On Convergence of the Parallel Schwarz Algorithm with Pseudo-Boundary and the Parallel Multisplitting Iterative Method

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1 Introduction

Suppose that we are given a linear system

$$ Ax = b $$

(1)

where $A \in \mathbb{R}^{n \times n}$ is a nonsingular matrix, $x, b \in \mathbb{R}^{n}$ are vectors. In order to compute the solution of (1) iteratively, O'Leary and White propose multisplitting methods in [6] which are based on several splittings of the matrix $A$. More precisely, in [6] a multisplitting of $A$ is defined as a collection of triples $(M_k, N_k, E_k)$, $k = 1, 2, \cdots, K$, such that for all $k$, $M_k$, $N_k$, $E_k$ are $n \times n$ matrices, each $M_k$ is nonsingular, $A = M_k - N_k$, and $E_k$ is a diagonal matrix with nonnegative entries satisfying $\sum_{k=1}^{K} E_k = I$. The corresponding multisplitting method to solve (1) is given by the iteration

$$ x^{m+1} = \sum_{k=1}^{K} E_k y^{m,k}, \quad m = 0, 1, \cdots $$

(2)

where

$$ M_k y^{m,k} = N_k x^{m} + b, \quad k = 1, 2, \cdots, K. $$

This multisplitting method has a natural parallelism, since the calculations of $y^{m,k}$ for various $k$ are independent and may therefore be performed in parallel. Moreover, the $i$-th component of $y^{m,k}$ need not be computed if the corresponding diagonal entry of $E_k$ is zero. This may result in considerable savings of computational time. Convergence results for method (2) were first given in [6]. Later, Neumann and Plemmons [5] obtained more qualitative results for one of cases considered in [6].

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2 Parallel Multisplitting TOR Method

Suppose that $A$ is a nonsingular $n \times n$ matrix, for $k = 1, 2, \cdots, K$, $L_k$, $F_k$, $U_k$, $E_k$ are $n \times n$ matrices, $L_k$ and $F_k$ are strictly lower triangular matrix satisfying

(1) $A = D - L_k - F_k - U_k$, where $D = \text{diag}(A)$ is an $n \times n$ and are diagonal matrix and nonsingular, and each $U_k$ is zero-diagonal matrix.

(2) $\sum_{k=1}^{K} E_k = I$ (n x n-identity matrix), where each $E_k$ is diagonal matrix and $E_k \geq 0$.

Then the collection of triples $(D - L_k - F_k, U_k, E_k)$ $(k = 1, 2, \cdots, K)$ is called a multisplitting of $A$.

For real numbers $\omega$, $\alpha$ and $\beta$, we define the following function $G_k: R^n \to R^n$, for $k = 1, 2, \cdots, K$

$$G_k(x) = [D - (\alpha L_k + \beta F_k)]^{-1} \left\{ [(1 - \omega)D + (\omega - \alpha)L_k + (\omega - \beta)F_k + \omega U_k]x + \omega b \right\}$$

Multisplitting TOR (MTOR) Method

For any starting vector $x^0 \in R^n$

$$x^{m+1} = \sum_{k=1}^{K} E_k G_k(x^m) \quad m = 0, 1, 2, \cdots$$

until convergence.

Now we define the matrix

$$T_{MTOR}(\omega, \alpha, \beta) = \sum_{k=1}^{K} E_k [D - \alpha L_k - \beta F_k]^{-1} [(1 - \omega)D + (\omega - \alpha)L_k + (\omega - \beta)F_k + \omega U_k]$$

and the vector

$$g_{MTOR}(\omega, \alpha, \beta) = \sum_{k=1}^{K} E_k [D - \alpha L_k - \beta F_k]^{-1} \omega b.$$ (3)

Then from the multisplitting TOR (MTOR) Method, we get

$$x^{m+1} = T_{MTOR}(\omega, \alpha, \beta)x^m + g_{MTOR}(\omega, \alpha, \beta) \quad m = 0, 1, 2, \cdots$$ (4)

For the MTOR method, corresponding to particular choices of the parameter set $(\omega, \alpha, \beta)$ to be $(1, 0, 0)$, $(1, 1, 1)$, $(\omega, 0, 0)$, $(\omega, \omega, \omega)$ and $(\omega, \gamma, \gamma)$, it naturally reduces to parallel multisplitting Jacobi (MP), Gauss-Seidel (MGS), JOR (MJOR), SOR (MSOR) and AOR (MAOR) method, where MSOR method is the relaxed parallel multisplitting method in [2]; MAOR method is the parallel multisplitting AOR algorithm in [9]. Thus the MTOR-method is a improvement and an generalization algorithm of [2] and [9]. Hence, a general series of parallel multisplitting method for solving the system of linear equation (1) is formed, which makes the new method more flexible and applicable.

3 Convergence of the MTOR Method

We first need to introduce several known concepts and useful lemmas.
A vector \( x \in \mathbb{R}^n \) is called nonnegative (positive), denoted \( x \geq 0 \) (\( x > 0 \)) if \( x_i \geq 0 \) (\( x_i > 0 \)) holds for all components of \( x = (x_1, x_2, \ldots, x_n)^T \).

Similarly, a matrix \( A \) is called nonnegative, if all of its entries are nonnegative.

For two matrices we write \( A \geq B \), when \( A - B \geq 0 \), and for two vectors \( x \geq y \), when \( x - y \geq 0 \) (\( x > y \)). Given a matrix \( A = (a_{ij}) \), we define its absolute value by \( |A| = (|a_{ij}|) \). It follows that \( |A| \geq 0 \) and that \( |AB| \leq |A||B| \) for any two matrices \( A \) and \( B \).

For any matrix \( A = (a_{ij}) \), such that \( a_{ij} \leq 0 \) for \( i \neq j \) and \( A^{-1} \geq 0 \), \( A \) is called a M-matrix (see [8]).

For any matrix \( A = (a_{ij}) \in \mathbb{R}^{n \times n} \), we define its comparison matrix \( \langle A \rangle = (\langle a_{ij} \rangle) \) by

\[
\langle a_{ij} \rangle = \begin{cases} |a_{ij}|, & \text{if } i = j \\
-|a_{ij}|, & \text{if } i \neq j 
\end{cases}
\]

A matrix \( A \) is called \( H \)-matrix if its comparison matrix \( < A > \) is an \( M \)-matrix.

Now we introduce several useful lemmas.

**Lemma 1** [2] Let \( A \) be an \( H \)-matrix, \( D = \text{diag}(A) \), and \( A = D - B \), then

1. \( A \) is nonsingular.
2. \( |A^{-1}| \leq \langle A \rangle^{-1} \)
3. \( |D| \) is nonsingular and \( \rho(|D|^{-1}|B|) < 1 \).

**Lemma 2** [8] Suppose \( A, B \) satisfy \( |A| \leq B \), then \( \rho(A) \leq \rho(B) \).

**Lemma 3** [8] Suppose that \( A \) is a nonnegative irreducible matrix. Then the spectral radius \( \rho(A) \) of \( A \) is an eigenvalue of \( A \) and the eigenvector \( x \) corresponding to \( \rho(A) \) satisfies \( x > 0 \).

**Theorem 1** Suppose that \( A \) is an \( H \)-matrix, with a multisplitting

\[
(D - L_k - F_k, U_k, E_k), \ k = 1, 2, \ldots, K
\]

such that

\[
\langle A \rangle = |D| - |L_k| - |F_k| - |U_k| = |D| - |B|
\]

where \( D = \text{diag}(A) \) is \( n \times n \), diagonal and nonsingular, each \( L_k \) and \( F_k \) is a strictly lower triangular matrix, each \( U_k \) is a zero-diagonal matrix. Then \( T_{MOR} \) method (5) converges for any starting vector \( x^0 \in \mathbb{R}^n \) provided that the parameters \( \omega, \alpha, \beta \) satisfy

\[
0 \leq \alpha, \beta \leq \omega, \ 0 < \omega < \frac{2}{1 + \rho(|D|^{-1}|B|)}.
\]

**Proof.** Since \( \rho(T_{MOR}(\omega, \alpha, \beta)) \leq \rho(|T_{MOR}(\omega, \alpha, \beta)|) \) by Lemma 2, where \( T_{MOR}(\omega, \alpha, \beta) \) is the iteration matrix given by (3), we only need to show that \( \rho(|T_{MOR}(\omega, \alpha, \beta)|) < 1 \).

As \( A \) is an \( H \)-matrix, \( D \) is a diagonal matrix, \( L_k \) and \( F_k \) are strictly lower triangular matrices, we easily see that \( D - \alpha L_k - \beta F_k \) are \( H \)-matrices for \( k = 1, 2, \ldots, K \). Using the result (2) of Lemma 1 and the definition of comparison matrix, we get

\[
|(D - \alpha L_k - \beta F_k)^{-1}| \leq \langle D - \alpha L_k - \beta F_k \rangle^{-1} = |D| - \alpha|L_k| - \beta|F_k|.
\]
First let the inequalities $0 \leq \alpha \leq \omega$, $0 \leq \beta \leq \omega$, $0 < \omega \leq 1$ hold. For $k = 1, 2, \cdots, K$, we define the matrices

$$M_k = |D| - \alpha|L_k| - \beta|F_k|,$$

and

$$N_k^1 = (1 - \omega)|D| + (\omega - \alpha)|L_k| + (\omega - \beta)|F_k| + \omega|U_k|.$$  

From (6), (7), we obtain

$$N_k^1 = M_k - \omega|D| - \omega|B| = M_k - \omega(|D| - |B|).$$  

We take absolute values of both sides of (3) and obtain

$$|T_{MTOR}(\omega, \alpha, \beta)| \leq \sum_{k=1}^{K} E_k M_k^{-1} N_k^1 I - \omega \sum_{k=1}^{K} E_k M_k^{-1} |D|(I - |D|^{-1}|B|).$$  

Let $\epsilon = [1, 1, \cdots, 1]^T \in R^a$. Since $|D|^{-1}|B|$ is nonnegative, the matrix $J_\epsilon = |D|^{-1}|B| + \epsilon \epsilon^T$ has only positive entries and is irreducible for any $\epsilon > 0$. By Lemma 3, we know that $\rho(J_\epsilon)$ is an eigenvalue of $J_\epsilon$ and the corresponding eigenvector $x_\epsilon \geq 0$ satisfying

$$J_\epsilon x_\epsilon = (|D|^{-1}|B| + \epsilon \epsilon^T)x_\epsilon = \rho(J_\epsilon)x_\epsilon.$$  

Moreover, since $0 < \omega \leq 1$, we have

$$1 - \omega + \omega \rho(|D|^{-1}|B|) < 1.$$  

By the continuity of the spectral radius, we also get

$$1 - \omega + \omega \rho(J_\epsilon) < 1$$  

if $\epsilon > 0$ is sufficient small.

By (9), we have

$$|T_{MTOR}(\omega, \alpha, \beta)| \leq I - \omega \sum_{k=1}^{K} E_k M_k^{-1} |D|[I - (|D|^{-1}|B| + \epsilon \epsilon^T)]$$

$$= I - \omega \sum_{k=1}^{K} E_k M_k^{-1} |D|(I - J_\epsilon)$$  

and by multiplying by $x_\epsilon$,

$$|T_{MTOR}(\omega, \alpha, \beta)|x_\epsilon \leq x_\epsilon - \omega \sum_{k=1}^{K} E_k M_k^{-1} |D|(1 - \rho(J_\epsilon))x_\epsilon.$$  

From the definition of $M_k$, the $M_k$ are $H$-matrices. By Lemma 1, we get

$$M_k \leq |D|, \ M_k^{-1} \geq |D|^{-1}.$$
By (10) and (12), we have

\[ |T_{MTOR}(\omega, \gamma, \tilde{\gamma})| x_\varepsilon \leq x_\varepsilon - \omega \sum_{k=1}^{K} E_k |D|^{-1} |D|(I - \rho(J_\varepsilon))x_\varepsilon \]

\[ = (1 - \omega + \omega \rho(J_\varepsilon))x_\varepsilon < x_\varepsilon. \] (13)

By exercise 2 of [8], p.48,

\[ \rho(|T_{MTOR}(\omega, \alpha, \beta)|) < 1 \]

holds.

Next let the inequalities \( 1 < \alpha \leq \omega, \ 1 < \beta \leq \omega, \ 1 < \omega < 2/(1 + \rho(|D|^{-1}|B|)) \) hold.

We define matrices

\[ N_k^2 = (\omega - 1)|D| + (\omega - \alpha)|L_k| + (\omega - \beta|F_k| + \omega|U_k|. \] (14)

From (6) and (14), then

\[ N_k^2 = M_k - [(2 - \omega)|D| - \omega|B|]. \] (15)

We take absolute values of both sides of (3) and have

\[ |T_{MTOR}(\omega, \gamma, \tilde{\gamma})| \leq \sum_{k=1}^{K} E_k M_k^{-1} N_k^2 \leq I - \sum_{k=1}^{K} E_k M_k^{-1} |D|[(2 - \omega)I - \omega|D|^{-1}|B|]. \] (16)

As in the previous proof, let \( e = [1, 1, \cdots, 1]^T \in \mathbb{R}^n \) and let \( x_\varepsilon > 0 \) denote the vector satisfying \( J_\varepsilon = (J + eee^T)x_\varepsilon = \rho(J_\varepsilon)x_\varepsilon \), where \( \varepsilon > 0 \) is sufficiently small such that \( \omega - 1 + \omega \rho(J_\varepsilon) < 1 \), since \( 1 < \omega < 2/(1 + \rho(|D|^{-1}|B|)) \).

From (16) we get

\[ |T_{MTOR}(\omega, \alpha, \beta)| \leq I - \sum_{k=1}^{K} E_k M_k^{-1} |D|[2 - \omega - \omega \rho(J_\varepsilon)] \]

and multiplying by \( x_\varepsilon \), then

\[ |T_{MTOR}(\omega, \alpha, \beta)| x_\varepsilon \leq x_\varepsilon - \sum_{k=1}^{K} E_k |D|^{-1} |D|[2 - \omega - \omega \rho(J_\varepsilon)]x_\varepsilon \]

\[ = x_\varepsilon - [2 - \omega - \omega \rho(J_\varepsilon)]x_\varepsilon = [\omega - 1 + \omega \rho(J_\varepsilon)]x_\varepsilon \]

\[ < x_\varepsilon. \]

Thus \( \rho(|T_{MTOR}(\omega, \alpha, \beta)|) < 1 \) follows again by exercise of [8], p.48.

Under the assumption of the theorem, this completes the proof.

Theorem 1 implies the following Corollaries

**Corollary 1** Under the conditions of Theorem 1, the MSOR method converges to the unique solution \( x^* \in \mathbb{R}^n \) of the system of weakly nonlinear equations (1) for any starting vector \( x^0 \in \mathbb{R}^n \) provided that the parameter \( \omega \) satisfies

\[ 0 < \omega < \frac{2}{1 + \rho(|D|^{-1}|B|)}. \] (18)
Corollary 2 Under the conditions of Theorem 1, the MAOR method converges to the unique solution \( x^* \in \mathbb{R}^n \) of the system of weakly nonlinear equations (1) for any starting vector \( x^0 \in \mathbb{R}^n \) provided that the parameter \( \omega \) satisfies
\[
0 \leq \gamma \leq \omega < \frac{2}{1 + \rho(|D|^{-1}|B|)}.
\] (19)

4 Block MTOR (BMTOR) Method

By splitting the number set \( \{1, 2, \cdots, n\} \) into \( K \) nonempty subset \( J_k \) \((k = 1, 2, \cdots, K)\), i.e.
\[
J_k \subset \{1, 2, \cdots, n\}, \quad \bigcup_{k=1}^{K} J_k = \{1, 2, \cdots, n\} \quad k = 1, 2, \cdots, K,
\]
we define the splitting matrices corresponding to the nonsingular matrix \( A \in \mathbb{R}^{n \times n} \) as follows:
\[
D = \text{diag}(A), \quad D \text{ is nonsingular}
\]
\[
L_k = (l^k_{ij}), \quad l^k_{ij} = \begin{cases} -a^k_{ij}, & 1 \leq j < \lfloor i/2 \rfloor, i, j \in J_k \\ 0, & \text{otherwise} \end{cases}
\] (20)
\[
F_k = (f^k_{ij}), \quad f^k_{ij} = \begin{cases} -a^k_{ij}, & [i/2] \leq j < i, i, j \in J_k \\ 0, & \text{otherwise} \end{cases}
\] (21)
\[
U_k = (u^k_{ij}), \quad u^k_{ij} = \begin{cases} 0, & j = i \\ -a^k_{ij} + f^k_{ij} + f^k_{ij}, & \text{otherwise} \end{cases}
\] (22)

with
\[
A = D - L_k - F_k - U_k, \quad k = 1, 2, \cdots, K.
\]

Here \([a]\) is used to denote the integer part of a positive number \(a\). The nonnegative diagonal matrices \( E_k \) \((k = 1, 2, \cdots, K)\) are introduced with \( e^k_i \geq 0 \) for \( i \in J_k \), \( e^k_i = 0 \) for \( i \notin J_k \), and \( \sum_{k=1}^{K} E_k = I \) (identity matrix).

With these matrices, a block multisplitting of the matrix \( A \) results and denoted by
\[
(D - L_k - F_k, U_k, E_k), \quad k = 1, 2, \cdots, K.
\]

Now we construct the block MTOR (BMTOR) method for solving the system of linear equations (1) as follows:

**BMTOR method**

For any starting vector \( x^0 \in \mathbb{R}^n \), for \( m = 0, 1, 2, \cdots \), until convergence
\[
x^{m+1} = \sum_{k=1}^{K} E_k x^{m,k}.
\]
where

\[ a_{ii}x_i^{m,k} - \alpha \sum_{1 \leq j \leq \lfloor i/2 \rfloor} t_{ij}x_j^{m,k} - \beta \sum_{\lfloor i/2 \rfloor < j \leq n} f_{ij}x_j^{m,k} \]

\[ = (1 - \omega)a_{ii}x_i^m + (\omega - \alpha) \sum_{1 \leq j \leq \lfloor i/2 \rfloor} t_{ij}x_j^m \]

\[ + (\omega - \beta) \sum_{\lfloor i/2 \rfloor < j \leq n} f_{ij}x_j^m + \omega \sum_{ij} u_{ij}x_j^m + \omega b_i, \ i \in J_k \]

\[ x_i^{m+1} = \sum_{k=1}^K c_{ij}^k x_i^{m,k}, \quad i = 1, 2, \ldots, n. \]

Here \( \alpha, \beta \geq 0 \) are relaxation factors and \( \omega > 0 \) is an acceleration parameter.

The BMTOR method is a block MTOR method for numerically solving the system of linear equation (1) in synchronous parallel environments. For different \( k \), the lower dimensional systems of equations (whose dimensions equal the number of elements included in the \( J_k \)) corresponding to the \( k \)-th splitting can be solved on the \( k \)-th processor of a multiprocessor system. A convergence theorem of the BMTOR method can be obtained in a similar way as for the MTOR method, so we will not demonstrate it here in detail.

**REFERENCES**


