Domain Decomposition Methods and Multilevel Preconditioners for Nonconforming and Mixed Methods for Partial Differential Problems

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ABSTRACT

In this paper we describe recent development of domain decomposition methods and multilevel preconditioners for solution of the discrete systems which arises from the application of nonconforming and mixed finite element methods to partial differential problems.

1. INTRODUCTION

A lot of numerical methods for solving partial differential problems have been developed in past years. Among the most popular and often used methods are Galerkin finite element methods, finite volume methods, and mixed finite element methods. The mixed methods have certain advantages over the Galerkin methods and the finite volume methods [**3**]. In particular, they preserve mass element by element, and produce a direct and accurate approximation for the vector unknown of the differential problems, which is the variable of primary interest in many applications such as the velocity field in the flow equation in highly heterogeneous media [**16**] and the electric field in the potential equation in semiconductor devices [**5, 7, 8**].

Due to their saddle point property, it is known that the linear systems arising from the mixed methods are generally harder to solve than those arising from comparable Galerkin methods. In particular, there has been little theory for constructing good preconditioners and developing efficient domain decomposition methods for solving the systems of algebraic equations arising from the mixed methods. An alternate approach was suggested by means of a nonmixed formulation. Namely,

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it has been shown that the mixed finite element methods are equivalent to a modification of nonconforming Galerkin methods [**1, 2, 4, 18**]. The nonconforming methods yield a symmetric and positive definite problem, which can be more easily solved using domain decomposition methods.

The purpose of this paper is to describe recent development of domain decomposition methods and multilevel preconditioners for solution of the discrete systems which arises from the application of nonconforming finite element methods to partial differential problems. Then, based on the equivalence mentioned above between the mixed and nonconforming methods, we discuss the domain decomposition methods and multilevel preconditioners for the mixed methods for the partial differential problems. In the next section we consider the domain decomposition methods, and then in the third section we present the multilevel preconditioners. In the final section we discuss the equivalence between the mixed and nonconforming finite element methods.

2. DOMAIN DECOMPOSITION METHODS

Let Ω be a bounded domain in $\mathbb{R}^d$, and let N_h be a nonconforming finite element space associated with a triangulation $\mathcal{E}_h$ of Ω, where h is the discretization parameter. We are concerned with the approximate problem: Find $u_h \in N_h$ such that

$$a_h(u_h, v) = (f, v), \quad \forall v \in N_h, \tag{2.1}$$

where the bilinear form $a_h(\cdot,\cdot)$ is assumed to be symmetric and positive definite, $(\cdot,\cdot)$ denotes the $L^2(\Omega)$ inner product (for simplicity), and $f \in L^2(\Omega)$. It is well known that the linear system arising from (2.1) is not well conditioned. The aim of this section is to develop an additive Schwarz algorithm for (2.1). For this, let $\{\Omega_j\}_{j=1}^J$ be an overlapping domain decomposition of Ω with the overlap parameter δ. The decomposition is assumed to align with the boundary $\partial\Omega$. Associated with each Ω_j, let N_h^j be a nonconforming finite element space of the same form as N_h, whose elements have support in Ω_j. The finite element space N_h is assumed to be represented as a sum of $J+1$ subspaces:

$$N_h = N_h^0 + N_h^1 + \ldots + N_h^J, \tag{2.2}$$

where N_h^0 is a so-called coarse space. We now define the operators $\Pi_j : N_h \to N_h^j$, $j = 0, 1, \ldots, J$, by

$$a_h(\Pi_i v, w) = a_h(v, w), \quad \forall w \in N_h^j, \tag{2.3}$$

and the operator $\Pi : N_h \to N_h$ by

$$\Pi = \sum_{j=0}^{J} \Pi_j. \tag{2.4}$$

In (2.3) we could use so-called inexact solvers; the analysis is the same. We now define a Schwarz algorithm for (2.1).

Additive algorithm. The additive Schwarz algorithm for (2.1) is given by

$$\Pi u_h = f_h, \quad f_h = \sum_{j=0}^{J} f_j, \tag{2.5}$$

where $f_j \in N_h^j$ satisfies

$$a_h(f_j, v) = (f, v), \quad \forall v \in N_h^j,\ j = 0, 1, \ldots, J.$$

Note that (2.1) and (2.5) have the same solution. The following abstract convergence result for bounds on the condition number of Π can be found in [**17**]. Assume that

(A1) There is a constant C such that every $v \in N_h$ can be represented by $v = \sum_{j=0}^{J} v_j$ with $v_j \in N_h^j$ satisfying

$$\sum_{j=0}^{J} a_h(v_j, v_j) \leq C a_h(v, v).$$

(A2) Let $\kappa = (\kappa_{ij})$ be a symmetric matrix with $\kappa_{ij} \geq 0$ satisfying

$$|a_h(v_i, v_j)| \leq \kappa_{ij} a_h(v_i, v_i)^{1/2} a_h(v_j, v_j)^{1/2}, \quad \forall v_i \in N_h^i,\ v_j \in N_h^j,\ i, j = 1, \ldots, J.$$

Then we have

$$\lambda_{\min}(\Pi) \geq C^{-1} \text{ and } \lambda_{\max}(\Pi) \leq \rho(\kappa) + 1,$$

where $\rho(\kappa)$ is the spectral radius of κ.

We now apply the above theory to analyze a couple of examples.

`Example 2.1.` Let N_h be the P_1 nonconforming finite element space of the second order elliptic problem in $\mathbb{R}^2$. That is,

$$N_h = \{v \in L^2(\Omega) : v|_E \in P_1(E),\ \forall E \in \mathcal{E}_h;\ v \text{ is continuous at the midpoints of interior sides and vanishes at the midpoints of sides on } \partial\Omega\},$$

where $\mathcal{E}_h$ is a partition of Ω into triangles. We need to assume a structure to our family of partitions. In the first step, let $\mathcal{E}_H$ be a coarse triangulation of Ω into nonoverlapping triangular substructures Ω_j', $j = 1, \ldots, J$. Then, in the second step we refine $\mathcal{E}_H$ into triangles to have a triangulation $\mathcal{E}_h$, $h < H$. Finally, let $\{\Omega_j\}_{j=1}^{J}$ be an overlapping domain decomposition of Ω by extending Ω_j' with the overlap parameter δ defined by

$$\delta = \min\{\text{dist}(\partial\Omega_j \setminus \partial\Omega, \partial\Omega_j' \setminus \partial\Omega), j = 1, \ldots, J\}.$$

Finally, define the coarse space

$$N_h^0 = \{v \in N_h : v = R_h\varphi,\ \varphi \in U_H\}, \tag{2.6}$$

where R_h is the nodal interpolation operator into N_h and U_H is the P_1-conforming space associated with $\mathcal{E}_H$. With these choices, it follows from the above abstract result [**15**] that the condition number $c(\Pi)$ of Π can be estimated by

$$c(\Pi) = O(1 + H/\delta). \tag{2.7}$$

`Example 2.2.` Let N_h be the rectangular nonconforming finite element space of the second order elliptic problem in $\mathbb{R}^2$. Namely,

$$N_h = \Big\{\xi : \xi|_E = a_E^1 + a_E^2 x + a_E^3 y + a_E^4(x^2 - y^2),\ a_E^i \in \mathbb{R},\ \forall E \in \mathcal{E}_h;$$
$$\text{if } E_1 \text{ and } E_2 \text{ share an edge } e, \text{ then } \int_e \xi|_{\partial E_1}\, ds = \int_e \xi|_{\partial E_2}\, ds;$$
$$\text{and } \int_{\partial E \cap \partial\Omega} \xi|_{\partial\Omega}\, ds = 0\Big\},$$

where $\mathcal{E}_h$ is a partition of Ω into rectangles. The overlapping domain decomposition $\{\Omega_j\}_{j=1}^J$ and spaces $\{N_h^j\}_{j=1}^J$ can be similarly defined as in Example 2.1. Finally, the coarse space is defined as follows. Let $\widehat{\mathcal{E}}_h$ be the triangulation of Ω into triangles obtained by connecting the two opposite vertices of the rectangles in $\mathcal{E}_h$. Associated with $\widehat{\mathcal{E}}_h$, let $\widehat{N}_h$ be the P_1 nonconforming finite element space in Example 2.1. Then we define the operator $\widehat{\mathcal{I}}_h : N_h \to \widehat{N}_h$ by the relation: If $v \in N_h$ and e is an edge of a triangle in $\widehat{N}_h$, then $\widehat{\mathcal{I}}_h v \in \widehat{N}_h$ is defined by

$$\frac{1}{|e|}(\widehat{\mathcal{I}}_h v, 1)_e = \frac{1}{|e|}(v, 1)_e.$$

Now we have

$$N_h^0 = \{v \in N_h : v = R_h\varphi,\ \varphi \in \widehat{N}_h^0\},$$

where $\widehat{N}_h^0$ is given as in (2.6), and R_h is the interpolation operator into N_h. The result (2.7) can be shown here by means of the operator $\widehat{\mathcal{I}}_h$ introduced above.

Other choices for N_h^0 in Examples 2.1 and 2.2 can be made [**12**]. It follows from (2.7) that if we use a generous overlapping, then the condition number of Π is uniformly bounded. With the same technique, two-level multiplicative algorithms and multilevel Schwarz algorithms can be developed [**12, 21**]. Moreover, exploiting the equivalence between the nonconforming and mixed methods [**6, 10, 12, 13**] (also see the fourth section), the Schwarz algorithms can be applied to the mixed methods [**3**]. Fourth order problems can be also analyzed using the present theory.

3. MULTILEVEL PRECONDITIONERS

Let

$$N_0 \to N_1 \to \cdots \to N_J$$

be a sequence of nonconforming finite element spaces associated with a increasing sequence of triangulations of Ω such that there is an intergrid transfer operator between two adjacent spaces:

$$I_j : N_{j-1} \to N_j, \quad j = 1, \dots, J.$$

We are again concerned with the approximate problem: Find $u_J \in N_J$ such that

$$a_J(u_J, v) = (f, v), \quad \forall v \in N_J, \tag{3.1}$$

where $a_J(\cdot,\cdot)$ is assumed to be equivalent to a discrete energy scalar product $(\cdot,\cdot)_{\mathcal{E}}$ on N_J:

$$a_J(v, v) \approx ||v||_{\mathcal{E}}^2, \quad \forall v \in N_J. \tag{3.2}$$

We define the operators $R_j : N_j \to N_J$ by

$$R_j = I_J I_{J-1} \cdots I_{j+1}, \quad j = 0, \dots, J-1.$$

Let $\{b_j(\cdot,\cdot)\}_{j=0}^J$ be a family of bilinear forms defined by

$$b_j(v, w) = \alpha_j(v, w), \quad \forall v, w \in N_j,$$

where the positive constants α_j are determined by the (inverse) inequalities

$$||v||_{\mathcal{E}}^2 \le \alpha_j ||v||^2, \quad \forall v \in N_j,\ j = 0, \dots, J.$$

Finally, introduce the operators $T^j : N_J \to N_j$ by

$$b_j(T^j v, w) = a_J(v, R_j w), \quad \forall w \in N_j,$$

and the elements $f^j \in N_j$ by

$$b_j(f^j, w) = (f, R_j w), \quad \forall w \in N_j.$$

Now it can be seen that the problem (3.1) is equivalent to

$$\mathcal{P}_J u_J = f_J, \tag{3.3}$$

where

$$\mathcal{P}_J = \sum_{j=0}^{J} R_j T^j, \quad f_J = \sum_{j=0}^{J} R_j f^j.$$

Assume that (3.2) and the following condition are satisfied:

$$b_j(v - I_j P_{j-1} v, v - I_j P_{j-1} v) \leq C_J ||v - I_j P_{j-1} v||_{\mathcal{E}}^2, \quad \forall v \in N_j,$$

where the operators $P_{j-1} : N_j \to N_{j-1}$ are given by

$$(P_{j-1} v, w)_{\mathcal{E}} = (v, w)_{\mathcal{E}}, \quad \forall v \in N_{j-1}, \, j = 1, \ldots, J.$$

Then we have the following abstract result on the condition number of $\mathcal{P}_J$ **[14, 19]**:

$$C_* \max_{j=0,\ldots,J} \tau_j \leq c(\mathcal{P}_J) \leq C^* C_J \sum_{j=0}^{J} \tau_j, \tag{3.4}$$

where the constants C_* and C^* are independent of j and

$$\tau_j = \sup\{||R_j v||_{\mathcal{E}}^2 / ||v||_{\mathcal{E}}^2, \, v \in N_j, \, v \neq 0\}.$$

We now again consider the two examples presented in §2. Let h_0 and $\mathcal{E}_{h_0} = \mathcal{E}_0$ be given. For each integer $0 < j \leq J$, let $h_j = 2^{-j} h_0$ and $\mathcal{E}_{h_j} = \mathcal{E}_j$ be constructed by connecting the midpoints of the edges of the triangle or rectangle in $\mathcal{E}_{j-1}$. Associated with each $\mathcal{E}_j$, let N_j be the nonconforming space defined as in Example 2.1 or Example 2.2. The intergrid transfer operators $I_j : N_{j-1} \to N_j$ are defined in the usual way by nodal or edge value averaging procedures **[6]**. Then, using the result (3.4), we have for both examples:

$$c(\mathcal{P}_J) = O(J).$$

Finally, thanks to the equivalence between the nonconforming and mixed finite element methods (see the next section), the theory presented in this section applies to the mixed methods **[11]**, where algebraic multilevel preconditioners for both nonconforming and mixed methods are also constructed. Fourth order differential problems can be also analyzed using the present theory. Due to the page limitation, numerical results are not shown here; please refer to **[11, 12, 14, 19]** for the numerical results on the domain decomposition methods and multilevel preconditioners.

4. MIXED FINITE ELEMENT METHODS

We concentrate on the model problem

$$\begin{aligned} -\nabla \cdot (a \nabla u) &= f \quad && \text{in } \Omega, \\ u &= 0 \quad && \text{on } \partial\Omega. \end{aligned} \tag{4.1}$$

The Raviart-Thomas space [20] over triangles is given by

$$\Lambda_h = \left\{ v \in (L^2(\Omega))^2 : v|_E = \left(a_E^1 + a_E^2 x,\ a_E^3 + a_E^2 y\right),\ a_E^i \in \mathbb{R},\ E \in \mathcal{E}_h \right\},$$
$$W_h = \left\{ w \in L^2(\Omega) : w|_E \text{ is constant for all } E \in \mathcal{E}_h \right\},$$
$$L_h = \left\{ \mu \in L^2(\partial\mathcal{E}_h) : \mu|_e \text{ is constant},\ e \in \partial\mathcal{E}_h;\ \mu|_e = 0,\ e \subset \partial\Omega \right\},$$

where $\partial\mathcal{E}_h$ denotes the set of all interior edges. Then the hybrid form of the mixed method for (4.1) is to seek $(\sigma_h, u_h, \lambda_h) \in \Lambda_h \times W_h \times L_h$ such that

$$\begin{aligned} \textstyle\sum_{E\in\mathcal{E}_h} (\nabla\cdot\sigma_h, w)_E &= (f,w), && \forall\, w \in W_h,\\ (\alpha\sigma_h, v) - \textstyle\sum_{E\in\mathcal{E}_h} \left[(u_h, \nabla\cdot v)_E - (\lambda_h, v\cdot\nu_E)_{\partial E}\right] &= 0, && \forall\, v \in \Lambda_h,\\ \textstyle\sum_{E\in\mathcal{E}_h} (\sigma_h\cdot\nu_E, \mu)_{\partial E} &= 0, && \forall\, \mu \in L_h, \end{aligned} \tag{4.2}$$

where ν_E denotes the unit outer normal to E and $\alpha = a^{-1}$. The solution σ_h is introduced to approximate the vector field

$$\sigma = -a\nabla u,$$

which is the variable of primary interest in many applications. Since σ lies in the space

$$H(\mathrm{div};\Omega) = \left\{ v \in (L^2(\Omega))^2 : \nabla\cdot v \in L^2(\Omega) \right\},$$

and we do not require that Λ_h be a subspace of $H(\mathrm{div};\Omega)$, the last equation in (4.2) is used to enforce that the normal components of σ_h are continuous across the interior edges in $\partial\mathcal{E}_h$, so in fact $\sigma_h \in H(\mathrm{div};\Omega)$.

There is no continuity requirement on the spaces Λ_h and W_h, so σ_h and u_h can be locally (element-by-element) eliminated from (4.2). In fact, from [6], (4.2) can be algebraically condensed to the symmetric, positive definite system for the Lagrange multiplier λ_h:

$$M_h \lambda_h = F_h, \tag{4.3}$$

where the contributions of the triangle E to the stiffness matrix M_h and the right-hand side F_h are

$$m_{ij}^E = \frac{\overline{\nu}_E^i\cdot\overline{\nu}_E^j}{(\alpha,1)_E}, \quad F_i^E = -\frac{(\alpha J_E^f, \overline{\nu}_E^i)_E}{(\alpha,1)_E} + (J_E^f, \nu_E^i)_{e_E^i}, \tag{4.4}$$

where ν_E^i denotes the outer unit normal to the edge e_E^i, $\overline{\nu}_E^i = |e_E^i|\nu_E^i$, $|e_E^i|$ is the length of e_E^i, $J_E^f = (f,1)_E(x,y)/(2|E|)$, and $|E|$ denotes the area of E.

Let P_h denote the $L^2(\Omega)$ projection operator onto W_h, $\alpha_h = P_h\alpha$, and $f_h = P_h f$. Also, set

$$\tilde{f}_h|_E = \frac{f_h}{2}\left(3 - \frac{\alpha}{\alpha_h}\right)|_E,$$

and

$$\tilde{a}_h(\psi,\varphi) = \sum_{E\in\mathcal{E}_h} (\alpha_h^{-1}\nabla\psi, \nabla\varphi)_E.$$

Then as shown in [6], the system (4.3) corresponds to the system arising from the triangular nonconforming finite element method: Find $\psi_h \in N_h$ such that

$$\tilde{a}_h(\psi_h,\varphi) = (\tilde{f}_h,\varphi), \quad \forall\varphi \in N_h, \tag{4.5}$$

where N_h is defined as in Example 2.1. Hence the previous methods can be used to solve (4.3), i.e., the mixed method (4.2). It should be also noted that the natural degrees of freedom, i.e., the values at the midpoint of edges of L_h and N_h are the same.

After the computation of λ_h, σ_h and u_h (if they are needed) can be recovered as follows. Set $\sigma_h|_E = (a_E + b_E x, c_E + b_E y)$ and $\overline{f}_E = f_h|_E$. Then it follows from [6] that

$$\begin{aligned} b_E &= \tfrac{\overline{f}_E}{2}, \\ a_E &= -\tfrac{1}{(\alpha,1)_E}\left(\textstyle\sum_{i=1}^3 |e_E^i|\nu_E^{i(1)}\lambda_h|_{e_E^i} + \tfrac{\overline{f}_E}{2}(\alpha,x)_E\right), \\ c_E &= -\tfrac{1}{(\alpha,1)_E}\left(\textstyle\sum_{i=1}^3 |e_E^i|\nu_E^{i(2)}\lambda_h|_{e_E^i} + \tfrac{\overline{f}_E}{2}(\alpha,y)_E\right), \end{aligned}$$

and

$$u_h|_E = \frac{1}{2|E|}\left((\alpha\sigma_h,(x,y))_E + \sum_{i=1}^3 \lambda_h|_{e_E^i}((x,y),\nu_E^i)_{e_E^i}\right), \quad E \in \mathcal{E}_h.$$

We now consider a modified version of the mixed method (4.2) in which the coefficient α is projected into the space W_h [9]: Find $(\sigma_h, u_h, \lambda_h) \in \Lambda_h \times W_h \times L_h$ such that

$$\begin{aligned} \textstyle\sum_{E\in\mathcal{E}_h}(\nabla\cdot\sigma_h, w)_E &= (f,w), && \forall\, w \in W_h, \\ (\alpha_h\sigma_h, v) - \textstyle\sum_{E\in\mathcal{E}_h}\left[(u_h,\nabla\cdot v)_E - (\lambda_h, v\cdot\nu_E)_{\partial E}\right] &= 0, && \forall\, v \in \Lambda_h, \\ \textstyle\sum_{E\in\mathcal{E}_h}(\sigma_h\cdot\nu_E,\mu)_{\partial E} &= 0, && \forall\, \mu \in L_h. \end{aligned}$$

Associated with this projected formulation, the linear system has the form in place of (4.4):

$$m_{ij}^E = \frac{\overline{\nu}_E^i\cdot\overline{\nu}_E^j}{(\alpha,1)_E}, \quad F_i^E = -\frac{(J_E^f,\overline{\nu}_E^i)_E}{|E|} + (J_E^f,\nu_E^i)_{e_E^i}, \quad E \in \mathcal{E}_h. \tag{4.6}$$

The corresponding nonconforming system becomes: Find $\psi_h \in N_h$ such that

$$\tilde{a}_h(\psi_h,\varphi) = (f_h,\varphi), \quad \forall\varphi \in N_h. \tag{4.7}$$

The present systems in (4.6) and (4.7) are simpler than the corresponding systems in (4.4) and (4.5). The advantage of the projected mixed formulation over the usual one is more obvious for the mixed finite element method over rectangles [6]. The three-dimensional case can be considered similarly.

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