A New Domain Decomposition Method for Convection–Dominated Problems

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Summary. In this article, we present a new domain decomposition method for solving convection–diffusion problems with dominant convection. We combine the sequential algorithm with a parallel strategy to adapt to the special properties of the equations. The sequential algorithm is used in the downwind direction while the parallel algorithm is applied in the crosswind direction. In both algorithms, an overlapping domain decomposition process is introduced. In each patch, we solve a local convection–dominated problem with artificial boundary conditions by the streamline diffusion finite element method. A globally continuous discrete solution is constructed from all the patch-wise solutions. It is proven that this global solution converges to the exact solution with an order of $O(h^{3/2})$ in the L^2–norm as long as the overlapping width is kept to $O(h^{3/4}|\log h|)$ in the crosswind direction and to $O(h|\log h|)$ in the downwind direction. It is emphasized that we do not require the patch size to be sufficiently small and do not iterate the whole process as usually done in other domain decomposition approaches.

1 Introduction

Since the domain decomposition method was proposed years ago, many papers have been published for solving symmetric positive definite elliptic problems. This method has been demonstrated to have a good condition number as an iteration method and can also be implemented in a parallel way. For nonstationary convection–dominated problems, a domain splitting method was proposed and analyed in [6]. The proper convergence order was there achieved without conditions on the macroelements. For stationary nonsymmetric problems, there are also some papers, e.g., [5] and [1], in which the convection term is treated as a perturbation of the elliptic part. In those

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papers, the coefficient of the diffusion is not allowed to be very small in comparison with that of the convection, i.e., the Peclet number should not be too large. They must enforce an inconvenient mesh condition that the coarse mesh must be fine enough. Looking at the analysis carefully, one can see that the coarse mesh size must be proportional to the inverse of the Peclet number. This is almost impossible for convection–dominated problems.

In Kapurkin and Lube [4], a modified Schwarz iteration method for convection–dominated problems was discussed. An overlapping domain decomposition technique was applied and a convergence proof was given for the continuous problems. Their numerical experiments already showed that the iteration number depends on the number of subdomains in the flow direction. From the theory of partial differential equations, it is obvious for convection–dominated problems that any change in the inflow boundary condition influences the solution at any point in the downwind direction decisively. The discrete solution in a subdomain must wait for its inflow boundary condition from the upstream–neighboring subdomain. Before the approximate inflow boundary condition comes from the neighboring subdomain, any artificial boundary condition in this subdomain gives only a false approximation and hence wastes time and effort.

For convection–dominated problems, we propose a new domain decomposition technique which avoids any unnecessary iterations. Simply speaking, we take sequential computations for subdomains in the downwind direction while we take parallel algorithms for subdomains in the crosswind direction. Since we cannot compute the discrete solution in the convection direction in a parallel way, the sequential algorithm in this direction is proposed only to reduce the size of the subproblems. In fact, we could use one whole strip from the inflow boundary to the outflow boundary if one would like to solve a comparatively bigger problem instead of several smaller ones.

In order to reduce the overlapping widths, we could also iterate the above process. One can investigate the dependence of the condition number on the iteration. This will be done in a forthcoming paper.

Test computations for model problems show that the proposed domain decomposition method works very well. The overlapping widths are usually limited to $3h$ in the downstream direction and to $h|\log h|$ in the crosswind direction. This logarithmic dependence occurs only in the crosswind direction. On the other hand, the test computations show that the overlapping sizes in the two directions are not influenced by the number of macro–elements, though we have linear dependence in our theoretic analysis. The discrete solution constructed by a global process converges usually with an order of $O(h^2)$. Since the sub–problems on each patch are small in comparison with the global one, the iterations for solving each algebraic system are significantly reduced.

2 A Domain Decomposition Technique

As an illustrative example, we consider the model problem

$$-\rho u_{xx} - \varepsilon u_{yy} + u_x + u = f, \qquad \text{in} \quad \Omega, \tag{2.1.a}$$

$$u = g, \qquad \text{on} \quad \partial\Omega, \tag{2.1.b}$$

where $\Omega = (0,1) \times (0,1)$ in R^2, ρ and ε are small non-negative parameters. If ρ or ε is zero, we should specify the boundary condition only on a part of the boundary $\partial\Omega$.

Since the standard finite element method gives an oscillating discrete solution if the boundary condition g or the source term f is not smooth, the streamline diffusion finite element method (SDFEM) was proposed to damp away the possible over– and undershootings. Let $\mathbf{V}_h \subset H^1(\Omega)$ be a finite element subspace, in which linear or bilinear functions are defined on each element. The functions in $\mathbf{V}_h^0$ have homogeneous boundary conditions. By modifying the test function as $W + \delta W_x$ with $\delta = O(h)$, the SDFEM reads: Find $V \in \mathbf{V}_h(\Omega)$, such that

$$\begin{aligned} \rho(V_x, W_x) + \varepsilon(V_y, W_y) + (V_x + V, W + \delta W_x) &= (f, W + \delta W_x), \quad \forall W \in \mathbf{V}_h^0(\Omega), \\ V &= G, \qquad\qquad\qquad \text{on } \partial\Omega, \end{aligned}$$

where G is some projection of g on $\mathbf{V}_h$ and the term $\rho V_{xx} + \varepsilon V_{yy}$ is neglected for a linear or bilinear ansatz. Since the diffusion coefficient ε is very small, or even zero, we introduce an artificial diffusion $\varepsilon_m = O(h^\alpha)$ with $\frac{3}{2} \le \alpha \le 2$, following an idea of Johnson et al. [3]. The order of ε_m depends on the localization property of the scheme, which will be determined later. Defining the bilinear form

$$B(V, W) = ((\rho + \delta)V_x, W_x) + \varepsilon_m(V_y, W_y) + (V_x, W) + (V, W + \delta W_x),$$

and a linear functional

$$L(W) = (f, W + \delta W_x),$$

we work with the discrete problem of finding $V \in \mathbf{V}_h$ such that

$$\begin{aligned} B(V, W) &= L(W), \qquad \forall W \in \mathbf{V}_h^0(\Omega), && (2.2.a) \\ V &= G, \qquad\quad \text{on } \partial\Omega. && (2.2.b) \end{aligned}$$

We associate the bilinear form $B(\cdot,\cdot)$ with the following energy norm

$$|||W|||^2 = \delta\|W_x\|^2 + \varepsilon_m\|W_y\|^2 + \|W\|^2. \tag{2.3}$$

It is easy to verify that

$$B(W, W) \ge \frac{1}{2}|||W|||^2, \quad \forall W \in \mathbf{V}_h^0(\Omega), \tag{2.4}$$

which gives the global stability of the SDFEM:

$$\|V\| \le 2\|f\| + C\|g\|_{\partial\Omega}. \tag{2.5}$$

To distinguish the type of boundaries, we introduce the following definitions for the domain Ω. Let $\mathbf{n} = (n_x, n_y)$ be the outer normal along the boundary $\partial\Omega$. Its inflow boundary $\partial\Omega_-$, outflow boundary $\partial\Omega_+$ and stationary boundary $\partial\Omega_0$ are defined by

$$\begin{aligned} \partial\Omega_- &= \{(x, y) \in \partial\Omega,\ n_x(x, y) < 0\}, \\ \partial\Omega_+ &= \{(x, y) \in \partial\Omega,\ n_x(x, y) > 0\}, \\ \partial\Omega_0 &= \{(x, y) \in \partial\Omega,\ n_x(x, y) = 0\}. \end{aligned}$$

The solution of problems with dominant convection has significantly different properties along these types of boundaries.

Problem (2.1) depicts a motion from the left to the right with small diffusion. A discretization method should cover this property of the equation. For parallel computation, we divide the domain Ω in patches with two families of lines which are parallel and orthogonal to the characteristics, respectively. In the present case, these two families of lines are $\{x = \text{const.}\}$ and $\{y = \text{const.}\}$. We let

$$\Omega_{i,j} = \Big\{(x,y):\ x_{i-1} \le x \le x_i,\ y_{j-1} \le y \le y_j\Big\},$$

for $i = 1, \ldots, M$ and $j = 1, \ldots, N$. Here, we have $x_0 = 0$, $x_M = 1$, $y_0 = 0$ and $y_N = 1$. Define $\Omega_j = \cup_{i=1}^{M} \Omega_{i,j}$.

The main idea for solving convection–dominated problem is as follows. In the crosswind direction we solve subproblems in a parallel manner, while in the downwind direction we solve subproblems in a sequential manner. To understand this scheme, we assume that there are N processors. The j–th processor solves subproblems in the "big" patch Ω_j. In the strip Ω_j, we solve subproblems for $i = 1, \ldots, M$ sequentially. We take overlaps in the crosswind and downwind direction, but with different overlapping sizes. Let d_x^i and d_y^i be the overlapping sizes of $\Omega_{i,j}$ in the downwind and crosswind direction, respectively. Thus, we define the overlapping patches (see Figure 1) as

$$\Omega_{i,j}^d = \Big\{(x,y):\ x_i \le x \le x_{i+1} + d_x^i,\ y_j - d_y^i \le y \le y_{j+1} + d_y^i\Big\} \cap \Omega.$$

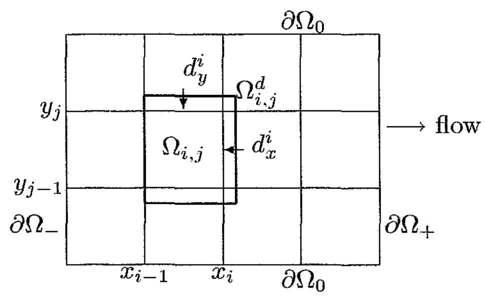

Figure 1 Subdomain $\Omega_{i,j}^d$

Now we can describe our domain decomposition scheme as follows.

Step 1: Assume that we have the discrete solution $U_{i-1,j}$ on the boundary $x = x_i$ with $U_{0,j} = G$. For $i = 1, \ldots, M$ and $j = 1, \ldots, N$, we find $U_{i,j} \in \mathbf{V}_h(\Omega_{i,j}^d)$, such that

$$B(U_{i,j}, W) = L(W), \qquad \forall W \in \mathbf{V}_h^0(\Omega_{i,j}^d), \tag{2.6.a}$$

$$U_{i,j} = U_{i-1,j}, \qquad \text{on } \partial(\Omega_{i,j}^d)_-, \tag{2.6.b}$$

$$U_{i,j} = 0, \qquad \text{on } \partial(\Omega_{i,j}^d)_+ \cup \partial(\Omega_{i,j}^d)_0. \tag{2.6.c}$$

Remark 1 *For $j = N$, $i = 1$ or $i = M$, we need to modify the homogeneous boundary conditions to the global boundary condition. We specify $U_{i,j}(x,y) = G(x,y)$ if (x,y) is located on $\partial\Omega$.*

Remark 2 *For a fixed i, we can compute the patchwise solutions in parallel with $j = 1, \dots, N$, but for a fixed j we must compute them sequentially with $i = 1, \dots, M$.*

Step 2: We assemble the patchwise solutions to a globally continuous solution $U \in \mathbf{V}_h(\Omega)$ by an averaging process in the nodal points z of T_h:

$$U(z) = U_{i,j}(z), \qquad \text{for } z \in \Omega_{i,j} \backslash \partial\Omega_{i,j}, \tag{2.7.a}$$
$$U(z) = \text{averaging}, \qquad \text{for } z \text{ on the common boundaries of } \Omega_{i,j} \tag{2.7.b}$$

3 Error Analysis for the DD Scheme

By Splitting $u - U = (u - V) + (V - U)$ with V defined in (2.2), we see that the estimate of the error $u - U$ between exact solution and the discrete solution of the domain decomposition scheme leads back to the error $U - V$, since the estimate of the first part can be found in a standard book (see, e.g., Johnson [2]). For the L^2–norm, we have

$$\|U - V\|^2 = \sum_{i,j} \|U - V\|^2_{\Omega_{i,j}} \leq C \sum_{i,j} \|U_{i,j} - V\|^2_{\Omega_{i,j}}. \tag{3.1}$$

Obviously, we have $\mathbf{V}^0_h(\Omega^d_{i,j}) \subset \mathbf{V}^0_h(\Omega)$. Setting $E_{i,j} = V - U_{i,j}$, we subtract equation (2.6) from equation (2.2) and get

$$B(E_{i,j}, W) = 0, \qquad \forall W \in \mathbf{V}^0_h(\Omega^d_{i,j}), \tag{3.2.a}$$
$$E_{i,j} = E_{i-1,j}, \qquad \text{on } \partial(\Omega^d_{i,j})_-, \tag{3.2.b}$$
$$E_{i,j} = V, \qquad \text{on } \partial(\Omega^d_{i,j})_+ \cup \partial(\Omega^d_{i,j})_0. \tag{3.2.c}$$

To estimate $E_{i,j}$ in (3.2), we state a lemma, the proof of which will be omitted.

Lemma 1 *Suppose that E satisfies*

$$B(E, W) = 0, \qquad \forall W \in \mathbf{V}^0_h(\omega^d), \tag{3.3.a}$$
$$E = G, \qquad \text{on } \partial\omega^d. \tag{3.3.b}$$

If the triangulation in $\omega^d \backslash \omega$ is quasi–uniform, then the choices of the streamline diffusion and the artificial diffusion must satisfy

$$\delta \geq Ch, \quad \varepsilon_m \geq Ch^{3/2}. \tag{3.4}$$

Then there exist constants $C > 0$ and $\theta > 0$, independent of h, ρ and ε, such that the function E in the subdomain ω admits the estimate

$$\|E\|^2_\omega \leq C \left(\|G\|^2_{\partial\omega^d_-} + e^{-\frac{\theta d_x}{\delta}} \|G\|^2_{\partial\omega^d_+} + e^{-\frac{\theta d_y}{\sqrt{\varepsilon_m}}} \|G\|^2_{\partial\omega^d_0} \right). \tag{3.5}$$

Applying this lemma to (3.2) on the subdomain $\Omega^d_{i,j}$ and noting that $\partial(\Omega_{i,j})^d_- \subset \Omega_{i-1,j}$, we have

$$\|E_{i,j}\|^2_{\Omega_{i,j}} \leq C \left(\|E_{i-1,j}\|^2_{\partial(\Omega_{i,j})^d_-} + e^{-\frac{\theta d^i_x}{\delta}} \|V\|^2_{\partial(\Omega_{i,j})^d_+} + e^{-\frac{\theta d^i_y}{\sqrt{\varepsilon_m}}} \|V\|^2_{\partial(\Omega_{i,j})^d_0} \right)$$

$$\leq C_0 h^{-1} \|E_{i-1,j}\|^2_{\Omega_{i-1,j}} + C_1 \left(e^{-\frac{\theta d_x^i}{\delta}} \|V\|^2_{\partial(\Omega_{i,j})_+^d} + e^{-\frac{\theta d_y^i}{\sqrt{\varepsilon_m}}} \|V\|^2_{\partial(\Omega_{i,j})_0^d} \right).$$

Taking into account that $E_{0,j} \equiv 0$ for $j = 1, \ldots, N$, and this into (3.1), we get

$$\|U - V\|^2 \leq C \sum_{i=1}^{M} h^{-(M-i)} \left(e^{-\frac{\theta d_x^i}{\delta}} + e^{-\frac{\theta d_y^i}{\sqrt{\varepsilon_m}}} \right) \sum_{j=1}^{N} \|V\|^2_{\Omega_{i,j}}.$$

For any $\nu \geq 3/2$, we can clearly choose d_x^i and d_y^i as

$$d_x^i = (M - i + 2\nu)\theta^{-1}\delta |\log h|, \quad d_y^i = (M - i + 2\nu)\theta^{-1}\sqrt{\varepsilon_m} |\log h|, \tag{3.6}$$

and we obtain by the global stability estimate (2.5)

$$\|U - V\|^2 \leq C h^{2\nu} \|V\|^2 \leq C h^{2\nu} \left(\|f\|^2 + \|g\|^2_{\partial\Omega} \right). \tag{3.7}$$

Summarizing the proof above and combining the standard estimate for $u - V$, we can state our convergence result.

Theorem 1 *Suppose that the triangulations $T_h(\Omega_{i,j})$ are quasi-uniform for any i, j, and that the diffusion coefficients satisfy*

$$\rho \leq Ch, \quad \varepsilon \leq Ch^{3/2}.$$

We specify the parameters δ and ε_m in the SDFEM by

$$\delta = Ch, \quad \varepsilon_m = Ch^{3/2}.$$

For any $\nu \geq 3/2$, we take the overlapping widths in the streamline and crosswind directions as

$$d_x^i = c(M - i + 2\nu)h|\log h|, \quad d_y^i = c(M - i + 2\nu)h^{3/4}|\log h|.$$

Then the discrete solution achieved by the domain decomposition scheme (2.6) *and* (2.7) *admits the error estimate*

$$\|u - U\| \leq Ch^{3/2}\|u\|_{H^2(\Omega)} + Ch^{\nu} \left(\|f\| + \|g\|_{\partial\Omega} \right). \tag{3.8}$$

Remark 3 *If we orient the quadrilateral mesh in the streamline direction and take it almost equidistant in the direction, we can improve the convergence order in the L^2-norm to $O(h^2)$ and reduce the overlapping width in the crosswind direction to $O(h|\log h|)$ (see [7] for the techniques of the proof).*

4 Numerical Results

The theoretical results of the preceding sections have been verified through various test computations. It turns out that the overlapping widths d_x and d_y actually do not depend on the number of the macro elements, though theoretical convergence results show such a dependence. All tests have been performed on the unit square and the equidistant $M \times M$ macro–elements are used. To verify the order of convergence, we choose a smooth solution.

Example 1: We consider the model problem

$$-\varepsilon\Delta u + u_x + 2u = f, \quad \text{in } \Omega, \tag{4.1.a}$$
$$u = 0, \quad \text{on } \partial\Omega, \tag{4.1.b}$$

with the exact solution $u = 4x(1-x)y(1-y)$. We take $\varepsilon = 10^{-10}$ for all the computations.

Test computations show that the number of overlapping elements in the downstream direction (here, x) is independent of the size of the fine meshes, though we theoretically have a logarithmic dependence on the fine mesh size. At each refinement level, for fixed overlapping size in the crosswind direction, the error does not decrease when d_x goes over $3h_x$. This means that the choice $d_x = 3h$ suffices to control the error of overlapping in the streamline direction. However, this logarithmic dependence on the fine mesh size is visible in the number of overlapping elements in the crosswind direction. While refining the fine mesh, we must increase the crosswind–overlapping size d_y logarithmically in order to control the error of the overlappings. In the theoretical analysis we have also that d_x and d_y are proportional to the number of macro-elements in the streamline direction. But in the test computations, we have not seen any relations between the overlapping sizes and the number of macro–elements.

In order to illustrate the error of the domain decomposition method, we first give the L^2–error and the corresponding order of convergence with the SDFEM without domain decompositions. We set $h = 2^{-N}$ with the refinement level N.

N	5	6	7	8	9	10	11
Error	4.68(-4)	1.18(-4)	2.95(-5)	7.39(-6)	1.85(-6)	4.63(-7)	1.16(-7)
Order		1.99	2.00	2.00	2.00	2.00	2.00

Table 1 The error and the convergence order without DD

In the following tables, we specify the overlapping size $d_y = Lh$ in the crosswind direction with the constant L. Table 2 represents the error for different refinement levels. The constant L shown in the tables is the smallest necessary number of overlapping elements to suppress the error of the domain decomposition.

N	5	6	7	8	9	10
L	4	4	5	6	7	8
Error	4.68(-4)	1.18(-4)	2.95(-5)	7.40(-6)	1.85(-6)	4.63(-7)

Table 2 Results for a 2×2 decomposition

Table 3 shows that the error does not depend on the number of macro-elements and that the number of overlapping elements in the crosswind direction increases

N	7	8	9	10	11
L	5	7	7	8	8
Error	2.97(-5)	7.40(-6)	1.85(-6)	4.63(-7)	1.17(-7)

Table 3 Results for a 32 × 32 decomposition

logarithmically with the refinement level. Comparing the errors in Tables 2 and 3 with those given in Table 1 for a fixed N, we conclude that the extra error caused by the domain decomposition is fully suppressed by appropriate overlapping sizes d_x and d_y and that the convergence is of second order. We can surely construct a special triangular mesh which gives an $O(h^{3/2})$–convergence. This was discussed in a previous paper [8].

The domain decomposition method has another great advantage. One can refine a macro–element to cope with the local property of the solution. For some solutions with boundary layers or interior layers, we must use locally refined mesh to reduce the global error. This is complicated to implement if one uses only one mesh over the domain. Using the domain decomposition method, it is easily perform by specifying a much finer mesh in some subregions.

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