Domain Decomposition Methods and Multilevel Preconditioners for Nonconforming and Mixed Methods for Partial Differential Problems

ZHANGXIN CHEN AND RICHARD E. EWING

ABSTRACT

In this paper we describe recent development of domain decomposition methods and multilevel preconditioners for solution of the discrete systems which arises from the application of nonconforming and mixed finite element methods to partial differential problems.

1. INTRODUCTION

A lot of numerical methods for solving partial differential problems have been developed in past years. Among the most popular and often used methods are Galerkin finite element methods, finite volume methods, and mixed finite element methods. The mixed methods have certain advantages over the Galerkin methods and the finite volume methods [3]. In particular, they preserve mass element by element, and produce a direct and accurate approximation for the vector unknown of the differential problems, which is the variable of primary interest in many applications such as the velocity field in the flow equation in highly heterogeneous media [16] and the electric field in the potential equation in semiconductor devices [5, 7, 8].

Due to their saddle point property, it is known that the linear systems arising from the mixed methods are generally harder to solve than those arising from comparable Galerkin methods. In particular, there has been little theory for constructing good preconditioners and developing efficient domain decomposition methods for solving the systems of algebraic equations arising from the mixed methods. An alternate approach was suggested by means of a nonmixed formulation. Namely,
it has been shown that the mixed finite element methods are equivalent to a modification of nonconforming Galerkin methods [1, 2, 4, 18]. The nonconforming methods yield a symmetric and positive definite problem, which can be more easily solved using domain decomposition methods.

The purpose of this paper is to describe recent development of domain decomposition methods and multilevel preconditioners for solution of the discrete systems which arises from the application of nonconforming finite element methods to partial differential problems. Then, based on the equivalence mentioned above between the mixed and nonconforming methods, we discuss the domain decomposition methods and multilevel preconditioners for the mixed methods for the partial differential problems. In the next section we consider the domain decomposition methods, and then in the third section we present the multilevel preconditioners. In the final section we discuss the equivalence between the mixed and nonconforming finite element methods.

2. DOMAIN DECOMPOSITION METHODS

Let $\Omega$ be a bounded domain in $\mathbb{R}^d$, and let $N_h$ be a nonconforming finite element space associated with a triangulation $\mathcal{T}_h$ of $\Omega$, where $h$ is the discretization parameter. We are concerned with the approximate problem: Find $u_h \in N_h$ such that

\begin{equation}
(2.1) \quad a_h(u_h, v) = (f, v), \quad \forall v \in N_h,
\end{equation}

where the bilinear form $a_h(\cdot, \cdot)$ is assumed to be symmetric and positive definite, $(\cdot, \cdot)$ denotes the $L^2(\Omega)$ inner product (for simplicity), and $f \in L^2(\Omega)$. It is well known that the linear system arising from (2.1) is not well conditioned. The aim of this section is to develop an additive Schwarz algorithm for (2.1). For this, let \{\Omega_j\}_{j=1}^J be an overlapping domain decomposition of $\Omega$ with the overlap parameter $\delta$. The decomposition is assumed to align with the boundary $\partial \Omega$. Associated with each $\Omega_j$, let $N^j_h$ be a nonconforming finite element space of the same form as $N_h$, whose elements have support in $\Omega_j$. The finite element space $N_h$ is assumed to be represented as a sum of $J + 1$ subspaces:

\begin{equation}
(2.2) \quad N_h = N^0_h + N^1_h + \ldots + N^J_h,
\end{equation}

where $N^0_h$ is a so-called coarse space. We now define the operators $\Pi_j : N_h \to N^j_h$, $j = 0, 1, \ldots, J$, by

\begin{equation}
(2.3) \quad a_h(\Pi_j v, w) = a_h(v, w), \quad \forall w \in N^j_h,
\end{equation}

and the operator $\Pi : N_h \to N_h$ by

\begin{equation}
(2.4) \quad \Pi = \sum_{j=0}^J \Pi_j.
\end{equation}

In (2.3) we could use so-called inexact solvers; the analysis is the same. We now define a Schwarz algorithm for (2.1).

**Additive algorithm.** The additive Schwarz algorithm for (2.1) is given by

\begin{equation}
(2.5) \quad \Pi u_h = f_h, \quad f_h = \sum_{j=0}^J f_j,
\end{equation}
where $f_j \in N^j_h$ satisfies
\[ a_h(f_j, v) = (f_j, v), \quad \forall v \in N^j_h, \quad j = 0, 1, \ldots, J. \]

Note that (2.1) and (2.5) have the same solution. The following abstract convergence result for bounds on the condition number of $\Pi$ can be found in [17]. Assume that

(A1) There is a constant $C$ such that every $v \in N_h$ can be represented by $v = \sum_{j=0}^{J} v_j$ with $v_j \in N^j_h$ satisfying
\[ \sum_{j=0}^{J} a_h(v_j, v_j) \leq C a_h(v, v). \]

(A2) Let $\kappa = (\kappa_{ij})$ be a symmetric matrix with $\kappa_{ij} \geq 0$ satisfying
\[ |a_h(v_i, v_j)| \leq \kappa_{ij} a_h(v_i, v_i)^{1/2} a_h(v_j, v_j)^{1/2}, \quad \forall v_i \in N^i_h, \quad v_j \in N^j_h, \quad i, j = 1, \ldots, J. \]

Then we have
\[ \lambda_{\min}(\Pi) \geq C^{-1} \quad \text{and} \quad \lambda_{\max}(\Pi) \leq \rho(\kappa) + 1, \]
where $\rho(\kappa)$ is the spectral radius of $\kappa$.

We now apply the above theory to analyze a couple of examples.

**Example 2.1.** Let $N_h$ be the $P_1$ nonconforming finite element space of the second order elliptic problem in $\mathbb{R}^2$. That is,
\[ N_h = \{ v \in L^2(\Omega) : v|_E \in P_1(E), \quad \forall E \in \mathcal{E}_h; \quad v \text{ is continuous at the midpoints of interior sides and vanishes at the midpoints of sides on } \partial \Omega \}, \]
where $\mathcal{E}_h$ is a partition of $\Omega$ into triangles. We need to assume a structure to our family of partitions. In the first step, let $\mathcal{E}_H$ be a coarse triangulation of $\Omega$ into nonoverlapping triangular substructures $\Omega_j^h$, $j = 1, \ldots, J$. Then, in the second step we refine $\mathcal{E}_H$ into triangles to have a triangulation $\mathcal{E}_h$, $h < H$. Finally, let $\{ \Omega_j \}_{j=1}^{J}$ be an overlapping domain decomposition of $\Omega$ by extending $\Omega_j^h$ with the overlap parameter $\delta$ defined by
\[ \delta = \min \{ \text{dist}(\partial \Omega_j \setminus \partial \Omega, \partial \Omega_j^h \setminus \partial \Omega), j = 1, \ldots, J \}. \]

Finally, define the coarse space
\[ n_h^0 = \{ v \in N_h : v = R_h \varphi, \quad \varphi \in U_H \}, \]
where $R_h$ is the nodal interpolation operator into $N_h$ and $U_H$ is the $P_1$-conforming space associated with $\mathcal{E}_H$. With these choices, it follows from the above abstract result [15] that the condition number $c(\Pi)$ of $\Pi$ can be estimated by
\[ c(\Pi) = O(1 + H/\delta). \]

**Example 2.2.** Let $N_h$ be the rectangular nonconforming finite element space of the second order elliptic problem in $\mathbb{R}^2$. Namely,
\[ N_h = \left\{ \xi : \xi|_E = a_E^x + a_E^y x + a_E x y + a_E^4 (x^2 - y^2), \quad a_E \in \mathbb{R}, \quad \forall E \in \mathcal{E}_h; \right. \]
\[ \left. \text{if } E_1 \text{ and } E_2 \text{ share an edge } e, \text{ then } \int_e \xi|_{\partial E_1} ds = \int_e \xi|_{\partial E_2} ds; \right. \]
\[ \text{and } \int_{\partial E \cap \partial \Omega} \xi|_{\partial \Omega} ds = 0, \right\}, \]
where $\mathcal{E}_h$ is a partition of $\Omega$ into rectangles. The overlapping domain decomposition $\{\Omega_j\}_{j=1}^J$ and spaces $\{N_h^j\}_{j=1}^J$ can be similarly defined as in Example 2.1. Finally, the coarse space is defined as follows. Let $\hat{\mathcal{E}}_h$ be the triangulation of $\Omega$ into triangles obtained by connecting the two opposite vertices of the rectangles in $\mathcal{E}_h$. Associated with $\hat{\mathcal{E}}_h$, let $\hat{N}_h$ be the $P_1$ nonconforming finite element space in Example 2.1. Then we define the operator $\hat{T}_h : N_h \rightarrow \hat{N}_h$ by the relation: If $v \in N_h$ and $e$ is an edge of a triangle in $\hat{N}_h$, then $\hat{T}_hv \in \hat{N}_h$ is defined by

$$\frac{1}{|e|} \langle \hat{T}_hv, 1 \rangle_e = \frac{1}{|e|} \langle v, 1 \rangle_e.$$  

Now we have

$$N_h^0 = \{ v \in N_h : v = R_h \varphi, \varphi \in \hat{N}_h^0 \},$$  

where $\hat{N}_h^0$ is given as in (2.6), and $R_h$ is the interpolation operator into $N_h$. The result (2.7) can be shown here by means of the operator $\hat{T}_h$ introduced above.

Other choices for $N_h^0$ in Examples 2.1 and 2.2 can be made [12]. It follows from (2.7) that if we use a generous overlapping, then the condition number of $\Pi$ is uniformly bounded. With the same technique, two-level multiplicative algorithms and multilevel Schwarz algorithms can be developed [12, 21]. Moreover, exploiting the equivalence between the nonconforming and mixed methods [6, 10, 12, 13] (also see the fourth section), the Schwarz algorithms can be applied to the mixed methods [3]. Fourth order problems can be also analyzed using the present theory.

3. MULTILEVEL PRECONDITIONERS

Let

$$N_0 \rightarrow N_1 \rightarrow \cdots \rightarrow N_J$$

be a sequence of nonconforming finite element spaces associated with a increasing sequence of triangulations of $\Omega$ such that there is an intergrid transfer operator between two adjacent spaces:

$$I_j : N_{j-1} \rightarrow N_j, \quad j = 1, \ldots, J.$$  

We are again concerned with the approximate problem: Find $u_J \in N_J$ such that

$$a_J(u_J, v) = (f, v), \quad \forall v \in N_J,$$

where $a_J(\cdot, \cdot)$ is assumed to be equivalent to a discrete energy scalar product $(\cdot, \cdot)_E$ on $N_J$:

$$a_J(v, v) \approx ||v||_E^2, \quad \forall v \in N_J.$$  

We define the operators $R_j : N_J \rightarrow N_j$ by

$$R_j = I_j I_{j-1} \cdots I_{j+1}, \quad j = 0, \ldots, J - 1.$$  

Let $\{b_j(\cdot, \cdot)\}_{j=0}^J$ be a family of bilinear forms defined by

$$b_j(v, w) = \alpha_j(v, w), \quad \forall v, w \in N_j,$$

where the positive constants $\alpha_j$ are determined by the (inverse) inequalities

$$||v||_E^2 \leq \alpha_j ||v||^2, \quad \forall v \in N_j, \quad j = 0, \ldots, J.$$  

Finally, introduce the operators $T_j^j : N_J \rightarrow N_j$ by

$$b_j(T_j^j v, w) = a_j(v, R_j w), \quad \forall v \in N_j,$$
and the elements \( f_j \in N_j \) by
\[
 b_j(f_j, w) = (f_j, R_j w), \quad \forall w \in N_j.
\]
Now it can be seen that the problem (3.1) is equivalent to
\[
\mathcal{P}_J u_J = f_J,
\]
where
\[
\mathcal{P}_j = \sum_{j=0}^{J} R_j T_j, \quad f_j = \sum_{j=0}^{J} R_j f_j.
\]
Assume that (3.2) and the following condition are satisfied:
\[
 b_j(v - I_j P_{j-1} v, v - I_j P_{j-1} v) \leq C_J \|v - I_j P_{j-1} v\|_E^2, \quad \forall v \in N_j,
\]
where the operators \( P_{j-1} : N_j \rightarrow N_{j-1} \) are given by
\[
(P_{j-1} v, w)_E = (v, w)_E, \quad \forall v \in N_{j-1}, j = 1, \ldots, J.
\]
Then we have the following abstract result on the condition number of \( \mathcal{P}_j [14, 19] \):
\[
C_* \max_{j=0, \ldots, J} \tau_j \leq c(\mathcal{P}_J) \leq C^* C_j \sum_{j=0}^{J} \tau_j,
\]
where the constants \( C_* \) and \( C^* \) are independent of \( j \) and \( \tau_j = \sup\{||R_j v||_E^2 / ||v||_E^2, v \in N_j, v \neq 0\} \).

We now again consider the two examples presented in §2. Let \( h_0 \) and \( \mathcal{E}_{h_0} = \mathcal{E}_0 \) be given. For each integer \( 0 < j < J \), let \( h_j = 2^{-j} h_0 \) and \( \mathcal{E}_{h_j} = \mathcal{E}_j \) be constructed by connecting the midpoints of the edges of the triangle or rectangle in \( \mathcal{E}_{j-1} \). Associated with each \( \mathcal{E}_j \), let \( N_j \) be the nonconforming space defined as in Example 2.1 or Example 2.2. The intergrid transfer operators \( I_j : N_{j-1} \rightarrow N_j \) are defined in the usual way by nodal or edge value averaging procedures [6]. Then, using the result (3.4), we have for both examples:
\[
c(\mathcal{P}_J) = O(J).
\]

Finally, thanks to the equivalence between the nonconforming and mixed finite element methods (see the next section), the theory presented in this section applies to the mixed methods [11], where algebraic multilevel preconditioners for both nonconforming and mixed methods are also constructed. Fourth order differential problems can be also analyzed using the present theory. Due to the page limitation, numerical results are not shown here; please refer to [11, 12, 14, 19] for the numerical results on the domain decomposition methods and multilevel preconditioners.

### 4. MIXED FINITE ELEMENT METHODS

We concentrate on the model problem
\[
-\nabla \cdot (a \nabla u) = f \quad \text{in } \Omega,
\]
\[
u = 0 \quad \text{on } \partial \Omega.
\]
The Raviart-Thomas space [20] over triangles is given by
\[ \Lambda_h = \{ v \in (L^2(\Omega))^2 : v|_E = (a_1^E + a_2^E x, a_3^E + a_4^E y), \ a_i^E \in \mathbb{R}, \ E \in \mathcal{E}_h \}, \]
\[ W_h = \{ w \in L^2(\Omega) : w|_E \text{ is constant for all } E \in \mathcal{E}_h \}, \]
\[ L_h = \{ \mu \in L^2(\partial\mathcal{E}_h) : \mu|_e \text{ is constant, } e \in \partial\mathcal{E}_h; \ \mu|_e = 0, \ e \subset \partial\Omega \}, \]
where \( \partial\mathcal{E}_h \) denotes the set of all interior edges. Then the hybrid form of the mixed method for (4.1) is to seek \((\sigma_h, u_h, \lambda_h) \in \Lambda_h \times W_h \times L_h \) such that
\[
\begin{align*}
\sum_{E \in \mathcal{E}_h} (\nabla \cdot \sigma_h, w)_E &= (f, w), \quad \forall w \in W_h, \\
(\sigma_h, v) - \sum_{E \in \mathcal{E}_h} ([u_h, \nabla \cdot v)_E - (\lambda_h, v \cdot \nu_E)_\partial E] &= 0, \quad \forall v \in \Lambda_h, \\
\sum_{E \in \mathcal{E}_h} (\sigma_h, \nu_E, \mu)_{\partial E} &= 0, \quad \forall \mu \in L_h,
\end{align*}
\]
where \( \nu_E \) denotes the unit outer normal to \( E \) and \( \alpha = a^{-1} \). The solution \( \sigma_h \) is introduced to approximate the vector field
\[ \sigma = -a \nabla u, \]
which is the variable of primary interest in many applications. Since \( \sigma \) lies in the space
\[ H(\text{div}; \Omega) = \{ v \in (L^2(\Omega))^2 : \nabla \cdot v \in L^2(\Omega) \}, \]
and we do not require that \( \Lambda_h \) be a subspace of \( H(\text{div}; \Omega) \), the last equation in (4.2) is used to enforce that the normal components of \( \sigma_h \) are continuous across the interior edges in \( \partial\mathcal{E}_h \), so in fact \( \sigma_h \in H(\text{div}; \Omega) \).

There is no continuity requirement on the spaces \( \Lambda_h \) and \( W_h \), so \( \sigma_h \) and \( u_h \) can be locally (element-by-element) eliminated from (4.2). In fact, from [6], (4.2) can be algebraically condensed to the symmetric, positive definite system for the Lagrange multiplier \( \lambda_h \):
\[
M_h \lambda_h = F_h,
\]
where the contributions of the triangle \( E \) to the stiffness matrix \( M_h \) and the right-hand side \( F_h \) are
\[
m_{i,j}^E = \frac{\nu_i^E \cdot \nu_j^E}{(\alpha, 1)_E}, \quad F_i^E = -\frac{(\alpha J_f^E, \nu_i^E)_E}{(\alpha, 1)_E} + (J_f^E, \nu_i^E)e_i^E,
\]
where \( \nu_E^i \) denotes the outer unit normal to the edge \( e_i^E \), \( \nu_i^E = |e_i^E| \nu_i^E \), \( |e_i^E| \) is the length of \( e_i^E \), \( J_f^E = (f, 1)_E(x,y)/(2|E|) \), and \(|E|\) denotes the area of \( E \).

Let \( P_h \) denote the \( L^2(\Omega) \) projection operator onto \( W_h \), \( \alpha_h = P_h \alpha \), and \( f_h = P_h f \). Also, set
\[ f_h|_E = \frac{f_h}{2} \left( 3 - \frac{\alpha}{\alpha_h} \right) |E|, \]
and
\[ a_h(\psi, \varphi) = \sum_{E \in \mathcal{E}_h} (\alpha_h^{-1} \nabla \psi, \nabla \varphi)_E. \]
Then as shown in [6], the system (4.3) corresponds to the system arising from the triangular nonconforming finite element method: Find \( \psi_h \in N_h \) such that
\[
a_h(\psi_h, \varphi) = (f_h, \varphi), \quad \forall \varphi \in N_h,
\]
where \( N_h \) is defined as in Example 2.1. Hence the previous methods can be used to solve (4.3), i.e., the mixed method (4.2). It should be also noted that the natural degrees of freedom, i.e., the values at the midpoint of edges of \( L_h \) and \( N_h \) are the same.
After the computation of $\lambda_h, \sigma_h$ and $u_h$ (if they are needed) can be recovered as follows. Set $\sigma_{E} = (a_{E} + b_{E}x, c_{E} + b_{E}y)$ and $\bar{f}_{E} = f_{h}|_{E}$. Then it follows from \textcolor{red}{[6]} that
\[
\begin{align*}
 b_{E} &= \frac{\bar{f}_{E}}{2}, \\
 a_{E} &= -\frac{1}{(\alpha_{i})_{E}} \left( \sum_{i=1}^{2} \| \sigma_{i}^{(1)} \|_{E}^{2} \lambda_{h}(x_{i})_{E} + \frac{\bar{f}_{E}(x,y)}{2} \right), \\
 c_{E} &= -\frac{1}{(\alpha_{i})_{E}} \left( \sum_{i=1}^{2} \| \sigma_{i}^{(2)} \|_{E}^{2} \lambda_{h}(x_{i})_{E} + \frac{\bar{f}_{E}(x,y)}{2} \right),
\end{align*}
\]
and
\[
\begin{align*}
u_{h}|_{E} &= \frac{1}{2|E|} \left( (\alpha_{h}, (x,y))_{E} + \sum_{i=1}^{3} \lambda_{h} \sigma_{i}^{(1)}(x,y) \nu_{i} \right), \quad E \in \mathcal{E}_{h}.
\end{align*}
\]

We now consider a modified version of the mixed method (4.2) in which the coefficient $\alpha$ is projected into the space $W_{h}$ \textcolor{red}{[9]}: Find $(\sigma_{h}, u_{h}, \lambda_{h}) \in \Lambda_{h} \times W_{h} \times L_{h}$ such that
\[
\begin{align*}
(\alpha_{h}, \nu, v) - \sum_{E \in \mathcal{E}_{h}} (\nabla \cdot \sigma_{h}, w)_{E} = (f, w), \quad \forall \ w \in W_{h}, \\
(\alpha_{h}, \nu, v) - (\lambda_{h}, 0, \nu_{E})_{\partial E} = 0, \quad \forall \ v \in \Lambda_{h}, \\
(\lambda_{h}, 0, \nu_{E})_{\partial E} = 0, \quad \forall \ \mu \in L_{h}.
\end{align*}
\]
Associated with this projected formulation, the linear system has the form in place of (4.4):
\[
\begin{align*}
m_{ij}^{E} &= \frac{\nu_{E} \cdot \nu_{E}}{(\alpha_{i})_{E}}, \quad \bar{f}_{i}^{E} = -\frac{(J_{E}, \nu_{E})_{E}}{|E|} + (J_{E}, \nu_{E})_{E}, \quad E \in \mathcal{E}_{h}.
\end{align*}
\]
The corresponding nonconforming system becomes: Find $\psi_{h} \in N_{h}$ such that
\[
(\alpha_{h}, \psi_{h}, \varphi) = (f_{h}, \varphi), \quad \forall \varphi \in N_{h}.
\]
The present systems in (4.6) and (4.7) are simpler than the corresponding systems in (4.4) and (4.5). The advantage of the projected mixed formulation over the usual one is more obvious for the mixed finite element method over rectangles \textcolor{red}{[6]}. The three-dimensional case can be considered similarly.

References


