Some Recent Developments in Domain Decomposition Methods with Nonconforming Finite Elements

Jinsheng Gu [1] and Xiancheng Hu [2]

1 Introduction

Domain decomposition methods have recently become an important focus in the field of computational mathematics due to the development of parallel computers. The nonconforming finite element methods are effective for solving partial differential equations derived from mechanics and engineering [3, 4, 5, 14]. But, there has been no extensive study of domain decomposition methods with nonconforming finite elements which lack global continuity. Therefore, a rather systematic investigation on domain decomposition methods with nonconforming elements is presented in this paper.

It is well–known that extension theorems play key role in domain decomposition analysis, especially in the case of nonoverlapping subregions. We also know that extension theorems hold for conforming elements [1, 16, 17]. Hence the domain decomposition analysis can be performed for the conforming finite element discrete problems. When domain decomposition methods with nonconforming elements are studied, a core question in domain decomposition analysis is "Do the extension theorems hold for nonconforming finite elements?"

For this reason, we have established extension theorems for nonconforming elements based on conforming interpolation operators and further error estimates of

[1] Faculty 404, Department of Jet Propulsion, Beijing University of Aeronautics & Astronautics, Beijing 100083, P.R. China.
[2] Department of Applied Mathematics, Tsinghua University, Beijing 100084, P.R. China.

Domain Decomposition Methods in Sciences and Engineering, edited by R. Glowinski *et al.*

nonconforming finite elements solution under weak conditions [6, 10, 12, 13]. The originality of the design of the nonoverlapping domain decomposition algorithms with *nonconforming* elements which are continuous at the midpoints of the element edges (such as the Crouzeix-Raviart elements), is that the internal cross points do not need to be handled. This leads to simplicity and high parallelism of our algorithms [6].

For the second order elliptic problems discretized by nonconforming element methods, all the domain decomposition algorithms, nonoverlapping or overlapping, are as efficient as their counterparts in the conforming cases, and even easier in implementation [6, 7]. For the Stokes problems discretized by the nonconforming mixed element methods [5, 14], several algorithms are presented and discussed in [11]. For fourth order elliptic problems, in the conforming case and in the Morley nonconforming discrete case, a series of algorithms have been developed. Many numerical results are consistent with the theoretical analysis; see [6].

The remainder of this paper consists of three sections. Extension theorems for nonconforming elements are presented in Sect. 2. Their applications and other results are indicated concisely in Sect. 3. Conclusions are given in Sect. 4.

2 Main results

We mainly consider the linear, selfadjoint Dirichlet elliptic problem of order $2m$ in variational form given by

$$u \in H_0^m(\Omega): \ a_\Omega(u,v) = (f,v), \quad \forall\, v \in H_0^m(\Omega). \tag{2.1}$$

Here, $m = 1, 2,\ \Omega \subset \Re^2$ is an open polygonal bounded domain and

$$(f,v) = \int_\Omega fv.$$

We assume that the bilinear form $a_\Omega(\cdot,\cdot)$, over Ω, satisfies the following standard conditions [17]:

$$\left\{ \begin{array}{l} a_\Omega(w,v) = a_\Omega(v,w) \\ a_\Omega(v,\ v) \geq c\|v\|^2_{H^m(\Omega)} \\ a_\Omega(w,v) \leq C\|w\|_{H^m(\Omega)}\|v\|_{H^m(\Omega)} \end{array} \right. \tag{2.2}$$

where c and C are positive constants.

Let $\Omega_h = \{e\}$ be a quasi–uniform mesh of Ω and let V_h be a finite element space associated with Ω_h. For (2.1), in the case $m = 1$, V_h is the Crouzeix–Raviart element space [5], or the piecewise quartic nonconforming rectangular element space [14], or the Wilson element space, or the Carey element space [3] or another nonconforming membrane element space. For (2.1), in the case $m = 2$, V_h is the Morley element space, or the Ziekenwicz element space, or the Adini element space, or another nonconforming plate element space[4]. Let

$$A(w,v) = \sum_{e\in\Omega_h} a_e(w,v)$$

and let

$$V_h^0 = \Big\{ v \in V_h :\ \text{the freedom of } v \text{ vanishes at each interpolation point } x \in \partial\Omega \Big\}.$$

Then the nonconforming finite element discrete problem for (2.1) is

$$u_h \in V_h^0 : \ A(u_h, v) = (f, v), \ \forall\, v \in V_h^0. \tag{2.3}$$

Let Γ be an open line segment in Ω such that $\Gamma \cap e = \emptyset$, $\forall\, e \in \Omega_h$ and let Ω be decomposed by Γ into two open subdomains, denoted by Ω_1 and Ω_2, which satisfies

$$\Omega_1 \bigcap \Omega_2 = \emptyset, \ \Omega_1 \bigcup \Omega_2 \bigcup \Gamma = \Omega, \ \overline{\Omega_1} \bigcup \overline{\Omega_2} = \overline{\Omega}.$$

For $k = 1, 2$, we make the following definitions:

$$A_k(w, v) = \sum_{e \in \Omega_k} a_e(w, v),$$

$$V_h^k = \Big\{ v \in V_h^0 : \text{ the degrees of freedom of } v \text{ vanish at each nodal point } x \in \Omega \backslash \overline{\Omega_k} \Big\},$$

$$V_h^{k,0} = \Big\{ v \in V_h^0 : \text{ the degrees of freedom of } v \text{ vanish at each nodal point } x \in \Omega \backslash \Omega_k \Big\},$$

$$\Phi_h = \Big\{ (v_1, v_2) : \ v_k \in V_h^k, \ A_k(v_k, w) = 0, \ \forall\, w \in V_h^{k,0}, \ k = 1, 2,$$
$$\text{the degrees of freedom of } \ v_1 \text{ and } v_2 \text{ are equal at each nodal point } x \in \Gamma \Big\}.$$

Theorem 2.1 ([6, 10, 12, 13]). *For the quasi-uniform mesh Ω_h, there exist two positive constants σ, τ, independent of the mesh parameter h, such that*

$$\tau A_2(v_2, v_2) \leq A_1(v_1, v_1) \leq \sigma A_2(v_2, v_2), \ \forall\, (v_1, v_2) \in \Phi_h$$

We can prove Theorem 2.1 by using a trace theorem, the regularity estimate of the elliptic problem, the additional finite element error estimate, the inverse inequality and using a conforming interpolation operator which forms a bridge between the nonconforming element space and the corresponding properly selected conforming element space. We omit its proof here. The interested reader are referred to [6, 10, 12, 13].

Theorem 2.1 plays a key role in the analysis of any two–subdomain nonoverlapping domain decomposition method for (2.3), see [7]. For example, the Dirichlet–Neumann alternating method, also known as the Marini–Quarteroni algorithm, can be applied to (2.3) with the same expression of the convergence factor, with σ, τ, in Theorem 2.1 as that in [15].

Theorem 2.1 is the so–called extension theorem in the two-subdomain case when (2.1) is a second order or fourth order problem. The extension theorems in the multi–subdomain case (with crosspoints), analogous to Lemma 3.5 [1] or Lemma 3.2 [17], have been established for second order problem, i.e. (2.1) with $m = 1$. It states that the strain energy of the discrete harmonic extension function of a subdomain does not exceed the sum of those of its adjacent (neighbouring) subdomains by at most a factor relevant to the maximum subdomain diameter (the coarse mesh parameter) H and the fine mesh parameter h. When V_h is a finite dimensional space whose elements (a function) are continuous at the mesh nodes, such as the Wilson element space or Carey element space (which are called *first kind nonconforming finite element space* for convenience), the factor is $c(1 + \ln \frac{H}{h})^2$. When V_h is a finite dimensional space whose element (a function) are continuous at each edge midpoint of $e \in \Omega_h$, such as the Crouzeix–Raviart space (which are called *second kind nonconforming finite element space*), the factor is $c(1+\ln \frac{H}{h}) \max(1+H^{-2}, 1+\ln \frac{H}{h})$. For the first kind nonconforming element space V_h, preconditioners can be constructed by substructuring which are as efficient as their counterparts in the conforming element case [6]. For the second kind nonconforming finite element space V_h, it is unnecessary and in fact impossible to calculate the values at the internal crosspoints. For this case, a number of simple and high parallel preconditioners and iterative domain decomposition algorithms have been developed [6].

3 Other results

1. In the two-subdomain nonoverlap case, the existing domain decomposition algorithms designed for the conforming finite element discrete problems (for (2.1) with $m = 1$) can be extended to the nonconforming case (for (2.1) with $m = 1, 2$) only after that the description has been changed [6,7,13]. In the multi–subdomain nonoverlapping case, we can construct preconditioners and iterative domain decomposition algorithms for the nonconforming finite element discrete problems (for (2.1) with $m = 1$ only) by revising properly those for the conforming element discrete problems [6]. Based on the extension theorems given above and other estimates, we can show that they are as efficient as their counterparts in the conforming element discrete case, and even easier in implementation.
2. The overlapping domain decomposition method (the parallel Schwarz alternating algorithm) for (2.3) has been studied when (2.1) is a second order or a fourth order problem. In each iteration, a coarse mesh problem is introduced and solved simultaneously with all the subproblems posed on the subdomains [8]. Its convergence is proved by using projection operator theory. When (2.1) is a second order problem, the convergence factor is independent of the fine mesh parameter h, and even of the coarse mesh parameter H, when the coarse mesh is properly employed.
3. The overlapping domain decomposition methods for second order and two dimensional nonselfadjoint elliptic problems discretized by the Crouzeix–Raviart elements have also been considered. We have shown that its convergence by the discrete maximum principle[9] under the condition that each internal angle of the triangular elements is no larger than $\frac{\pi}{2} - \alpha$ (α is a positive constant).
4. The nonconforming mixed finite element method has been applied to solving the following two dimensional stationary incompressible Stokes model problem with the kinetic viscosity coefficient 1

$$\begin{cases} -\Delta \mathbf{u} + \nabla p = f & \text{in } \Omega \\ \text{div } \mathbf{u} = 0 & \text{in } \Omega \\ \mathbf{u} = 0 & \text{on } \partial\Omega, \end{cases} \tag{3.1}$$

where Ω is a bounded polygonal domain, $\mathbf{u}$ is the velocity and p the pressure. The variational form of (3.1) is a saddle point problem. To discretize it, a piecewise constant element space and the second kind nonconforming finite element space are employed to approximate the pressure field and the velocity field, respectively;

see [5, 14]. An extension theorem for the discrete Stokes problem, analogous to Theorem 2.1, has been established based on the Brezzi–Babuska condition, the Stokes extension operator and Theorem 2.1. Then the two–subdomain nonoverlapping domain decomposition algorithms can be developed and analyzed correspondingly. As for the multi–subdomain cases, it can be handled using the techniques of Bramble, et al. [2] and the domain decomposition methods for the Laplace equation discretized by second kind nonconforming finite elements [11].

4 Conclusions

Through the systematic study of domain decomposition methods with nonconforming finite elements, we see that although the nonconforming finite element function lacks global continuity, a theoretical foundation can be established. The existing domain decomposition methods, developed in the conforming finite element discrete case, can be revised properly and extended to the nonconforming finite element discrete case. It is notable that some algorithms for the second kind nonconforming finite element discrete problems are simple and easy in implementation. But it remains an open problem how to design and analyze the nonoverlapping domain decomposition methods for fourth order elliptic problem in the multi–subdomain case.

REFERENCES

[1] Bramble J. H., Pasciak J. E. and Schatz A. H. (1986) The construction of preconditioners for elliptic problems by substructuring. I. *Math. Comp.* 47, 103–134.

[2] Bramble J. H., Pasciak J. E. and Schatz A. H. (1986) A preconditioning technique for indefinite systems resulting from mixed approximations of elliptic problems. *Math. Comp.* 50, 1–17.

[3] Carey G. F. (1976) An analysis of finite element equations and mesh subdivision. *Comput. Methods Appl. Mech. Engrg.* 9, 165–179.

[4] Ciarlet P. G. (1978) *The Finite Element Method for Elliptic Problems.* North–Holland, Amsterdam.

[5] Crouzeix M. and Raviart P. A. (1973) Conforming and nonconforming finite element methods for solving the stationary Stokes equations. *RAIRO Numer. Anal.* 7–R3, 33–76.

[6] Gu J. (1993) *Domain Decomposition Methods with Nonconforming Finite Elements.* Ph. D. thesis. (in Chinese). Dept. of Applied Math., Tsinghua University, China.

[7] Gu J. and Hu X. (1995) On domain decomposition methods in two–subdomain nonoverlap cases. *Chinese J. Num. Math. & Appl.* 17:1, 78–94.

[8] Gu J. and Hu X. (1994) On the overlap domain decomposition methods for elliptic problems in multi–subdomain case. *J. Tsinghua University.* 34:6, 50–57 (in Chinese).

[9] Gu J. and Hu X. (1995) Overlapping domain decomposition method for nonselfadjoint elliptic problems discretized by Crouzeix–Raviart elements. *Chinese J. Num. Math. & Appl.* 17:4.

[10] Gu J. and Hu X. (1995) Some estimates with nonconforming elements in the analysis of domain decomposition methods. UCD/CCM Report No. 39.

[11] Gu J. and Hu X. (1995) Domain decomposition methods for Stokes problem discretized by nonconforming mixed finite elements. *J. Beijing University of Aeronautics and Astronautics.* 21:4 (in Chinese).

[12] Gu J. and Hu X. (1994) On an essential estimate in the analysis of domain decomposition methods. *J. Comput. Math.* 12:2, 132–137.

[13] Gu J. and Hu X. (1994) Extension theorems for plate elements with applications. (Submitted to Math. Comp.). UCD/CCM Report No. 46.

[14] Han H. (1984) Nonconforming elements in the mixed finite element method. *J. Comput. Math.* 2:3, 223–233.

[15] Marini L. D. and Quarteroni A. (1989) A relaxation procedure for domain decomposition methods using finite elements. *Numer. Math.* 55, 575–598.

[16] Widlund O. B. (1987) An extension theorem for finite element spaces with three applications. In *Numerical Techniques in Continuum Mechanics*, Vol. 16. Braaunschweig Wiesbaden.

[17] Widlund O. B. (1988) Iterative substructuring methods: algorithms and theory for elliptic problems in the plane. in *Proceedings of 1st International Symposium on DDM.*