative algorithm, 2 subdomains

<i>Tot.</i>	<i>Mom.</i>	<i>Pres.</i>
128.0	31.6	48.9

Table 2 Single-block solution times

Table 2 lists the computing times for single block solution. Comparing Tables 2 and 1, we see that solution time for the momentum equations comes close to single block solution time. For the pressure equations, solution time is still a factor 2 – 3 larger. With accurate subdomain solution this was approximately a factor 7.

Because of the significant reduction in computing time for the pressure equations, the total computing time shows a reduction of a factor of 2 by inaccurate subdomain solution. Still, the computing time with multi-block solution is almost twice as large as with single-block solution, and is dominated by the pressure solution time. A coarse grid correction, e.g. [BPS89, BS92], could be implemented for the pressure equations to reduce computing time further.

Inaccurate solution of subdomain problems combined with GCR acceleration removes the restriction inherent in GMRES solution of interface equations (8) that subdomain problems should be solved accurately (enough). The GCR based algorithm is therefore in general more reliable than the GMRES algorithm for solving interface equations.

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