On Convergence of the Parallel Schwarz Algorithm with Pseudo-Boundary and the Parallel Multisplitting Iterative Method

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1 Introduction

Suppose that we are given a linear system

$$Ax = b \tag{1}$$

where $A \in R^{n\times n}$ is a nonsingular matrix, $x, b \in R^n$ are vectors. In order to compute the solution of (1) iteratively, O'Leary and White propose multisplitting methods in [6] which are based on several splittings of the matrix A. More precisely, in [6] a multisplitting of A is defined as a collection of triples $(M_k,\ N_k,\ E_k)$, $k = 1, 2, \cdots, K$, such that for all k, M_k, N_k, E_k are $n \times n$ matrices, each M_k is nonsingular, $A = M_k - N_k$, and E_k is a diagonal matrix with nonnegative entries satisfying $\sum_{k=1}^{K} E_k = I$. The corresponding multisplitting method to solve (1) is given by the iteration

$$x^{m+1} = \sum_{k=1}^{K} E_k y^{m,k}, \quad m = 0, 1, \cdots \tag{2}$$

where

$$M_k y^{m,k} = N_k x^m + b, \quad k = 1, 2, \cdots, K.$$

This multisplitting method has a natural parallelism, since the calculations of $y^{m,k}$ for various k are independent and may therefore be performed in parallel. Moreover, the i-th component of $y^{m,k}$ need not be computed if the corresponding diagonal entry of E_k is zero. This may result in considerable savings of computational time. Convergence results for method (2) were first given in [6]. Later, Neumann and Plemmons [5] obtained more qualitative results for one of cases considered in [6].

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2 Parallel Multisplitting TOR Method

Suppose that A is a nonsingular $n \times n$ matrix, for $k = 1, 2, \cdots, K$, L_k, F_k, U_k, E_k are $n \times n$ matrices, L_k and F_k are strictly lower triangular matrix satisfying
(1) $A = D - L_k - F_k - U_k$, where $D = \text{diag}(A)$ is an $n \times n$ and are diagonal matrix and nonsingular, and each U_k is zero-diagonal matrix.
(2) $\sum_{k=1}^{K} E_k = I$ ($n \times n$-identity matrix), where each E_k is diagonal matrix and $E_k \geq 0$.

Then the collection of triples $(D - L_k - F_k,\ U_k,\ E_k)$ $(k = 1, 2, \cdots, K)$ is called a multisplitting of A.

For real numbers ω, α and β, we define the following function G_k: $R^n \longrightarrow R^n$, for $k = 1, 2, \cdots, K$

$$G_k(x) = [D - (\alpha L_k + \beta F_k)]^{-1}\{[(1-\omega)D + (\omega - \alpha)L_k + (\omega - \beta)F_k + \omega U_k]x + \omega b\}$$

Multisplitting TOR (MTOR) Method

For any starting vector $x^0 \in R^n$

$$x^{m+1} = \sum_{k=1}^{k=K} E_k G_k(x^m) \qquad m = 0, 1, 2, \cdots$$

until convergence.

Now we define the matrix

$$T_{MTOR}(\omega, \alpha, \beta) = \sum_{k=1}^{K} E_k[D - \alpha L_k - \beta F_k]^{-1}[(1-\omega)D + (\omega - \alpha)L_k + (\omega - \beta)F_k + \omega U_k] \tag{3}$$

and the vector

$$g_{MTOR}(\omega, \alpha, \beta) = \sum_{k=1}^{K} E_k[D - \alpha L_k - \beta F_k]^{-1}\omega b.$$

Then from the multisplitting TOR (MTOR) Method, we get

$$x^{m+1} = T_{MTOR}(\omega, \alpha, \beta)x^m + g_{MTOR}(\omega, \alpha, \beta)\ ,\ \ m = 0, 1, 2, \cdots. \tag{4}$$

For the MTOR method, corresponding to particular choices of the parameter set (ω, α, β) to be (1, 0, 0), (1, 1, 1), $(\omega, 0, 0)$, (ω, ω, ω) and (ω, γ, γ), it naturally reduces to parallel multisplitting Jacobi (MP), Gauss-Seidel (MGS), JOR (MJOR), SOR (MSOR) and AOR (MAOR) method, where MSOR method is the relaxed parallel multisplitting method in [2]; MAOR method is the parallel multisplitting AOR algorithm in [9]. Thus the MTOR-method is a improvement and an generalization algorithm of [2] and [9]. Hence, a general series of parallel multisplitting method for solving the system of linear equation (1) is formed, which makes the new method more flexible and applicable.

3 Convergence of the MTOR Method

We first need to introduce several known concepts and useful lemmas.

A vector $x \in R^n$ is called nonnegative (positive), denoted $x \geq 0$ $(x > 0)$ if $x_i \geq 0$ $(x_i > 0)$ holds for all components of $x = (x_1, x_2, \cdots, x_n)^T$.

Similarly, a matrix A is called nonnegative, if all of its entries are nonnegative.

For two matrices we write $A \geq B$, when $A - B \geq 0$, and for two vectors $x \geq y (x > y)$, when $x - y \geq 0 (x - y > 0)$. Given a matrix $A = (a_{ij})$, we define its absolute value by $|A| = (|a_{ij}|)$. It follows that $|A| \geq 0$ and that $|AB| \leq |A||B|$ for any two matrices A and B

For any matrix $A = (a_{ij})$, such that $a_{ij} \leq 0$ for $i \neq j$ and $A^{-1} \geq 0$, A is called a M-matrix (see [8]).

For any matrix $A = (a_{ij}) \in R^{n \times n}$, we define its comparison matrix $\langle A \rangle = (\langle \alpha_{ij} \rangle)$ by

$$\langle \alpha_{ij} \rangle = \begin{cases} |a_{ij}|, & \text{if} \quad i = j \\ -|a_{ij}|, & \text{if} \quad i \neq j \end{cases}$$

A matrix A is called H-matrix if its comparison matrix $< A >$ is an M-matrix.

Now we introduce several useful lemmas.

Lemma 1 [2] *Let A be an H-matrix, $D = diag(A)$, and $A = D - B$, then*
(1) A is nonsingular.
(2) $|A^{-1}| \leq \langle A \rangle^{-1}$
(3) $|D|$ is nonsingular and $\rho(|D|^{-1}|B|) < 1$.

Lemma 2 [8] *Suppose A, B satisfy $|A| \leq B$, then $\rho(A) \leq \rho(B)$.*

Lemma 3 [8] *Suppose that A is a nonnegative irreducible matrix. Then the spectral radius $\rho(A)$ of A is an eigenvalue of A and the eigenvector x corresponding to $\rho(A)$ satisfies $x > 0$.*

Theorem 1 *Suppose that A is an H-matrix, with a multisplitting*

$$(D - L_k - F_k, U_k, E_k),\ k = 1, 2, \cdots, K$$

such that

$$\langle A \rangle = |D| - |L_k| - |F_k| - U_k| = |D| - |B|$$

where $D = diag(A)$ is $n \times n$, diagonal and nonsingular, each L_k and F_k is a strictly lower triangular matrix, each U_k is a zero-diagonal matrix. Then MTOR method (5) converges for any starting vector $x^0 \in R^n$ provided that the parameters ω, α, β satisfy

$$0 \leq \alpha\ ,\ \beta \leq \omega\ \ ,\ \ 0 < \omega < \frac{2}{1 + \rho(|D|^{-1}|B|)}. \tag{5}$$

Proof. Since $\rho(T_{MTOR}(\omega, \alpha, \beta)) \leq \rho(|T_{MTOR}(\omega, \alpha, \beta)|)$ by Lemma 2, where $T_{MTOR}(\omega, \alpha, \beta)$ is the iteration matrix given by (3), we only need to show that $\rho(|T_{MTOR}(\omega, \alpha, \beta)|) < 1$.

As A is an H-matrix, D is a diagonal matrix, L_k and F_k are strictly lower triangular matrices, we easily see that $D - \alpha L_k - \beta F_k$ are H-matrices for $k = 1, 2, \cdots, K$. Using the result (2) of Lemma 1 and the definition of comparison matrix, we get

$$|(D - \alpha L_k - \beta F_k)^{-1}| \leq \langle D - \alpha L_k - \beta F_k \rangle^{-1} = |D| - \alpha |L_k| - \beta |F_k|.$$

First let the inequalities $0 \le \alpha \le \omega$, $0 \le \beta \le \omega$, $0 < \omega \le 1$ hold.
For $k = 1, 2, \cdots, K$, we define the matrices

$$M_k = |D| - \alpha|L_k| - \beta|F_k|, \tag{6}$$

and

$$N_k^1 = (1-\omega)|D| + (\omega - \alpha)|L_k| + (\omega - \beta)|F_k| + \omega|U_k|. \tag{7}$$

From (6), (7), we obtain

$$N_k^1 = M_k - \omega|D| - \omega|B| = M_k - \omega(|D| - |B|). \tag{8}$$

We take absolute values of both sides of (3) and obtain

$$|T_{MTOR}(\omega, \alpha, \beta)| \le \sum_{k=1}^{K} E_k M_k^{-1} N_k^1 I - \omega \sum_{k=1}^{K} E_k M_k^{-1} |D|(I - |D|^{-1}|B|). \tag{9}$$

Let $e = [1, 1, \cdots, 1]^T \in R^n$. Since $|D|^{-1}|B|$ is nonnegative, the matrix $J_\epsilon = |D|^{-1}|B| + \epsilon ee^T$has only positive entries and is irreducible for any $\epsilon > 0$. By Lemma 3, we known that $\rho(J_\epsilon)$ is an eigenvalue of J_ϵ and the corresponding eigenvector $x_\epsilon \ge 0$ satisfying

$$J_\epsilon x_\epsilon = (|D|^{-1}|B| + \epsilon ee^T)x_\epsilon = \rho(J_\epsilon)x_\epsilon.$$

Moreover, since $0 < \omega \le 1$, we have

$$1 - \omega + \omega\rho(|D|^{-1}|B|) < 1.$$

By the continuity of the spectral radius, we also get

$$1 - \omega + \omega\rho(J_\epsilon) < 1 \tag{10}$$

if $\epsilon > 0$ is sufficient small.
By (9), we have

$$\begin{aligned} |T_{MTOR}(\omega, \alpha, \beta)| &\le I - \omega \sum_{k=1}^{K} E_k M_k^{-1} |D|[I - (|D|^{-1}|B| + \epsilon ee^T)] \\ &= I - \omega \sum_{k=1}^{K} E_k M_k^{-1} |D|(I - J_\epsilon) \end{aligned} \tag{11}$$

and by multiplying by x_ϵ,

$$|T_{MTOR}(\omega, \alpha, \beta)|x_\epsilon \le x_\epsilon - \omega \sum_{k=1}^{K} E_k M_k^{-1} |D|(1 - \rho(J_\epsilon))x_\epsilon. \tag{12}$$

From the definition of M_k, the M_k are H-matrices. By Lemma 1, we get

$$M_k \le |D| \ , \ M_k^{-1} \ge |D|^{-1}.$$

By (10) and (12), we have

$$\begin{aligned}|T_{MTOR}(\omega,\gamma,\bar{\gamma})|x_\epsilon &\le x_\epsilon - \omega\sum_{k=1}^{K} E_k|D|^{-1}|D|(I-\rho(J_\epsilon))x_\epsilon \\ &= (1-\omega+\omega\rho(J_\epsilon))x_\epsilon < x_\epsilon. \qquad (13)\end{aligned}$$

By exercise 2 of [8], p.48,

$$\rho(|T_{MTOR}(\omega,\alpha,\beta)|) < 1$$

holds.

Next let the inequalities $1 < \alpha \le \omega$, $1 < \beta \le \omega$, $1 < \omega < 2/(1+\rho(|D|^{-1}|B|)$ hold. We define matrices

$$N_k^2 = (\omega-1)|D| + (\omega-\alpha)|L_k| + (\omega-\beta|F_k| + \omega|U_k|. \qquad (14)$$

From (6) and (14), then

$$N_k^2 = M_k - [(2-\omega)|D| - \omega|B|]. \qquad (15)$$

We take absolute values of both sides of (3) and have

$$|T_{MTOR}(\omega,\gamma,\bar{\gamma})| \le \sum_{k=1}^{K} E_k M_k^{-1} N_k^2 \le I - \sum_{k=1}^{K} E_k M_k^{-1}|D|[(2-\omega)I - \omega|D|^{-1}|B|]. \qquad (16)$$

As in the previous proof, let $e = [1,1,\cdots,1]^T \in R^n$ and let $x_\epsilon > 0$ denote the vector satisfying $J_\epsilon = (J+\epsilon ee^T)x_\epsilon = \rho(J_\epsilon)x_\epsilon$, where $\epsilon > 0$ is sufficiently small such that $\omega - 1 + \omega\rho(J_\epsilon) < 1$, since $1 < \omega < 2/(1+\rho(|D|^{-1}|B|)$.

From (16) we get

$$|T_{MTOR}(\omega,\alpha,\beta)| \le I - \sum_{k=1}^{K} E_k M_k^{-1}|D|[(2-\omega)I - \omega J_\epsilon] \qquad (17)$$

and multiplying by x_ϵ, then

$$\begin{aligned}|T_{MTOR}(\omega,\alpha,\beta)|x_\epsilon &\le x_\epsilon - \sum_{k=1}^{K} E_k|D|^{-1}|D|[2-\omega-\omega\rho(J_\epsilon)]x_\epsilon \\ &= x_\epsilon - [2-\omega-\omega\rho(J_\epsilon)]x_\epsilon = [\omega-1+\omega\rho(J_\epsilon)]x_\epsilon \\ &< x_\epsilon.\end{aligned}$$

Thus $\rho(|T_{MTOR}(\omega,\alpha,\beta)|) < 1$ follows again by exercise of [8], p.48.

Under the assumption of the theorem, this completes the proof.

Theorem 1 implies the following Corollaries

Corollary 1 *Under the conditions of Theorem 1, the MSOR method converges to the unique solution $x^* \in R^n$ of the system of weakly nonlinear equations (1) for any starting vector $x^0 \in R^n$ provided that the parameter ω satisfies*

$$0 < \omega < \frac{2}{1+\rho(|D|^{-1}|B|)}. \qquad (18)$$

Corollary 2 *Under the conditions of Theorem 1, the MAOR method converges to the unique solution $x^* \in R^n$ of the system of weakly nonlinear equations (1) for any starting vector $x^0 \in R^n$ provided that the parameter ω satisfies*

$$0 \le \gamma \le \omega,\ 0 < \omega < \frac{2}{1+\rho(|D|^{-1}|B|)}. \tag{19}$$

4 Block MTOR (BMTOR) Method

By splitting the number set $\{1, 2, \cdots, n\}$ into K nonempty subset J_k $(k = 1, 2, \cdots, K)$, i.e.

$$J_k \subset \{1, 2, \cdots, n\}\,,\ \cup_{k=1}^{K} J_k = \{1, 2, \cdots, n\}\ \ k = 1, 2, \cdots, K,$$

we define the splitting matrices corresponding to the nonsingular matrix $A \in R^{n\times n}$ as follows:

$$D = \mathrm{diag}(A)\,,\ \ D\ \text{ is nonsingular}$$

$$L_k = (l_{ij}^k)\ \ l_{ij}^k = \begin{cases} -a_{ij}^k, 1 \le j < [i/2]\ \ i, j \in J_k \\ 0,\ \ \text{otherwise} \end{cases} \tag{20}$$

$$F_k = (f_{ij}^k)\ \ f_{ij}^k = \begin{cases} -a_{ij}^k,\ \ [i/2] \le j < i\ \ i, j \in J_k \\ 0,\ \ \text{otherwise} \end{cases} \tag{21}$$

$$U_k = (u_{ij}^k)\ \ u_{ij}^k = \begin{cases} 0\ ,\ \ j = i \\ -(a_{ij} + l_{ij}^k + f_{ij}^k)\ \ ,\ \ \ \text{otherwise} \end{cases} \tag{22}$$

with

$$A = D - L_k - F_k - U_k,\ k = 1, 2, \cdots, K.$$

Here $[a]$ is used to denote the integer part of a positive number a. The nonnegative diagonal matrices E_k $(k = 1, 2, \cdots, K)$ are introduced with $e_i^k \ge 0$ for $i \in J_k$, $e_i^k = 0$ for $i \notin J_k$, and $\sum_{k=1}^{K} E_k = I$ (identity matrix).

With these matrices, a block multisplitting of the matrix A results and denoted by

$$(D - L_k - F_k, U_k, E_k),\ \ k = 1, 2, \cdots, K.$$

Now we construct the block MTOR (BMTOR) method for solving the system of linear equations (1) as follows:

BMTOR method

For any starting vector $x^0 \in R^n$, for $m = 0, 1, 2, \cdots$, until convergence

$$x^{m+1} = \sum_{k=1}^{K} E_k x^{m,k}$$

where

$$a_{ii}x_i^{m,k} - \alpha \sum_{1\le j\le [i/2]} l_{ij}^k x_j^{m,k} - \beta \sum_{[i/2]\le j\le n} f_{ij}^k x_j^{m,k}$$

$$= (1-\omega)a_{ii}x_i^m + (\omega - \alpha) \sum_{1\le j\le [i/2]} l_{ij}^k x_j^m$$

$$+ (\omega - \beta) \sum_{[i/2]\le j\le n} f_{ij}^k x_j^m + \omega \sum_{j \neq i} u_{ij}^k x_j^m + \omega b_i \ , \ i \in J_k$$

$$x_i^{m+1} = \sum_{k=1}^{K} e_i^k x_i^{m,k} \quad , \quad i = 1, 2, \cdots, n.$$

Here $\alpha, \beta \ge 0$ are relaxation factors and $\omega > 0$ is an acceleration parameter.

The BMTOR method is a block MTOR method for numerically solving the system of linear equation (1) in synchronous parallel environments. For different k, the lower dimensional systems of equations (whose dimensions equal the number of elements included in the J_k) corresponding to the k-th splitting can be solved on the k-th processor of a multiprocessor system. A convergence theorem of the BMTOR method can be obtained in a similar way as for the MTOR method, so we will not demonstrate it here in detail.

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