Solution of Singular Boundary Element Equations based on Domain Splitting

Ke CHEN

(*Department of Mathematics, University of Liverpool, England.*)

1 Introduction

We consider the efficient solution of dense linear systems $A\underline{u} = \underline{f}$ by a preconditioned iterative method, where A is a $n \times n$ dense and unsymmetric matrix, and show that domain decomposition leads to a construction of practical preconditioners. Such linear systems arise from the solution of boundary element equations.

The boundary element methods (BEM), as a powerful alternative to finite element methods (FEM) and finite difference methods (FDM) for solving partial differential equations (PDE), are usually applied to PDE's with known free-space Green's functions; see [3]. However recent developments suggest that BEM can also be successfully applied to the solution of more general PDE's; see [11].

As far as the iterative solution is concerned, its success largely depends on the spectral properties of the integral operator or of the matrices of discrete linear systems. More precisely, if the underlying operator is smooth and compact, iterative methods can be very efficient without preconditioning; see [1], [4] and [5]. This paper addresses the case of a non-compact operator, where preconditioning is essential for iterative methods.

In the literature, work on preconditioning singular boundary element equations has mostly been based on 'algebraic considerations'. The underlying ideas fall into two main categories, a) design a preconditioner that can be inverted or solved easily, and contains or somehow represents the dominant part of contributions due to singular integrals; b) design a preconditioner that is sparse and somehow 'close' to the inverse of the coefficient matrix A. Essentially efficiency is the chief consideration and naturally the preconditioners suggested are often sparse; refer to [2], [6], [7], [9] and [10] among others.

In this paper, we propose to construct preconditioners for the singular integral

---

Domain Decomposition Methods in Sciences and Engineering, edited by R. Glowinski *et al.*

operator equation first, based on boundary domain splitting, and then proceed with a construction of preconditioners for the dense linear systems. Using this new and general framework, we can show that several preconditioners in the literature are related. Our theory provides a justification for these preconditioners and points out a way toward new designs and further modifications. Some numerical results are reported.

2 Boundary elements and dense linear systems

Let $\Omega \in R^2$ denote a closed domain that may be interior and bounded, or exterior and unbounded, and $\Gamma = \partial\Omega$ be its (finite part) boundary that can be parameterised by $p = (x, y) = (x(s), y(s)), a \le s \le b$. Then a boundary integral equation that usually arises from reformulating a PDE in Ω can be written as

$$u(p) - \int_\Gamma \bar{k}(p,q)u(q)dS_q = f(p), \qquad p \in \Gamma, \tag{1}$$

or
$$u(s) - \int_a^b k(s,t)u(t)dt = f(s), \qquad s \in [a,b], \tag{2}$$

or simply
$$(I - \mathcal{K})u = f. \tag{3}$$

To solve the above equation numerically, we divide the boundary Γ (interval $[a, b]$) into m boundary elements (non-intersecting subintervals $E_i = [s_{i-1}, s_i]$). On each interval E_i, we may either approximate the unknown u by an interpolating polynomial of order r that leads to a collocation method, or apply a quadrature method of r nodes that gives rise to the Nyström method. Both discretization methods approximate equation (3) by

$$(I - \mathcal{K}_n)u_n = f, \tag{4}$$

where we can write

$$\mathcal{K}_n u = \mathcal{K}_n u_n = \sum_{j=1}^{m}\left[\sum_{i=1}^{r} w_i k(s, t_{ji})u_{ji}\right], \qquad u_n(t_{ji}) = u(t_{ji}) = u_{ji}, \quad \text{and} \quad n = mr.$$

We use vector $\underline{u}$ to denote u_{ji}'s at all nodes. By a collocation step in equation (4), we obtain a linear system of equations

$$(I - K)\underline{u} = \underline{f}, \qquad \text{or} \qquad A\underline{u} = \underline{f}, \tag{5}$$

where matrices K and A are dense and unsymmetric (in general). The conditioning of A depends on the smoothness of kernel function $k(s,t)$. A strong singularity (as $t \to s$) leads to non-compactness of operator $\mathcal{K}$ and renders equation (5) difficult to solve by iterative methods without preconditioning.

To gain some insight into preconditioning, we may describe a BEM procedure as an interaction of three stages, each respectively characterized by

(1). Integral Operator $\mathcal{K}$
(2). Approximating Integrals $(\mathcal{K}_n u_n)(s)$
(3). Matrix Elements of K

Then stages $1 \to 2 \to 3$ represent the well-known (standard) BEM procedure. The simple fact is that any undesirable spectral properties of K (stage 3) must stem from the integral operator $\mathcal{K}$ (stage 1), although we need preconditioners for A (in stage 3) and we see that singularities tend to dominate integrals (in stage 2). Therefore we propose to exploit stages $3 \to 2 \to 1$ in search of a better preconditioned problem to work with or a general theoretical framework.

We remark that most preconditioners in the literature exploit stages $3 \to 2$ only.

3 Preconditioning techniques

We shall review three important preconditioners for solving $A\underline{u} = \underline{f}$ and illustrate each case by a 9×9 system.

3.1 Two grid based sparse column preconditioners

The success of this preconditioner, due to Yan [10], follows from the facts that a $n \times n$ sparse column matrix B with m nonzero long columns has its inverse of the identical sparsity and a system such as $B\underline{x} = \underline{y}$ for $x, y \in R^n$ can be solved in only $(n-m)m$ operations. For example, with $n = 9$,

$$B = \begin{pmatrix} \times & & \times & & & \times & & & \times \\ & \times & \times & & & \times & & & \times \\ & & \times & & & \times & & & \times \\ & & \times & \times & & \times & & & \times \\ & & \times & & \times & \times & & & \times \\ & & \times & & & \times & & & \times \\ & & \times & & & \times & \times & & \times \\ & & \times & & & \times & & \times & \times \\ & & \times & & & \times & & & \times \end{pmatrix}.$$

Formally, let $n = \eta m$ with η, m integers. Then from matrix $K = (k_1, k_2, \cdots, k_n)$, construct column vectors by

$$k_i' = \begin{cases} k_i, & \text{if } i = \ell\eta, \quad 1 \le \ell \le m, \\ 0, & \text{otherwise}, \end{cases}$$

and define a new matrix by $K' = (k_1', k_2', \cdots, k_n')$. Then we use $A' = (I + \eta K')$ as a preconditioner. It may be expected that A' is 'close' to $A = A_n$ because $A' \approx A_m$ which is the corresponding discrete matrix with m nodes.

3.2 Mesh neighbour based approximate inverses

The starting point in the mesh neighbour preconditioner of Vavasis [9] is that we hope to find a matrix such that $PA \approx I$ and this P should have its diagonal elements and immediate neighbours possess more importance as A usually comes from singular integral equations. In particular, assume that P is a quasi-tridiagonal matrix; for $n = 9$

we have

$$P = \begin{pmatrix} \times & \times & & & & & & & \times \\ \times & \times & \times & & & & & & \\ & \times & \times & \times & & & & & \\ & & \times & \times & \times & & & & \\ & & & \times & \times & \times & & & \\ & & & & \times & \times & \times & & \\ & & & & & \times & \times & \times & \\ & & & & & & \times & \times & \times \\ \times & & & & & & & \times & \times \end{pmatrix}. \tag{6}$$

Then to find $P^T = (p_1, p_2, \cdots, p_n)$ in terms of $A = (a_1, a_2, \cdots, a_n)$, a direct approach is used. Let $PA = I \iff A^T P^T = I \iff A^T p_i = e_i$. Then, as p_i only has three nonzero positions $p_{i_1}, p_{i_2}, p_{i_3}$, an approximate solution is obtained by solving a 3×3 system

$$\begin{pmatrix} \times & \times & \times \\ \times & \times & \times \\ \times & \times & \times \end{pmatrix} \begin{pmatrix} p_{i_1} \\ p_{i_2} \\ p_{i_3} \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}.$$

3.3 Sparse LU decompositions

Similar to the idea of §3.2, the work of [2] assumes that preconditioner B should have its diagonal elements and immediate neighbours possess more importance for the same reason that A usually comes from singular integral equations. When $n = 9$, such a matrix B is of the same sparsity as P in equation (6).

As B^{-1} cannot maintain any sparsity property of B, it is proposed, in order to solve $B\underline{x} = \underline{y}$ for $x, y \in R^n$, to decompose B as $B = LU$, where L and U are sparse triangular matrices. To illustrate, for $n = 9$, we have

$$B = LU = \begin{pmatrix} P_{1,1} & & & & & & & & \\ P_{2,1} & P_{2,2} & & & & & & & \\ & P_{3,2} & P_{3,3} & & & & & & \\ & & P_{4,3} & P_{4,4} & & & & & \\ & & & P_{5,4} & P_{5,5} & & & & \\ & & & & P_{6,5} & P_{6,6} & & & \\ & & & & & P_{7,6} & P_{7,7} & & \\ & & & & & & P_{8,7} & P_{8,8} & \\ P_{9,1} & P_{9,2} & P_{9,3} & P_{9,4} & P_{9,5} & P_{9,6} & P_{9,7} & P_{9,8} & P_{9,9} \end{pmatrix}$$
$$\times \begin{pmatrix} 1 & P_{1,2} & & & & & & & P_{1,9} \\ & 1 & P_{2,3} & & & & & & P_{2,9} \\ & & 1 & P_{3,4} & & & & & P_{3,9} \\ & & & 1 & P_{4,5} & & & & P_{4,9} \\ & & & & 1 & P_{5,6} & & & P_{5,9} \\ & & & & & 1 & P_{6,7} & & P_{6,9} \\ & & & & & & 1 & P_{7,8} & P_{7,9} \\ & & & & & & & 1 & P_{8,9} \\ & & & & & & & & 1 \end{pmatrix}.$$

A general algorithm is given as follows **(Calculation of LU)**

$$
\begin{array}{|l|}
\hline
P_{1,1} = B_{1,1}, \quad P_{2,1} = B_{2,1}, \quad P_{n,1} = B_{n,1}, \\
P_{1,2} = B_{1,2}/B_{1,1}, \quad P_{1,n} = B_{1,n}/B_{1,1}, \quad P_{2,2} = B_{2,2} - B_{2,1}P_{1,2}, \quad P_{n,2} = -B_{n,1}P_{1,2}, \\
s = B_{n,1}P_{1,n} \\
\hline
\text{FOR } i = 2 : n-2 \\
\quad \begin{array}{|ll|}
\hline
P_{i,i+1} = B_{i,i+1}/P_{i,i}, & P_{i,n} = -B_{i,i-1}P_{i-1,n}/P_{i,i} \\
P_{i+1,i+1} = B_{i+1,i+1} - B_{i+1,i}P_{i,i+1}, & P_{n,i+1} = -P_{n,i}P_{i,i+1} \\
s = s + P_{n,i}P_{i,n} & \\
\hline
\end{array} \\
\text{END} \\
P_{n,n-1} = P_{n,n-1} + B_{n,n-1}, \quad P_{n-1,n} = (B_{n-1,n} - B_{n-1,n-2}P_{n-2,n})/P_{n-1,n-1}, \\
P_{n,n} = B_{n,n} - P_{n,n-1}P_{n-1,n} - s. \\
\hline
\end{array}
$$

A related algorithm, needed at a preconditioning step, for solving a system $P\underline{x} = \underline{y}$ is

Forward substitution :

$$
\begin{array}{|lll|}
\hline
 & z_1 = y_1/P_{1,1}, & s = P_{n,1}, \\
\text{FOR } i = 2 : n-1 & & \\
 & z_i = (y_i - P_{i,i-1}z_{i-1})/P_{i,i}, & s = s + P_{n,i}z_i \\
\text{END} & & \\
 & z_n = (y_n - s)/P_{n,n}. & \\
\hline
\end{array}
$$

Backward substitution :

$$
\begin{array}{|ll|}
\hline
x_n = z_n, & x_{n-1} = z_{n-1} - P_{n-1,n}x_n, \\
\text{FOR } i = n-2 : -1 : 1 & \\
 & x_i = z_i - P_{i,i+1}x_{i+1} - P_{i,n}x_n, \\
\text{END} & \\
\hline
\end{array}
$$

In this paper, one of our aims has been to offer a new understanding of the above otherwise heuristic sparse preconditioners appeared in the literature. This is to be achieved by use of operator splittings or domain decomposition.

4 Operator and domain splitting

Here we introduce the idea of operator splitting that was originated in [7]. Use the partition of §2, $[a,b] = \bigcup_{i=1}^{m} E_i$. Accordingly we can partition variable u and vector $\underline{u}$ as follows $u = (u_1, u_2, \cdots, u_m)^T$ and $\underline{u} = (\underline{u}_1, \underline{u}_2, \cdots, \underline{u}_m)^T$.

Similarly operator $\mathcal{K}$ is partitioned into a matrix form and further we can observe that all singularities of $\mathcal{K}$ are contained in the following operator

$$
\bar{\mathcal{K}} = \begin{pmatrix}
\mathcal{K}_{1,1} & \mathcal{K}_{1,2} & & & \mathcal{K}_{1,m} \\
\mathcal{K}_{2,1} & \mathcal{K}_{2,2} & \mathcal{K}_{2,3} & & \\
 & \mathcal{K}_{3,2} & \ddots & \ddots & \\
 & & \ddots & \ddots & \mathcal{K}_{m-1,m} \\
\mathcal{K}_{m,1} & & & \mathcal{K}_{m,m-1} & \mathcal{K}_{m,m}
\end{pmatrix}.
$$

The corresponding matrix out of K is

$$\bar{K} = \begin{pmatrix} K_{1,1} & K_{1,2} & & & K_{1,m} \\ K_{2,1} & K_{2,2} & K_{2,3} & & \\ & K_{3,2} & \ddots & \ddots & \\ & & \ddots & \ddots & K_{m-1,m} \\ K_{m,1} & & & K_{m,m-1} & K_{m,m} \end{pmatrix}.$$

Define operators $\mathcal{D} = I - \bar{\mathcal{K}}$ and $\mathcal{C} = \mathcal{K} - \bar{\mathcal{K}}$, and matrices $D = I - \bar{K}$ and $C = K - \bar{K}$. Then it can be shown that operator $\mathcal{D}$ is bounded and $\mathcal{C}$ is compact. So operator $\mathcal{D}^{-1}\mathcal{C}$ is also compact. Thus the solution of $A\underline{u} = \underline{f}$ is reduced to that of $[I - D^{-1}C]\underline{u} = D^{-1}\underline{f}$. Here D is in general a block quasi-tridiagonal matrix and the solution of $D\underline{x} = \underline{y}$ should similarly follow §3.3.

While it is now natural to use D as a preconditioner, we may conclude that an efficient preconditioner should contain D or its close approximation. Following [7], we may further show that $D_1 = diag(D)$ is also an efficient preconditioner.

5 Interpretation of preconditioners

We now use the theory of the above section to identify the operator splittings implied in the preconditioners of §3.

Firstly we see that the preconditioner of [10] uses the following splitting $\mathcal{K} = \mathcal{D} + \mathcal{C}$, where $\mathcal{D} = I - \tilde{\mathcal{K}}$ with $\tilde{\mathcal{K}} = (\tilde{\mathcal{K}}_{ij})$ and $\tilde{\mathcal{K}}_{ij} = \eta \int_{E_j/\eta} k(s,t)u_j(t)dt \approx \int_{E_j} k(s,t)u_j(t)dt = \mathcal{K}_{ij}$. Theoretically $\mathcal{C}$ is not compact but $\mathcal{C}$ should approach 0 if $m, n \to \infty$. Practically η should not be too large (then the method becomes more expensive if $m \approx n$).

Secondly, for the preconditioners of [9] and [2], the underlying splittings would be identical to that in the last section for piecewise constant approximations or the panel method (mid-point rule). The reason is that both were proposed based on assumptions on stage 2 (regarding singular integrals rather than operators).

6 Numerical results

The first problem to be tested has a weak singularity (see [10])

$$\text{Problem 1 :} \qquad u(s) + \gamma \int_{-\pi}^{\pi} u(s)b(s,t)dt = f(s), \qquad s \in [-\pi, \pi]$$

which arises from the solution of the exterior Neumann's problem of the Laplace equation (when $\gamma = 1$) over an elliptic boundary $p(s) = (\cos(s), sin(s)/4)$. Here the kernel function is

$$b(s,t) = \frac{(p(t) - p(s)) \cdot n(p(t))|p'(t)|}{\pi |p(t) - p(s)|^2} = \frac{4}{\pi(17 - 15\cos(t+s))}.$$

The parameter γ is included to vary the difficulty of the problem. We specifically choose $f(s) = |\sin(s)| + 2\gamma\{4\cos(s)\log[(17 + 15\cos(s))/(17 - 15\cos(s))] + 17\sin(s)\tan^{-1}(15\sin(s)/8)\}/(15\pi)$ so that $u(s) = |\sin(s)|$.

Table 1 Convergence results of Problem 1 ($\gamma = 10$)

Method	Diagonal [7]	Inverse [9]	LU [2]	Mod LU [2]	Two grid [10]	Mod TG [10]
N = 16	6	6	23	26	7	13
N = 32	12	10	10	38	17	31
N = 64	17	15	10	45	34	65
N =128	18	19	13	45	41	80
N =256	19	20	13	41	50	87
N =512	18	20	14	40	53	89
N =1024	18	19	16	39	55	92

Table 2 Convergence results of Problem 2

Method	Diagonal [7]	Inverse [9]	LU [2]	Mod LU [2]	Two grid [10]	Mod TG [10]
N = 16	9	14	10	12	25	16
N = 32	10	18	19	16	50	22
N = 64	11	20	36	26	98	30
N =128	11	23	71	36	194	41
N =256	13	26	144	48	386	56
N =512	14	27	291	57	770	75
N =1024	14	30	599	62	1538	93

The second problem possesses a Cauchy singularity (see [7])

$$\text{Problem 2 :} \qquad \begin{cases} \frac{1}{\pi}\int_{-1}^{1} \frac{w(t)\phi(t)}{t-x}dt + \int_{-1}^{1} \frac{(t^2-x^2)^2}{t^2+x^2} w(t)\phi(t)dt = f(x), \\ \frac{1}{\pi}\int_{-1}^{1} w(t)\phi(t)dt = 0, \qquad\qquad x \in (-1,1), \end{cases}$$

which has the exact solution $\phi(x) = x|x|$.

Both problems are discretized by the Nyström method using uniform nodes for Problem 1 and Chebyshev nodes for Problem 2. The conjugate gradient iterative method to the normal equation (CGN) is adopted; see [8]. The tolerance for residual errors is set to be TOL $= 10^{-J}$ where $J = 1 + \log(N)/\log(2)$. Results are shown in Tables 1–2 of comparisons of six preconditioners, where 'Mod' means a modified version and the modifications are based on discussions of last two sections.

Although further and more extensive tests are needed to compare these preconditioners, our preliminary conclusion is that for singular boundary element equations, the most robust preconditioner is that based on boundary domain decomposition [7].

Acknowledgement

The support for this research work from the Nuffield Foundation (UK) is gratefully acknowledged.

References

Amini S. and Chen K. (1989) Conjugate gradient method for second kind integral equations – applications to the exterior acoustic problem. *Engineering Analysis with Boundary Elements*, **6**(2), 72–77.

Amini S. and Maines N. (1994) Iterative solutions of boundary integral-equations. In : *Proceedings of the 14th conference on Boundary Element Methods*, ed. C. A. Brebbia, Ch.65, 193–200.

Brebbia C. A. et al (1984) *Boundary element techniques – theory and applications*, Springer-Verlag.

Hemker P. W. and Schippers H. (1981) Multigrid methods for the solution of Fredholm integral equations of the second kind. *Math. Comp.*, **36**, 215–232.

Hackbusch W. (1985) *Multigrid methods and applications.* Springer-Verlag.

Canning F. (1992) Sparse approximation for solving integral equations with oscillatory kernels. *SIAM J. Sci. Stat. Comp.*, **13**, 71–87.

Chen K. (1994) Efficient iterative solution of linear systems from discretizing singular integral equations. *Elec. Tran. Numer. Anal.*, **2**, 76–91.

Nachtigal N. M. et al (1992) How fast are nonsymmetric matrix iterations?. *SIAM J. Matr. Anal. Appl.*, **13**(3), 778–795.

Vavasis S. (1992) Preconditioning for boundary integral equations. *SIAM J. Matr. Anal. Appl.*, **13**(3), 905–925.

Yan Y. (1994) Sparse preconditioned iterative methods for dense linear systems. *SIAM J. Sci. Comp.*, **15**(5), 1190–1200.

Partridge P.W., Brebbia C. A. and Wrobel L. C. (1992) *The dual reciprocity boundary element method.* CMP Southampton, UK.