Convergence Analysis of Parallel Domain Decomposition Algorithm for Navier-Stokes Equations

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1 Introduction

In the paper [1], we developed a Schwarz's domain decomposition algorithm and a convergence analysis for the stationary incompressible Navier-Stokes problem. But this is a serial algorithm. In this paper, we discuss a class of parallel algorithms. We consider the stationary Navier-Stokes equations with the Dirichlet boundary condition:

$$\begin{cases}
-\nu \Delta u + \sum_{j=1}^{N} u \frac{\partial u}{\partial x_j} + \text{grad}\, p = f \\
\text{div}\, u = 0 \quad \text{in} \ \Omega \\
u u|_{\partial \Omega} = 0,
\end{cases} \quad (1)$$

where $\Omega$ is a bounded domain of $\mathbb{R}^N (N = 2, 3)$ with a Lipschitz-continuous boundary $\partial \Omega$. The vector $u = \{u_i\}_{i=1}^{N}$ is the velocity of the fluid, $\nu > 0$ is its kinematic viscosity (assumed to be constant), $p$ its pressure and the vector $f = \{f_i\}_{i=1}^{N} \in [H^{-1}(\Omega)]^N$ the density of the body forces per unit mass. We introduce the following spaces:

$$X = [H^1_0(\Omega)]^N, \quad V = \{v|v \in X; \text{div}\, v = 0, \text{in} \ \Omega\},$$

$$M = L^2_0(\Omega) = \{q|q \in L^2(\Omega); \int_{\Omega} q\, dx = 0\}.$$

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The weak formulation of (1) is

\[(P)\]
\[
\begin{cases}
\text{Find } u \in V \text{ such that } \\
a_0(u,v) + a_1(u; u, v) = \langle f, v \rangle, \forall v \in V,
\end{cases}
\]
or

\[(Q)\]
\[
\begin{cases}
\text{Find } \{u, p\} \in X \times M \text{ such that } \\
a_0(u, v) + a_1(u; u, v) - b(p, v) = \langle f, v \rangle, \forall v \in X \\
b(q, u) = 0, \quad \forall q \in M,
\end{cases}
\]

where

\[a_0(u, v) = \nu(\text{grad} u, \text{grad} v), \quad a_1(u; v, w) = \sum_{i,j=1}^{N} \int_{\Omega} \frac{\partial u_i}{\partial x_j} u_j v_i dx,
\]

\[b(p, v) = (p, \text{div} v), \quad \langle f, v \rangle = \int_{\Omega} f \cdot v dx, \quad \forall u, v, w \in X.
\]

Problem (P) is equivalent to problem (Q) and there exists at least one solution[2,3].

2 The Parallel Algorithm

Assume that \(\Omega\) is split into \(m\) subdomains \(\Omega_j\):

\[\Omega = \bigcup_{j=1}^{m} \Omega_j \quad (m \geq 2),\]

satisfying

\[\exists \xi_j \in C_0^{+\infty}(\Omega_j), 0 \leq \xi_j \leq 1, \text{in } \Omega_j; \xi_j = 0, \text{in } \Omega \setminus \Omega_j,
\]
such that

\[\sum_{j=1}^{m} \xi_j = 1, \text{in } \Omega. \tag{2}\]

Later on, we denote by \(r(x)\) the number of subdomains to which the point \(x\) belongs.

Let

\[V_j = \{u \in [H_0^1(\Omega_j)]^N; \text{div} u = 0, \text{in } \Omega_j\} \quad (j = 1, 2, \cdots, m).
\]

and consider \(V_j\) as a closed subspace of \(V\) by extending its elements to \(\Omega \setminus \Omega_j\) by 0. Then we have the following simple result.

**Lemma 1** ([1]). *Under the condition (2), we have*

\[V = V_1 + V_2 + \cdots + V_m,
\]

*and, for all \(v \in V\), there exist \(v_j \in V_j\) such that*

\[v = \sum_{j=1}^{m} v_j \quad \text{and} \quad \max_{1 \leq j \leq m} \| v_j \|_1 \leq C_0 \| v \|_1,
\]
where $C_0$ is some positive constant.

The algorithm is designed as follows:

**Step 0.** Choose an initial value $u^0 \in V$.

**Step 1.** For $j = 1, 2, \cdots, m$ and $n \geq 0$, solve in parallel the following subproblems:

\[
\begin{cases}
\text{Find } u_j^{n+1} \in V_j \text{ and } p_j^{n+1} \in L_0^2(\Omega_j) \text{ such that } \\
a_0(u_j^{n+1}, v)_{\Omega_j} + a_1(u_j^{n+1}; u_j^{n+1}, v)_{\Omega_j} - b(p_j^{n+1}, v)_{\Omega_j} \\
= <f, v>_{\Omega_j}, \quad \forall v \in [H_0^1(\Omega_j)]^N, \\
b(q, u_j^{n+1})_{\Omega_j} = 0, \quad \forall q \in L_0^2(\Omega_j), \\
p_j^{n+1} = 0, \quad \text{in } \Omega \setminus \Omega_j. 
\end{cases}
\tag{3}
\]

**Step 2.** Choose $\theta_j \in (0, 1)$ such that $\sum_{j=1}^{m} \theta_j = 1$ and for $n \geq 0$ set

\[
u^{n+1} = \sum_{j=1}^{m} \theta_j u_j^{n+1}, \quad p^{n+1}(x) = \frac{1}{r(x)} \sum_{j=1}^{m} p_j^{n+1}.
\tag{4}
\]

Set $n = n + 1$, go to Step 1.

Here the notations are given by:

\[
a_0(u, v)_{\Omega_j} = \nu \int_{\Omega_j} \nabla u \cdot \nabla v \, dx, \quad a_1(u; v, w)_{\Omega_j} = \sum_{i,j=1}^{N} \int_{\Omega_j} u_i \frac{\partial v_i}{\partial x_j} w_j \, dx, \\
b(p, v)_{\Omega_j} = \int_{\Omega_j} p \nabla v \, dx, \quad <f, v>_{\Omega_j} = \int_{\Omega_j} f \cdot v \, dx.
\]

3 **The Convergence**

Let $H$ be a Hilbert space, $F$ a differentiable mapping from $H$ into $H'$ (the dual space of $H$), $DF(\cdot)$ its derivative, and let $u \in H$ be a solution of the equation $F(u) = 0$. We say that $u$ is a nonsingular solution if there exists a constant $\gamma_0 > 0$ such that

$$
\|DF(u) \cdot v\|_H \geq \gamma_0 \|v\|_H \quad \forall v \in H.
$$

For the Navier-Stokes problem, we define a $C^2$-mapping $F(\cdot) : V \to V'$ as follows:

$$
<F(u), v> = a_0(u, v) + a_1(u; u, v) - <f, v> \quad \forall u, v \in V.
$$

Clearly, $F(\cdot)$ is infinitely differentiable in $V$ and its derivative $DF(u) \in L(V; V')$ is given by:

$$
<F(u), v, w> = a_0(v, w) + a_1(v; u, w) + a_1(v; u, w).
$$

As a consequence, problem (P) can be rewritten as:

\[
\begin{cases}
\text{Find } u \in V \text{ such that } \\
<F(u), v> = 0, \quad \forall v \in V.
\end{cases}
\tag{5}
\]
In addition, we introduce the abstract Stokes operator: \( A \in \mathcal{L}(V; V') \) defined by

\[
< Au, v > = a_0(u, v), \quad \forall u, v \in V.
\]

Obviously, \( A \) is symmetric, \( V \)-elliptic, and there exists \( A^{-1} \in \mathcal{L}(V'; V) \). Furthermore, the mapping \( f \in V' \rightarrow \| f \|_{V'} = < A^{-1} f, f >^\frac{1}{2} \) is a norm on \( V' \) that is equivalent to the dual norm \( \| \cdot \|_V \).

We define the functional

\[
J(v) = \frac{1}{2} \| F(v) \|_{V'}^2.
\]

Then problem (5) is equivalent to:

\[
\begin{align*}
\text{Find } & u \in V \text{ such that } \\
J(u) & = \inf_{v \in V} J(v).
\end{align*}
\]

**(Lemma 2.** Let \( u^* \) be a nonsingular solution of Navier-Stokes problem (1). Then the functional

\[
J(v) = \frac{1}{2} < A^{-1} F(v), F(v) >
\]

is strictly convex in a neighborhood of \( u^* \) and weakly lower semicontinuous. This means that there exist two constants, \( \rho > 0 \) and \( \alpha > 0 \), such that

\[
D^2 J(v) \cdot (w, w) \geq \alpha \| w \|_V^2, \quad \forall v \in S(u^*; \rho), \forall w \in V.
\]

Here

\[
S(u^*; \rho) = \{ v \in V; \| v - u^* \|_V \leq \rho \}.
\]

**Proof.** The strict convexity of \( J(v) \) in a neighborhood of a nonsingular solution \( u^* \) can be found in [3].

Now, we will prove the weak lower semicontinuity. To do this, let \( \{ v_i \} \) be a weakly convergent sequence in \( V \). Assume that \( v_i \rightarrow v^* \) weakly in \( V \). We then have

\[
\lim_{i \rightarrow \infty} < F(v_i), v > = < F(v^*), v > \quad \forall v \in V.
\]

(see chapter 9 in [4]). Let \( A^{-1} F(v_* - g_i) = g_* - g_i, A^{-1} F(v_i) = g_i \), i.e.

\[
Ag_* = F(v_*), \quad Ag_i = F(v_i), \\
< A(g_* - g_i), v > = < F(v_* - F(v_i), v > = a_0(g_* - g_i, v).
\]

Hence

\[
\lim_{i \rightarrow \infty} a_0(g_* - g_i, v) = \lim_{i \rightarrow \infty} < F(v_* - F(v_i), v > = < F(v_* - F(v_*), v = 0 \quad \forall v \in V.
\]

This implies \( g_i \rightarrow g_* \) weakly in \( V \).

Also, with \( Ag = F(v) \)

\[
J(v) = < A^{-1} F(v), F(v) > = < g, Ag > = a_0(g, g),
\]
since
\[ a_0(g_i - g_*, g_i - g_*) \geq 0, \quad a_0(g_i, g_i) \geq 2a_0(g_i, g_*) - a_0(g_*, g_*), \]
\[ \lim_{i \to \infty} \inf_i a_0(g_i, g_i) \geq a_0(g_*, g_*). \]

It follows that
\[ \lim_{i \to \infty} \inf_i J(v_i) \geq J(v). \]
\[ J(v_i) \text{ is weakly lower semicontinuous.} \]

Let \( v^0 \) be an initial value in \( S(u^*, \rho) \), and let \( R = J(v^0) \). It is clear that \( D = \{ v \in V; J(v) \leq R \} \cap \gamma S(u^*, \rho) \) is not empty because of \( v^0 \in S(u^*, \rho) \), and functional \( J(v) \) is strictly convex in \( D \). Its derivative \( DJ(v) \) is uniformly continuous in \( D \). Therefore there exist constants \( \alpha, C > 0 \) such that (7)
\[ J(v) - J(u) - \langle DJ(u), v - u \rangle \geq \alpha \| v - u \|_1^2, \quad \forall u, v \in D \] (8),
and
\[ \| DJ(v) - DJ(u) \|_{V'} \leq C \| v - u \|_1 \quad \forall u, v \in D. \] (9)

Now, we consider a local minimization problem:
\[
\begin{cases}
\text{Find } u \in D \\
\text{such that } J(u) = \inf_{v \in D} J(v)
\end{cases}
\] (10)

**Theorem 1.** There exists a unique solution \( u^* \) of (10) which satisfies (6).

**Proof.** By using lemma 2, \( J(v) \) is weakly lower semicontinuous in \( V \) and \( D \) is a closed subset in \( V, J(v) \geq 0 \). Then the generalized Weierstrass theorem shows that (10) has a unique solution \( u^* \). \( \square \)

Clearly, the domain decomposition algorithm for (10) consists of
\[
(P_n) \begin{cases}
\text{Find } u^{n+1}_j \in \{ u^n + V_j \} \cap D \\
\text{such that } a_0(u^{n+1}_j, v)_{\Omega_j} + a_1(u^{n+1}_j, u^{n+1}_j, v)_{\Omega_j} = \langle f, v \rangle_{\Omega_j} \quad \forall v \in V_j.
\end{cases}
\] (11)

\((P_n)\) is equivalent to
\[
\begin{cases}
u^{n+1}_j = \arg\inf_{v \in V_j} J(u^n + v) \quad \text{in } \Omega_j \quad v^n + v \in D \\
u^{n+1}_j = 0 \quad \text{in } \Omega \setminus \Omega_j \\
u^{n+1}_j = u^n + v^{n+1}_j \quad \text{in } \Omega.
\end{cases}
\]

Similarly, (11) has a unique solution sequence \( \{u^n_j\} \).

**Theorem 2.** Assume that \( \{u^n, p^n\} \) is an isolated solution of the N-S Eqs. Then the sequence \( \{u^n, p^n\} \), defined by (11) and (4), converges strongly in \( X \times M \) to \( \{u^*, p^*\} \).

**Proof** It follows from (11) that
\[ J(u^{n+1}_j) \leq J(u^n), \quad u^{n+1}_j \in D \quad (n \geq 0, j = 1, 2, \ldots, m). \]
Since $J(v)$ is convex in $D$, we have

$$J(u^{n+1}) = J\left(\sum_{j=1}^{m} \theta_j u_j^{n+1}\right) \leq \sum_{j=1}^{m} \theta_j J(u_j^{n+1}) \leq J(u^n), \quad (n \geq 0).$$

(12)

Therefore there exists a $q \in R^1$ such that $J(u^n) \to q$, as $n \to +\infty$. Thus, \{u^n\} $\subset D$. Furthermore, (11) yields

$$\begin{cases}
< DJ(u_j^{n+1}), v_j >= 0 & \forall v_j \in V_j \\
u_j^{n+1} - u^n \in V_j & (j = 1, 2, \cdots, m; n \geq 0).
\end{cases}$$

(13)

Combining (11) with (7), we deduce

$$J(u^n) - J(u_j^{n+1}) \geq \alpha \| u^n - u_j^{n+1} \|_1^2.$$  

For $\theta_j \in (0, 1)$ satisfying $\sum_{j=1}^{m} \theta_j = 1$, we have

$$\alpha \sum_{j=1}^{m} \theta_j \| u^n - u_j^{n+1} \|_1^2 \leq J(u^n) - \sum_{j=1}^{m} \theta_j J(u_j^{n+1})$$

$$\leq J(u^n) - J(u^{n+1}) \to 0, \text{ as } n \to +\infty.$$  

Therefore,

$$\| u^n - u_j^{n+1} \|_1 \to 0 \quad (j = 1, 2, \cdots, m), \quad \text{as } n \to +\infty.$$  

By using Lemma 1, for each $v \in V$, there exist $v_j \in V_j$ such that

$$v = \sum_{j=1}^{m} v_j \quad \text{and} \quad \max_{1 \leq j \leq m} \| v_j \|_1 \leq C_0 \| v \|_1.$$  

Combining this with (7) and (8), we deduce

$$| < DJ(u^n), v > | = | \sum_{j=1}^{m} < DJ(u^n), v_j > | = | \sum_{j=1}^{m} < DJ(u^n) - DJ(u_j^{n+1}), v_j > |$$

$$\leq C_0 \sum_{j=1}^{m} \| DJ(u^n) - DJ(u_j^{n+1}) \|_V \| v \|_1 \leq C_0 C \sum_{j=1}^{m} \| u^n - u_j^{n+1} \|_1 \| v \|_1.$$  

Hence

$$\| DJ(u^n) \|_V \leq C_0 C \sum_{j=1}^{m} \| u^n - u_j^{n+1} \|_1 \to 0, \quad \text{as } n \to +\infty.$$  

Since $u^*$ is a solution of problem (6), we find

$$J(u^*) \leq J(u^n) \quad (n \geq 0).$$  

Using again (7), we get

$$\alpha \| u^n - u^* \|_1^2 \leq J(u^*) - J(u^n) + < DJ(u^n), u^n - u^* >.$$  

$$\| u^n - u^* \|_1^2 \leq J(u^*) - J(u^n) + < DJ(u^n), u^n - u^* >.$$  

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\[ \leq \| DJ(u^n) \|_V \| u^n - u^* \|_1. \]

Therefore
\[ \| u^n - u^* \|_1 \leq \frac{1}{\alpha} \| DJ(u^n) \|_V \to 0, \quad n \to +\infty. \]

Thus, the sequence \( \{u^n\} \) converges strongly in \( V \) to the isolated solution \( u^* \) of problem (6).

Next, we consider the convergence of the pressure sequence \( \{p^n\} \).

Let the pair \( \{u^*, p^*\} \) be a solution of Problem (Q). We know that
\[ \| u^n - u^* \|_1 \to 0, \| u_j^{n+1} - u^* \|_1 \to 0, \quad n \to +\infty (j = 1, 2, \cdots, m). \]

Moreover, using (Q), we have for any \( v_j \in [H^1_0(\Omega_j)]^N \) with \( v_j = 0 \) in \( \Omega \setminus \Omega_j \):
\[ b(p_j^{n+1} - p_j, v_j) = a_0(u_j^{n+1} - u^*, v_j) + a_1(u_j^{n+1} - u^*, v_j) \]
and
\[ -a_1(u^*, v_j) \to 0, \quad n \to +\infty. \]

Similarly, if \( \Omega_i \cap \Omega_j \neq \emptyset \), then we have for any \( v_{ij} \) in \( [C_0^\infty(\Omega_i \cap \Omega_j)]^N \) with \( v_{ij} = 0 \) in \( \Omega \setminus (\Omega_i \cap \Omega_j) \):
\[ b(p_i^{n+1} - p_j^{n+1}, v_{ij}) \to 0, \quad n \to +\infty. \]

Since \( C_0^\infty(\Omega_i \cap \Omega_j) \) is dense in \( L^2(\Omega_i \cap \Omega_j) \), we get
\[ b(p_i^{n+1} - p_j^{n+1}, v_{ij}) \to 0, \quad \forall v \in [H^1(\Omega_i \cap \Omega_j)]^N, \quad n \to +\infty. \]

Hence, we get for all \( v_j \in [H^1_0(\Omega_j)]^N \), with \( v_j = 0 \) in \( \Omega \setminus \Omega_j \):
\[ b(p^{n+1} - p^*, v_j)_{\Omega_j} = b(p_j^{n+1} - p^*, v_j)_{\Omega_j} + b(\frac{1}{r(x)} \sum_{k=1}^M p_k^{n+1} - p_j^{n+1}, v_j)_{\Omega_j} \]
\[ = b(p_j^{n+1} - p^*, v_j)_{\Omega_j} + \frac{1}{r(x)} \sum_{k=1}^m b(p_k^{n+1} - p_j^{n+1}, v_j)_{\Omega_j \cap \Omega_k} \to 0, \quad n \to +\infty. \]

Set \( v_j = \xi_j v \), for all \( v \in [H^1_0(\Omega)]^N \). We assert that
\[ b(p^{n+1} - p^*, v) = \sum_{j=1}^m b(p^{n+1} - p^*, v_j)_{\Omega_j} \to 0, \quad n \to +\infty, \]
i.e. \( \forall v \in [H^1_0(\Omega)]^N \), \( (p^{n+1} - p^*, \text{div} v) \to 0 \) as \( n \to +\infty \). Since \( \text{div} \) maps \([H^1_0(\Omega)]^N\) onto \( L^2_0(\Omega|[3]) \), we obtain
\[ p^n \rightharpoonup p^* \quad \text{weakly in} \quad M \quad \text{as} \quad n \to +\infty. \]

The inf-sup condition [3] yields
\[ p^n \rightharpoonup p^* \quad \text{strongly in} \quad M \quad \text{as} \quad n \to +\infty. \]

Hence, Theorem 2 is valid.
REFERENCES


