Variational Inequalities for Navier-Stokes Flows Coupled with Potential Flow through Porous Media

H.Kawarada*, H.Fujita** and H.Kawahara***

* University of Chiba
** Meiji University
*** Kawasaki Steel System Development

1. Introduction

The domain is made of two parts; In one part, the flow obeys the Navier-Stokes equations. In the second part, the flow is potential and driven by Darcy's law. On some parts of the interface between them, the fluid flows into both sides under the control of the threshold to the potential of friction type. This threshold describes the effect of surface tension of the fluid in capilaries which are regularly arrayed in the neighbourhood of the boundary. We show how this problem is formulated by a variational inequality: existence of solution, approximation by means of fictitious domain method via singular perturbations and numerical simulations are studied and presented.

2. Formulation

2.1. The geometry

The problem is discussed in R^2. The geometry is the rectangle $(0, L) \times (0, a)$ for the Navier-Stokes part and the rectangle $(0, L) \times (a, 2a)$ for the Darcy part, which we denote by Ω_0 and Ω_1 respectively. $(0, L) \times \{a\}$ is an interface between Ω_0 and Ω_1, which we denote by γ . The Darcy boundary is the upper horizontal one plus the halves of the vertical one, the union of which we denote by Γ_1.

The Navier-Stokes boundary is composed of the inflow boundary Γ_{in} and the outflow boundary Γ_{out}, which are the remaining two halves of the vertical boundaries, and the lower horizontal boundary Γ_w.

2.2. Notations

p : pressure ($p = p_k$ in $\Omega_k, k = 0, 1$)
u: velocity ($u = u_k$ in $\Omega_k, k = 0, 1$)
ϕ : potential for the flow in Ω_1 ($\phi = p_1 + \rho g x_2,\ \rho g = 1$)
n : unit outward normal vector to the boundary of Ω_0
n_1 : unit outward normal vector to the boundary of Ω_1
ν : effective viscosity (inverse of Reynolds number)
k : permeability coefficient of the flow in Ω_1
c : resistance coefficient of the flow in Ω_1 (inverse of k)

Domain Decomposition Methods in Sciences and Engineering, edited by R. Glowinski *et al.*

g_n : threshold parameter (> 0) controlling the occurence of penetration on γ
f : external force acted in Ω_0 ($f = f_0 - \nabla x_2$)
β : velocity profile of Poiseuille type defined on Γ_{in}
ϵ : singular perturbation parameter (> 0)
u_n : normal component of u
u_t : tangential component of u
$S_n = \nu \frac{\partial u_{0n}}{\partial n} - p_0 + p_1$: difference between normal stress of the flow in Ω_0 and pressure of the flow in Ω_1
$S_t = \nu \frac{\partial u_{0t}}{\partial n}$: tangential stress of the flow in Ω_0
χ : characteristic function of Ω_1 in Ω.

2.3. The model problem

The model problem is described as follows ;

Find

$$u = \begin{cases} u_0 & \text{in } \Omega_0, \\ u_1 & \text{in } \Omega_1 \end{cases}$$

and

$$p = \begin{cases} p_0 & \text{in } \Omega_0, \\ p_1 & \text{in } \Omega_1 \end{cases}$$

such that

$$-\nu\Delta u_0 + (u_0 \cdot \nabla) u_0 + \nabla p_0 = f \quad \text{in } \Omega_0 \tag{2.1}$$

$$\nabla \cdot u_0 = 0 \quad \text{in } \Omega_0 \tag{2.2}$$

$$c\, u_1 + \nabla p_1 = -\nabla x_2 \quad \text{in } \Omega_1 \tag{2.3}$$

$$\nabla \cdot u_1 = 0 \quad \text{in } \Omega_1 \tag{2.4}$$

$$|S_n| \leq g_n \quad \text{on } \gamma \tag{2.5}$$

$$g_n\, |u_n| + S_n \cdot u_n = 0 \quad \text{on } \gamma \tag{2.6}$$

$$S_t = 0 \quad \text{on } \gamma \tag{2.7}$$

$$u_{0n} = u_{1n} \quad \text{on } \gamma \tag{2.8}$$

$$u_0 = \beta \quad \text{on } \Gamma_{in} \tag{2.9}$$

$$u_0 = 0 \quad \text{on } \Gamma_w \tag{2.10}$$

$$\nu \frac{\partial u_{on}}{\partial n} - p_0 = 0 \quad \text{on } \Gamma_{out} \tag{2.11}$$

$$u_{ot} = 0 \quad \text{on } \Gamma_{out} \tag{2.12}$$

$$u_{1n_1} = 0 \quad \text{on } \Gamma_1. \tag{2.13}$$

Hereafter we denote (2.1) - (2.13) by (PDEF).

Remark 2.1. The physical meaning of (2.5) and (2.6) is following [1], [3] , [4];
If $|S_n| < g_n$, then $u_n = 0$ on γ ,
If $|S_n| = g_n$, then $u_n = 0$ or $u_n \neq 0, S_n \cdot u_n < 0$ on γ.

One easily check that
(i) $u_n \to 0$ as $g_n \to +\infty$;
(ii) $S_n \to 0$ as $g_n \to 0$.

Remark 2.2. Let us note that (2.2), (2.11) and (2.12) mean $\frac{\partial u_{0n}}{\partial n} = 0$. Then (2.11) becomes $p_0 = 0$.

Remark 2.3. Obviously we see

(2.14) $$u_1 = -k\nabla\phi \quad \text{in} \quad \Omega_1 \ .$$

Substituting (2.14) into (2.4), we have

(2.15) $$\Delta\ \phi = 0 \quad \text{in} \quad \Omega_1 \ .$$

(2.8) and (2.13) mean

(2.16) $$-k\frac{\partial \phi}{\partial n} = u_{on} \quad \text{on}\ \gamma$$

(2.17) $$\frac{\partial \phi}{\partial n_1} = 0 \quad \text{on}\ \Gamma_1 \ .$$

Furthermore, ϕ satisfies the related one with respect to (2.5) and (2.6) on γ . Here we should note that the solvability of the problem (2.15), (2.16) and (2.17) leads to $\int_\gamma u_{on}\ ds = 0$.

If one solves this problem, the solution ϕ_0 obtained has inevitably an uncertainty of the constant .

Let normalize ϕ_0 such that $\int_{\Omega_1} \phi_0\ dx = 0$ and let represent ϕ as the solution of the coupled system (PDEF) by $\phi = \phi_0 + d$, where d is an unknown constant to be determined according to (2.11). One method to fix d is shown in the algorithm solving $(\mathrm{NM})_1$.

3. Reformulation by variational inequalities

3.1. Coupled formulation

Define

$$a_k(u,v) = \int_{\Omega_k} \nabla u \cdot \nabla v\ dx,$$

$$(u,v)_k = \int_{\Omega_k} u \cdot v\ dx \quad \text{for}\ k = 0,1$$

and

$$J(v) = \int_\gamma g_n\ |v_n|\ d\Gamma \ .$$

Let $K_{0\sigma} = \{\ v \mid v = \beta\ \text{ on } \Gamma_{in},\ v = 0\ \text{ on } \Gamma_w,\ v_t = 0\ \text{ on } \Gamma_{out}\ \} \cap\ H^1_\sigma(\Omega_0)$

where $H^1_\sigma(\Omega_0)$ is the solenoidal subspace of $H^1(\Omega_0)$ for vector functions and $P = H^1(\Omega_1) = P_0 + \{1\}$ where $P_0 = \{ \ \eta \in H^1(\Omega_1) \mid \int_{\Omega_1} \eta \ dx = 0 \ \}$.

If we couple (2.1), (2.2) and (2.15) under the boundary conditions prescribed on γ, we have

Find $u \in K_{0\sigma}$ and $\phi = \phi_0 + d \in P$ ($\phi_0 \in P_0, \ d \in R$) such that

$$\nu \ a_0(u, v-u) + J(v) - J(u) + \Big((u \cdot \nabla)u, v-u\Big)_0 \tag{3.1}$$

$$\geq - \int_\gamma \phi(v_n - u_n) \ d\Gamma + \int_\gamma a(v_n - u_n) \ d\Gamma - (f, v-u)_0 \ , \quad \forall v \in K_{0\sigma}$$

$$k \ (\nabla\phi, \ \nabla\eta)_1 = \int_\gamma u_n \ \eta \ d\Gamma \ , \quad \forall \eta \in P \ . \tag{3.2}$$

We denote (3.1) and(3.2) by $(VIF)_1$.

Remark 3.1. When the Reynolds number is sufficiently small, the existence and uniqueness theorem on $(VIF)_1$ is obtained .

Remark 3.2. If we use the fictitious domain method stated in [2], (3.1) is approximated by singularly perturbed problem defined in Ω. In this case, Ω_1 plays a role of the fictitious domain in place of the Darcy part.

3.2. Uncoupled formulation

Let

$$K_\sigma = \{ \ v \in L^2_\sigma(\Omega) \cap H^1_\sigma(\Omega_0) \mid v = \beta \ \text{ on } \Gamma_{in}, \ v = 0 \ \text{ on } \Gamma_w,$$

$$v_t = 0 \ \text{ on } \Gamma_{out}, \ v_{n1} = 0 \ \text{ on } \Gamma_1 \ \} \ .$$

Where $L^2_\sigma(\Omega)$ is the solenoidal subspace of $L^2(\Omega)$ for vector functions.

Find $u \in K_\sigma$ such that

$$\nu \ a_0(u, \ v-u) + J(v) - J(u) + \Big((u \cdot \nabla)u, v-u\Big)_0 + (c \ u, v-u)_1 \tag{3.3}$$

$$\geq \ (f, v-u)_0 - (\nabla x_2, v-u)_1 \ , \quad \forall v \in K_\sigma \ .$$

We denote (3.3) by $(VIF)_2$.

Remark 3.3. One can check the equivalence relation among (PDEF), $(VIF)_1$ and $(VIF)_2$ and the unique existence of the solution of $(VIF)_2$ by use of the standard arguments in the theory of the variational inequalities [3] .

4. Singularly perturbed approximation for $(VIF)_2$

Define

$$J_\epsilon(v) = \int_\gamma g_n \ \psi_\epsilon(u_n) \ ds \ ,$$

where $\psi_\epsilon(t) = \int_0^t \ \tanh(\frac{s}{\epsilon}) \ ds$.

Let

$$\widetilde{K_\sigma} = \{ \ v \in H^1_\sigma(\Omega) \mid v = \beta \text{ on } \Gamma_{in}, \ v = 0 \ \text{ on } \Gamma_w, \ v_t = 0 \text{ on } \Gamma_{out}, \ v = 0 \text{ on } \Gamma_1 \ \} .$$

Find $u \in K_\sigma$ such that

$$\text{(4.1)} \quad \nu \ a_0(u, v-u) + \epsilon \ a_1(u, v-u) + \Big((u \cdot \nabla)u, v-u\Big)_0 + J_\epsilon(v) - J_\epsilon(u) + (c\,u, v-u)_1$$

$$\geq \ (f, v-u)_0 - (\nabla x_2, v-u)_1, \ \ \forall v \in \widetilde{K_\sigma}.$$

We denote (3.4) by $(VIF)_{2\epsilon}$ which reduces to the following formulation $(PDEF)_\epsilon$ [6] ;

$$\text{(4.2)} \qquad -\nu \ \Delta u_0 + (u_0 \cdot \nabla)u_0 + \nabla p_0 = f \quad \text{in } \Omega_0,$$

$$\text{(4.3)} \qquad \nabla \cdot \ u_0 = 0 \quad \text{in } \Omega_0,$$

$$\text{(4.4)} \qquad -\epsilon \ \Delta \ u_1 + c u_1 + \nabla p_1 = -\nabla x_2 \quad \text{in } \Omega_1,$$

$$\text{(4.5)} \qquad \nabla \cdot u_1 = 0 \quad \text{in } \Omega_1,$$

$$\text{(4.6)} \qquad \nu \cdot \frac{\partial u_{on}}{\partial n} - p_0 = \epsilon \cdot \frac{\partial u_{1n}}{\partial n} - p_1 + g_n \tanh(\frac{u_n}{\epsilon}) \quad \text{on } \gamma$$

$$\text{(4.7)} \qquad \nu \cdot \frac{\partial u_{ot}}{\partial n} = \epsilon \cdot \frac{\partial u_{1t}}{\partial n} \quad \text{on } \gamma$$

$$\text{(4.8)} \qquad u_0 = u_1 \quad \text{on } \gamma$$

$$\text{(4.9)} \qquad u_0 = \beta \quad \text{on } \Gamma_{in}$$

$$\text{(4.10)} \qquad u_0 = 0 \quad \text{on } \Gamma_w$$

$$\text{(4.11)} \qquad \nu \cdot \frac{\partial u_{0n}}{\partial n} - p_0 = 0 \quad \text{on } \Gamma_{out}$$

$$\text{(4.12)} \qquad u_{ot} = 0 \quad \text{on } \Gamma_{out}$$

$$\text{(4.13)} \qquad u_1 = 0 \quad \text{on } \Gamma_1.$$

Remark 4.1. By virtue of a well-known arguments, we see that $(VIF)_{2\epsilon}$ has a unique solution u_ϵ in $\widetilde{K_\sigma}$, when the Reynolds number is sufficiently small. Let $\epsilon \to 0$, then there exists u_0 in K_σ such that

$u_\epsilon \to u_0$ in $H^1_2(\Omega_0)$ weakly ,
$u_\epsilon \to u_0$ in $L^2(\Omega_1)$ weakly
and u_0 satisties $(VIF)_2$ $(see[2],[5],[7])$.

5. Numerical models

Here we present two types of numerical model for (PDEF).

5.1.

$(VIF)_1$ leads to the one numerical model by itself, which we denote by $(NM)_1$.

5.2.

If we apply the distribution theoretic approach to fictitious domain method for Neumann problems [2] to $(PDEF)_\epsilon$, then we have the other numerical model $(NM)_2$ as follows ;

$$(5.1)\quad -\nabla\cdot\{\nu(1-\chi)+\epsilon\chi\}\nabla u+(1-\chi)(u\cdot\nabla)u+c\chi u+g_n\cdot\tanh(\frac{u_n}{\epsilon})\cdot\nabla\chi+\nabla p$$

$$=(1-\chi)f-\chi\cdot\nabla x_2 \quad \text{in } \Omega ,$$

$$(5.2)\qquad \nabla\cdot u=0 \quad \text{in } \Omega$$

and u satisfies (4.9)-(4.13).

6. Numerical results

We state an algorithm to solve $(NM)_1$.

Algorithm

Step 0. Put the initial guess of ϕ to be $\phi^0\equiv 0$ in Ω_1 .

Step 1. Solve (3.1) of $(VIF)_1$ by putting $\phi=\phi_0$.

Let the solution obtained be u^0 and p^0 .

Step 2. Compute $\delta^0 = \frac{1}{L}\int_\gamma u_n^0\, ds$ (L is the length of γ).

Step 3. Solve (3.2) putting $u_n=u_n^0-\delta^0$ on γ .

Let the solution be ϕ^1 and normalize it as $\widetilde{\phi^1}\in P_0$.

Step 4. Replace ϕ in (3.1) by $\widetilde{\phi^1}+k^1$, where $k^1=\delta^0$.

Then return to Step 1.

Remark 6.1 In the n-th iteration, k^n is defined as follows ;

$$k^n = k^{n-1}+\delta^{n-1} = \sum_{j=0}^{n-1}\delta^j .$$

Where $\delta^j=\frac{1}{L}\int_\gamma u_n^j\, ds$, $(j=0,1,\cdots,n-1)$.

Numerical experiments showed the following asymphotic property of δ^n and k^n ;

$$\delta^n \to 0 \quad (n \to \infty) ,$$

and

$$k^n \to d \quad (n \to \infty)$$

where d is a constant to be found out.

Remark 6.2. As the resolution of (3.1) in $(VIF)_1$, we adopted to solve the approximate equations stated in the Remark 3.2.

Finally several numerical results based on $(NM)_1$ and $(NM)_2$ are shown in the following. Figures 1 and 2 show the numerical results obtained by use of $(NM)_1$ and $(NM)_2$, respectively. Figure 3 shows the numerical result of the case that there is a tiny hole at $(L,\ 2a)$, which was obtained by use of $(NM)_1$.

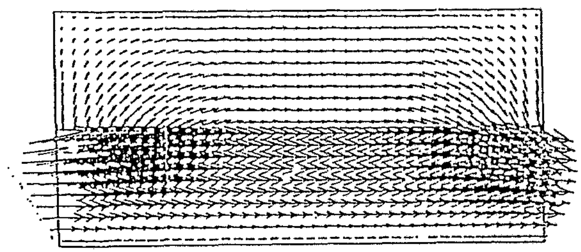

Figure 1.

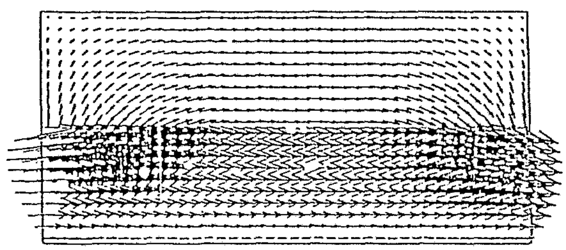

Figure 2.

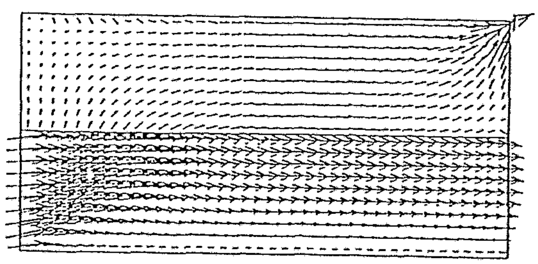

Figure 3.

References

1. G.Duvaut and J.L.Lions, Les Inequations en Mecanique et en Physique (Dunod, Paris 1972).
2. H.Fujita, H.Kawahara and H.Kawarada, Distribution theoretic approach to fictitious domain method for Neumann problems, East-West J.Numer.Math., Vol.3, No.2, pp.111-126 (1995).
3. R.Glowinski, J.L.Lions and R.Tremolieres, Numerical Analysis of Variational Inequalities (North-Holland, Amsterdam 1981).
4. R.Glowinski, Numerical Methods for Nonlinear Variational Problems (Springer, New York 1984).
5. J.L.Lions, Quelques methods de resolution des problems aux limites non lineares (Dunod Gauthier Villars, 1969).
6. J.L.Lions, Une Application de la Dualite aux Regularises et Penalises, Colloq. Rio-de-Janeiro,Aermann,Paris, pp.199-205 (1972).
7. R.Temam, Theory and Numerical Analysis of the Navier-stokes Equations (North-Holland, Amsterdam 1977).