Chaotic Iterative Methods by Space Decomposition and Subspace Correction

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1 Preface

In this paper, we first construct an abstract chaotic algorithm based on the fundamental framework proposed in [2], [18]. We then prove the convergence property of the algorithm under realistic much receivable conditions and further present some convergence rate estimates with respect to the structure of space decomposition and the parameters of the inexact solvers. Finally, we apply the abstract theory to a concrete chaotic algorithm called S-CR method [12,13] and show the convergence rate can be estimated by $1-(1-\sqrt{1-\frac{C\omega_1(2-\omega)}{(1+H^{-2})(1+log\frac{H}{h})^2}}\,)^{m-1}$, where m is the number of the subproblems, and ω_1 and ω, are the relaxation parameters, and H and h, the diameters of the coarse and finite element triangulation. These results guarantee the effectiveness of a number of chaotic algorithms when executed on message-passing distributed memory multiprocessor systems.

2 Description of the Algorithm

In the following, we shall discuss iterative algorithms to approximate the solution of a linear equation

$$Au = f \tag{2.1}$$

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where A is a symmetric positive definite (SPD) operator defined on a finite dimensional Hilbert space V with inner product $(\cdot,\cdot)$ and induced norm $\|\cdot\|$. (2.1) usually comes from the discretization of positive definite self-adjoint elliptic problem, with a proper boundary condition, by finite element method, finite difference method, etc.

Let V be decomposed into m subspaces $V_i \subset V$, $1 \leq i \leq m$, such that

$$V = \sum_{i=1}^{m} V_i, \tag{2.2}$$

and, for simplicity, let $(\cdot,\cdot)_A = (A\cdot,\cdot)$ denote the new inner product on V with induced norm $\|\cdots\|_A$. For each k, we define $Q_k, P_k : V \to V_k$ as the orthogonal projection operators onto V_k associated with the inner product $(\cdot,\cdot)$ and $(\cdot,\cdot)_A$ respectively, and define $A_k : V_k \to V_k$ by

$$(A_k u_k, v_k) = (A u_k, v_k), \qquad u_k, v_k \in V_k. \tag{2.3}$$

A_k can be regarded as a restriction of A on V_k and is SPD in the inner product $(\cdot\ ,\cdot)$. It follows from the above definitions that

$$A_k P_k = Q_k A. \tag{2.4}$$

Then, based on the framework in [2],[18], we construct the following chaotic algorithm: Algorithm A. $u^0 \in V$ is an arbitrary initial value, assume $u^{k-1} \in V$ has been obtained. Then u^k is defined by

$$u^k = u^{k-1} + R_{\tau(k)} Q_{\tau(k)} (f - A u^{k-1}). \tag{2.5}$$

Here we have enumerated the iterative solution in time order, R_i is the inexact solver of the subproblem defined on V_i, i.e. $R_i \simeq A_i^{-1}$, R_i is SPD in the inner product $(\cdot\ ,\cdot)$, and $\tau(k)$ denotes the subscript of the subproblem used at the kth step. We assume that $\{\tau(k)\}_1^\infty$ include i, infinitely many times; here i is an arbitrary natural number, $1 \leq i \leq m$. It guarantees that iterated sequence $\{\ u^k\}$ is corrected by using the kth subproblem infinitely many times, which satisfies an intuitive convergence requirement. The above condition is named the admissible condition by L. Elsner et al [7],[8].

Let $E^k = u^k - u$ represent the error, then from (2.5) we have

$$E^k = (I - R_{\tau(k)} A_{\tau(k)} P_{\tau(k)}) E^{k-1}. \tag{2.6}$$

3 Abstract Results

Let $\omega = \max_{1\leq k\leq m} \rho(R_k A_k)$, $\omega_1 = \min_{1\leq k\leq m} \rho(R_k A_k)$, where $\rho(A)$ denotes the spectral radius of A. In what follows, we assume that $0 < \omega < 2$. Then we have

Lemma 3.1 ([2],[18]) *Under the above condition,*

$$\|I - R_k A_k P_k\|_A \leq 1 \tag{3.1}$$

where $1 \leq k \leq m$.

Lemma 3.2 *Let $M_1 = V_l$ be any element of $\{V_k\}_{k=1}^m$ (here $\{V_k\}_{k=1}^m$ is a finite set with sum V_k, $k = 1, 2, \cdots, m$ as its elements) and let M_2 be the sumspace of any subset of $\{V_k\}_{k=1}^m$ (i.e., $M_2 = \sum_{i=1}^l V_{t_i}$ for a subset $\{V_{t_i}\}_{i=1}^l \subset \{V_k\}_{k=1}^m$). Let $C_{M_1M_2}$ denotes the positive constant, such that, for any $v \in M_1 + M_2$, there exist $v_i \in M_i$, $i = 1, 2$ satisfying that $v = v_1 + v_2$ and $[\|v_1\|_A^2 + \|v_2\|_A^2] \le C_{M_1M_2}^2 \|v\|_A^2$. Then we have*

$$\|v\|_A^2 \le C_{M_1M_2}^2 [\|P_1'v\|_A^2 + \|P_2'v\|_A^2], \qquad v \in M_1 + M_2,$$

where P_i' denote the orthogonal projection operators onto M_i with respect to inner product $(\cdot, \cdot)_A$, $i = 1, 2$, respectively.

This is just the variant of P.L. Lions' Lemma [15].

Lemma 3.3 *Let $M_1 = V_l$, M_2, P_2' and $C_{M_1M_2}$ be defined as in Lemma 3.2. Then we have*

$$\|[I - R_lA_lP_l][I - P_2']v\|_A \le \sqrt{1 - \frac{\omega_1(2-\omega)}{C_{M_1M_2}^2}} \|v\|_A, \quad v \in M_1 + M_2.$$

Lemma 3.4 *Let V_{k_1} and V_{k_2} be any two different elements of $\{ V_k\}_{k=1}^m$. Then we have*

$$\|[I - R_{k_1}A_{k_1}P_{k_1}][I - R_{k_2}A_{k_2}P_{k_2}]v\|_A \le \sqrt{1 - \frac{\omega_1(2-\omega)}{9C_{k_1k_2}^2}} \|v\|_A, \quad v \in V_{k_1} + V_{k_2},$$

where $C_{k_1k_2}$ denotes the constant $C_{M_1M_2}$ given in Lemma 3.2 for $M_i = V_{k_i}$, $i = 1, 2$.

Because of the finite number of choices of M_1, M_2, V_{k_1} and V_{k_2}, let $\sigma \in (0, 1)$ denote the maximum value of $\sqrt{1 - \frac{\omega_1(2-\omega)}{C_{M_1M_2}^2}}$ and $\sqrt{1 - \frac{\omega_1(2-\omega)}{9C_{k_1k_2}^2}}$ in all possible situations. Then from Lemma 3.3 and Lemma 3.4, we have

Lemma 3.5 *Let $M_1 = V_l$ be any element of $\{V_k\}_{k=1}^m$, M_2 be the sum of any subset of $\{V_k\}_{k=1}^m$, and let V_{k_1}, V_{k_2} be any two different elements of $\{V_k\}_{k=1}^m$. Then we have*

$$\begin{cases} \|[I - R_lA_lP_l][I - P_2']v\|_A \le \sigma\|v\|_A, & v \in M_1 + M_2, \\ \|[I - R_{k_1}A_{k_1}P_{k_1}][I - R_{k_2}A_{k_2}P_{k_2}]v\|_A \le \sigma\|v\|_A, & v \in V_{k_1} + V_{k_2}. \end{cases} \tag{3.2}$$

Lemma 3.6 *Let $\{t_1, t_2\}$ be an arbitrary subset of $\{1, 2, \cdots, m\}$, then for arbitrary K natural numbers $\alpha_k \in \{t_1, t_2\}, k = 1, 2, \cdots, K$, and $\{t_1, t_2\} \subset \{\alpha_1, \alpha_2, \cdots, \alpha_K\}$, we have*

$$\| \prod_{k=1}^K (I - R_{\alpha_k}A_{\alpha_k}P_{\alpha_k})v\|_A \le \sigma\|v\|_A, \qquad v \in V_{t_1} + V_{t_2}, \tag{3.3}$$

where σ is defined as that in Lemma 3.5.

Lemma 3.6 follows from Lemma 3.1 and Lemma 3.5 easily.

Lemma 3.7 *For any integer l ($2 \le l \le m$), there exists a constant $\sigma^l \in (0, 1)$ such that, for arbitrary subset $\{t_1, t_2, \cdots, t_l\} \subset \{ 1, 2, \cdots, m\}$, arbitrary $\alpha_k \in \{t_1, t_2, \cdots, t_l\}$, $k = 1, 2, \cdots, K$ with $\{t_1, t_2, \cdots, t_l\} \subset \{\alpha_1, \alpha_2, \cdots, \alpha_K\}$, we have*

$$\| \prod_{k=1}^K (I - R_{\alpha_k}A_{\alpha_k}P_{\alpha_k})v\|_A \le \sigma^l\|v\|_A, \quad v \in \sum_{k=1}^K V_{t_k}, \tag{3.4}$$

where $\sigma^2 = \sigma$ *and* $\sigma^{l+1} = \sigma + (1-\sigma)\sigma^l$.

Proof. By induction. As l =2, the result follows from Lemma 3.6. Assume the result is true for l $(2 \le l < m)$, we want to prove the correctness for $l+1$.

For arbitrary $\{t_1, t_2, \cdots, t_{l+1}\} \subset \{1, 2, \cdots, m\}$, arbitrary $\alpha_k \in \{t_1, t_2, \cdots, t_{l+1}\}$, $k = 1, 2, \cdots, K$ with $\{t_1, t_2, \cdots, t_{l+1}\} \subset \{\alpha_1, \alpha_2, \cdots, \alpha_K\}$, $v \in \sum_{k=1}^{l+1} V_{t_k}$, consider the estimate of $\prod_{k=1}^{K}(I - R_{\alpha_k} A_{\alpha_k} P_{\alpha_k})v$. Without loss of generality, we may assume that $\alpha_1 = t_1$, $t_1 \notin \{\alpha_2, \alpha_2, \cdots, \alpha_K\}$. Otherwise, through a search process, we can find some $i(1 \le i \le K)$, such that $\{t_1, t_2, \cdots, t_{l+1}\} \subset \{\alpha_i, \alpha_{i+1}, \cdots, \alpha_K\}$, and $\alpha_i \notin \{\alpha_{i+1}, \alpha_{i+2}, \cdots, \alpha_K\}$. We might as well suppose that $\alpha_i = t_1$. Then from Lemma 3.1

$$\| \prod_{k=1}^{K} (I - R_{\alpha_k} A_{\alpha_k} P_{\alpha_k}) v \|_A \le \| \prod_{k=i}^{K} [(I - R_{\alpha_k} A_{\alpha_k} P_{\alpha_k}) v \|_A$$

which is converted to the estimate of the case for which the assumption holds.

Let $W = \sum_{k=2}^{l+1} V_{t_k}$, $e = \prod_{k=2}^{K}(I - R_{\alpha_k} A_{\alpha_k} P_{\alpha_k})v$. Then from the induction assumption, we have

$$\|e - P_{W^\perp} v\|_A = \| \prod_{k=2}^{K} (I - R_{\alpha_k} A_{\alpha_k} P_{\alpha_k})(v - P_{W^\perp} v) \|_A \le \sigma^l \|v - P_{W^\perp} v\|_A, \qquad (3.5)$$

where $P_{W^\perp}$ denotes the orthogonal projection operator from V onto $W^\perp$ with respect to the inner product $(\cdot, \cdot)_A$.

Let $\eta = \frac{\|e - P_{W^\perp} v\|_A}{\|v - P_{W^\perp} v\|_A}$ ($\eta = 0$ as $v - P_{W^\perp} v = 0$), then $0 \le \eta \le \sigma^l$.

Introduce the auxiliary function

$$\begin{cases} v^* = P_{W^\perp} v + \dfrac{1}{\eta}(e - P_{W^\perp} v), & (\eta > 0), \\ v^* = v, \quad (\eta = 0). \end{cases} \qquad (3.6)$$

It is easy to see that

$$\begin{cases} \|v^*\|_A^2 = \dfrac{1}{\eta^2} \|e - P_{W^\perp} v\|_A^2 + \|P_{W^\perp} v\|_A^2 = \|v\|_A^2, \\ e = \eta v^* + (1 - \eta) P_{W^\perp} v^*. \end{cases} \qquad (3.7)$$

Then,

$$\begin{aligned} &\| \prod_{k=1}^{K} (I - R_{\alpha_k} A_{\alpha_k} P_{\alpha_k}) v \|_A = \|(I - R_{\alpha_1} A_{\alpha_1} P_{\alpha_1}) e\|_A \\ &= \|(I - R_{\alpha_1} A_{\alpha_1} P_{\alpha_1})[\eta v^* + (1 - \eta) P_{W^\perp} v*]\|_A \\ &\le \eta \|v\|_A + (1 - \eta)\sigma \|v\|_A \le [\sigma + (1 - \sigma)\sigma^l] \|v\|_A. \end{aligned}$$

The last inequalities follow from (3.7), Lemma 3.1 and Lemma 3.5. Here we also use the fact that $v^* \in V_{t_1} + W$. Lemma 3.7 is now proved.

Let $l = m$, we get the following result:

Lemma 3.8 *For arbitrary natural numbers $\alpha_i \in \{1,2,\cdots,m\}$, $i = 1,2,\cdots,K$, and $\{1,2,\cdots,m\} \subset \{\alpha_1,\alpha_2,\cdots,\alpha_K\}$,*

$$\|\prod_{k=1}^{K}(I - R_{\alpha_k}A_{\alpha_k}P_{\alpha_k})\|_A \le \sigma^m < 1, \tag{3.8}$$

where $\sigma^m = 1-(1-\sigma)^{m-1}$.

Theorem 3.9 *Assume that V is split into m subspaces $\{V_k\}_{k=1}^m$ such that $V = \sum_{k=1}^m V_k$ and the inexact solvers R_k are chosen to be SPD on V_k with respect to the inner product $(\cdot,\cdot)$ for respectively, $k = 1,2,\cdots,m$, which also satisfy*

$$0 < \max_{1\le k\le m} \rho(R_kA_k) < 2.$$

Then Algorithm A is convergent. Furthermore, we have the following convergence rate estimates, i.e. if the iteration is corrected in each processor at least once in some iteration section, then the error will decrease according to the factor $1-(1-\sigma)^{m-1}$ after this iteration section, which is independent of the corrected times K and the concrete structure of this section.

Theorem 3.9 follows easily from Lemma 3.8.

4 An Application

Let Ω be a polygonal domain in R^2, for ease of exposition, and consider the problem

$$\begin{cases} -\sum_{i,j=1}^{2} \partial_i(a_{ij}\partial_j u) = f, & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega. \end{cases} \tag{4.1}$$

Here $\partial_i = \frac{\partial}{\partial x_i}$, and we assume that the matrix of $(a_{ij})_{2\times 2}$ with continuously differential coefficients, is symmetric for each $x \in \Omega$ and there exist two constants C_0, C_1 such that

$$C_0|\xi|^2 \le \sum_{i,j=1}^{2} a_{ij}(x)\xi_i\xi_j \le C_1|\xi|^2, \quad x \in \bar{\Omega}, \quad \xi \in R^2. \tag{4.2}$$

The variational form of (4.1) is

$$\begin{cases} u \in H_0^1(\Omega), \\ a(u,v) = (f,v), \qquad v \in H_0^1(\Omega). \end{cases} \tag{4.3}$$

Here $a(u,v) = \sum_{i,j=1}^{2}\int_\Omega a_{ij}\partial_i u \partial_j v dx$, and $(\cdot\ ,\cdot)$ denotes the L^2-inner product on Ω with induced norm $\|.\|$.

Let $\{E_i\}_{i=1}^m$ be the coarse quasi-uniform triangulation of Ω with size H, i.e., there exist constants C_2, C_3 not depending on H such that each triangle E_i is contained in (respectively, contains) a disk of radius C_3H (respectively C_2H). We further divide each E_i into smaller triangles to form a quasi-uniform finite element triangulation $\{K\}_{K\in T_h}$ of Ω with h as its diameter. Define

$$S^h(\Omega) = \{v \in C^0(\bar{\Omega}) : v|_K \text{ is linear, } K \in T_h\}, \qquad S_0^h(\Omega) = S^h(\Omega) \cap H_0^1(\Omega).$$

Then the finite element approximation of (4.3) reads as follows:

$$\begin{cases} u_h \in S_0^h(\Omega) \equiv V, \\ a(u_h, v) = (f, v), \qquad v \in V. \end{cases} \tag{4.4}$$

To give the chaotic algorithm in detail, we proceed to construct the following subdomains $\{\Omega_i\}_{i=1}^m$:

$$\Omega_i = \{\cup E_j : \bar{E}_j \cap \bar{E}_i \neq \emptyset, j = 1, 2, \cdots, m\}, \quad i = 1, 2, \cdots, m,$$

and define $V_i = H_0^1(\Omega_i) \cap V$, $i = 1, 2, \cdots, m$. Clearly, $V = \sum_{i=1}^m V_i$. Then we can apply Algorithm A of section 2 to solve (4.4), this is referred to as the S-CR method[12],[13]at this time. Here, the linear operator A in (2.1) is defined by $(Au, v) = a(u, v),\ u,\ v \in V$.

Lemma 4.1 *For the S-CR method, the constant $C_{M_1 M_2}$ given in Lemma 3.2 can be estimated as follows:*

$$C_{M_1 M_2}^2 \le C(1 + H^{-2})(1 + log\frac{H}{h})^2, \tag{4.5}$$

where the constant C depends only on the constants C_0, C_1, C_2, C_3 given before and the domain Ω.

Proof. Without loss of generality, assume that $M_1 = V_1 = S_0^h(\Omega_1)$, $M_2 = \sum_{i=1}^l V_{t_i}$ $= S_0^h(\tilde{\Omega}_1)$, where $\tilde{\Omega}_1 = \cup V_{t_i}$, for any open set F, $S^h(F) \equiv S^h(\Omega) \cap H_0^1(F)$, $S_0^h(F) \equiv S^h\ (F) \cap H_0^1(F)$. Then $M_1 + M_2 = S_0^h(\Omega_1 \cup \tilde{\Omega}_1)$. Let $\Omega' = \Omega_1 \cap \tilde{\Omega}_1$. Obviously Ω' is formed by some coarse triangles in $\{E_i\}_{i=1}^m$. For any $v \in M_1 + M_2$, let

$$v_1|_{\bar{\Omega}_1 \backslash \bar{\Omega}'} = v|_{\bar{\Omega}_1 \backslash \bar{\Omega}'}, \qquad v_2|_{\bar{\tilde{\Omega}}_1 \backslash \bar{\Omega}'} = v|_{\bar{\tilde{\Omega}}_1 \backslash \bar{\Omega}'}.$$

The construction of v_i on $\bar{\Omega}'$ is relatively complex. Define F^0 as the interior points set of F. For any coarse triangle E in Ω', if $\bar{E} \subset \bar{\Omega}' \setminus (\partial\Omega_1 \cap \partial\Omega')^0$, let $v_1\ |_{\bar{E}} = v$; otherwise, E has at least one vertex on $\partial\Omega_1 \cap \partial\Omega'$, we then define E^i, $i = 1, 2, 3$ as its three vertices, E_{12}, E_{13}, E_{23}, the related edges; thus we have the following two cases:

(1). There exist two vertices, e.g. E^2, E^3 on $\partial\Omega_1 \cap \partial\Omega'$. We first construct two linear functions $g(x), g_1(x)$ on E, such that, $g(E^i) = v(E^i), i = 1,2,3$, $g_1(E^1) = v(E^1)$, $g_1(E^2)$ $= g_1(E^3) = 0$; and then construct the following an auxiliary function $\tilde{v}\ (x) \in S^h(E)$ such that

$$\begin{cases} a(\tilde{v}, w) = 0, \qquad w \in S_0^h(E), \\ \tilde{v}|_{E_{12}, E_{13}} = v - g, \\ \tilde{v}|_{E_{23}} = 0. \end{cases} \tag{4.6}$$

Finally, we define

$$v_1|_E = g_1 + \tilde{v}. \tag{4.7}$$

(2). There exists just one vertex e.g. E^3 on $\partial\Omega_1 \cap \partial\Omega'$. Let g_1 be the linear function on E such that $g_1(E^i) = v(E^i)$, $i = 1, 2$, $g_1(E^3) = 0$, and let g be defined as above. Let $\tilde{v}$ be the auxiliary function in $S^h(E)$ such that

$$\begin{cases} a(\tilde{v}, w) = 0, & w \in S_0^h(E), \\ \tilde{v}|_{\partial E} = v - g. \end{cases} \tag{4.8}$$

Then we define v_1 on E as (4.7) too. Thus, we get the construction of v_1 on $\bar{\Omega}'$. The function v_2 on $\bar{\Omega}'$ is defined as $v - v_1$. From the definitions above, we know that $v_i \in M_i$, $i = 1, 2$, and $v = v_1 + v_2$.

Now we proceed with the estimate of $C_{M_1M_2}$. Clearly it suffices to consider the term

$$\int_E [\sum_{i,j=1}^{2} a_{ij}\partial_i v_1 \partial_j v_1] dx$$

for E belonging to the above two cases, since $v_1|_E$ =v for other coarse triangle E. We only give the estimate for case (1). From (4.2), (4.6), (4.7), the extension Lemma and maximum norm estimates given in [1], [6], we know that

$$\int_E [\sum_{i,j=1}^{2} a_{ij}\partial_i v_1 \partial_j v_1] dx \leq C[\|\nabla g_1\|_{0,E}^2 + \|v - g\|^2_{H_{00}^{1/2}(E_{12})} + \|v - g\|^2_{H_{00}^{1/2}(E_{13})}], \tag{4.9}$$

$$\|\nabla g_1\|_{0,E}^2 \leq C|v(E_1)|^2 \leq C[(1 + log\frac{H}{h})\|\nabla v\|_{0,E}^2 + H^{-2}\|v\|_{0,E}^2], \tag{4.10}$$

and

$$\|v - g\|^2_{H_{00}^{1/2}(E_{12})} + \|v - g\|^2_{H_{00}^{1/2}(E_{13})} \leq C(1 + log\frac{H}{h})^2\|\nabla v\|_{0,E}^2. \tag{4.11}$$

Here the definitions of the Sobolev's norms are the same as those in [1], [6]. From (4.8)-(4.10), we obtain

$$\int_E [\sum_{i,j=1}^{2} a_{ij}\partial_i v_1 \partial_j v_1] dx \leq C[(1 + log\frac{H}{h})^2\|\nabla v\|_{0,E}^2 + H^{-2}\|v\|_{0,E}^2]. \tag{4.12}$$

For case (2), the application of the above technique will lead to the same estimate (4.12). Hence, from (4.2), (4.12) and Poincare's inequality [1],[6], we finally have

$$\|v_1\|_A^2 = a(v_1, v_1) \leq C(1 + log\frac{H}{h})^2(1 + H^{-2})\|v\|_A^2.$$

The Lemma then follows.

From Theorem 3.9 and Lemma 4.1, we easily have the following result:

Theorem 4.2 *Let the subspaces V_k be chosen as above, and the inexact solvers R_k be SPD with respect to the inner product $(\cdot, \cdot)$(L^2-inner product) which also satisfies*

$$0 < \omega_1 = \min_{1 \leq k \leq m} \rho(R_k A_k) \leq \omega = \max_{1 \leq k \leq m} \rho(A_k P_k) < 2.$$

Then the S-CR method is convergent. Furthermore, we have the following error estimate, i.e. if the iteration is corrected at each processor at least once in some iteration section, then the error will have decreased according to the factor $1 - (1 - \sqrt{1 - \frac{C\omega_1(2-\omega)}{(1+H^{-2})(1+log\frac{H}{h})^2}})^{m-1}$ *after this iteration section. Here, the constant* C *is independent of* H, h, ω_1, ω *and* m.

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