Splitting Extrapolation Method for Solving Multidimensional Problems in Parallel

C. B. Liem, T. M. Shih and T. Lu

ABSTRACT. The splitting extrapolation method is an important method in numerical solution of multidimensional problems. Two types of splitting extrapolation algorithms with their applications in solving partial differential equations are disscussed. Numerical experiments show that the method is superior than the Richardson extrapolation method in the sense of parallelism, computational complexity and computer storage needed.

INTRODUCTION

Many mathematical models of scientific and engineering problems are described by partial differential equations. Despite the significant progress made in digital computers over the last two decades, the solution of high dimensional problems with complicated domains still remains difficult. It is due to the fact that the computational complexity and computer storage required increase exponentially with respect to the dimension. In order to overcome this difficulty, it is necessary to develop parallel algorithms with high accuracy.

In recent years, promising progress has been made in parallel computational methods. The following three types of methods have a common characteristic: large scale multidimensional problems are subdivided into smaller problems.

(a) Domain decomposition methods, including multilevel methods and the fast adaptive composite grid methods [4];
(b) Sparse grid combination techniques [7]; and
(c) Splitting extrapolation methods [1, 2, 3, 5, 6].

The splitting extrapolation, which are also called the multivariate Richardson extrapolation, was first established in 1983 by Q. Lin and T. Lu [1]. It is an ideal method for dealing with the so called "dimensional effect" arised in solving multidimensional problems. The most recent development of the method can be found in the monograph [8]. A comprehensive review on splitting extrapolation methods and sparse grid combination techniques can be found in [7].

THE PRINCIPLE OF SPLITTING EXTRAPOLATION

In order to find the approximate solution of a continuous problem, one first choose a suitable grid parameter h and an appropriate discretization scheme, so as

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to convert the continuous problem into a set of algebraic equations and then obtain the numerical solution $u(h)$. The accuracy of $u(h)$ depends on h. For instance, a finite difference scheme for a s-dimensional problem can naturally include s independent grid parameters, i.e., $h = (h_1, h_2, \cdots, h_s)$. However, independent grid parameters can be chosen according to the scale as well as the geometry of the domain. The number of these parameters can even be larger than the dimension s.

For many continuous problems, it can be shown that under certain assumptions, there is an asymptotic expansion of the error between the numerical solution $u(h)$ and the exact solution u:

$$u(h) = u + \sum_{1 \le |\alpha| \le m} C_\alpha h^{2\alpha} + O\left(h_0^{2m+1}\right), \tag{1}$$

where $h = (h_1, h_2, \cdots, h_s)$, $\alpha = (\alpha_1, \alpha_2, \cdots, \alpha_s)$, $|\alpha| = \alpha_1 + \alpha_2 + \cdots + \alpha_s$, $h^{2\alpha} = h_1^{2\alpha_1} \cdots h_s^{2\alpha_s}$ and $h_0 = \max_{1 \le i \le s} h_i$, here $h_1, h_2, \cdots, h_s$ are independent grid parameters. If $h_1 = h_2 = \cdots = h_s$, (1) is the classical Richardson asymptotic expansion. Obviously, (1) indicates that the error of $u(h)$ is $O\left(h_0^2\right)$. It is expected that, based on (1), a cheaper and more accurate solution can be obtained by the method of splitting extrapolation, i.e., instead of taking a global refinement in all directions as suggested by the classical Richardson extrapolation, one needs only to carry out some unidirectional refinements. In 1990, we proposed two types of unidirectional refinements [3].

Type 1. Given an initial grid parameter $h = (h_1, \cdots, h_s)$, we choose successively the refined grid parameters $\frac{h}{2^\beta} = \left(\frac{h_1}{2^{\beta_1}}, \cdots, \frac{h_s}{2^{\beta_s}}\right)$, $0 \le |\beta| \le m$, and obtain the corresponding approximate solution $u\left(\frac{h}{2^\beta}\right)$.

Type 2. Choose successively the refined grid parameter $\frac{h}{(1+\beta)} = \left(\frac{h_1}{1+\beta_1}, \cdots, \frac{h_s}{1+\beta_s}\right)$, $0 \le |\beta| \le m$, and the corresponding approximate solution is denoted by $u\left(\frac{h}{1+\beta}\right)$.

$u\left(\frac{h}{2^\beta}\right)$(or $u\left(\frac{h}{(1+\beta)}\right)$), $0 \le |\beta| \le m$, can be evaluated in parallel. Furthermore, by using the extrapolation coefficients $\{a_\beta$ or $\tilde{a}_\beta$, $0 \le |\beta| \le m\}$, we can obtain the following approximations with m splits.

Type 1.

$$u_m(h) = \sum_{0 \le |\beta| \le m} a_\beta u\left(\frac{h}{2^\beta}\right), \tag{2}$$

and

Type 2.

$$\tilde{u}_m(h) = \sum_{0 \le |\beta| \le m} \tilde{a}_\beta u\left(\frac{h}{2^\beta}\right). \tag{3}$$

Both $u_m(h)$ and $\tilde{u}_m(h)$ are of order $2m+1$.

It is known that the extrapolation coefficients a_β satisfy the following equations

$$(4)\qquad \begin{cases} \sum\limits_{0\le|\beta|\le m} a_\beta = 1 \\ \sum\limits_{0\le|\beta|\le m} \frac{a_\beta}{2^{2(\beta,\alpha)}} = 0, 0 \le |\alpha| \le m \end{cases}.$$

Similar relations also hold for $\tilde{a}_\beta$.

The exact extrapolation coefficients can be computed in advance. In the following, for a two-dimensional case and $0 \le |\beta| \le 3$, terms in the asymptotic expansion (1), the corresponding approximate solutions and the extrapolation coefficients a_β are arranged in triangular patterns:

$$\begin{array}{llll} 1 & h_x^2 & h_x^4 & h_x^6 \quad \cdots \\ h_y^2 & h_x^2h_y^2 & h_x^4h_y^2 & \cdots \\ h_y^4 & h_x^2h_y^4 & \cdots & \\ h_y^6 & \cdots & & \\ \cdots & & & \end{array}$$

$$\begin{array}{lllll} u(h_x, h_y) & u\left(\frac{h_x}{2}, h_y\right) & u\left(\frac{h_x}{4}, h_y\right) & u\left(\frac{h_x}{8}, h_y\right) & \cdots \\ u\left(h_x, \frac{h_y}{2}\right) & u\left(\frac{h_x}{2}, \frac{h_y}{2}\right) & u\left(\frac{h_x}{4}, \frac{h_y}{2}\right) & \cdots & \\ u\left(h_x, \frac{h_y}{4}\right) & u\left(\frac{h_x}{2}, \frac{h_y}{4}\right) & \cdots & & \\ u\left(h_x, \frac{h_y}{8}\right) & \cdots & & & \\ \cdots & & & & \end{array}$$

$$\begin{array}{lllll} -\frac{97}{567} & \frac{148}{135} & -\frac{64}{27} & \frac{4096}{2835} & \cdots \\ \frac{148}{135} & -\frac{80}{27} & \frac{256}{135} & \cdots & \\ -\frac{64}{27} & \frac{256}{135} & \cdots & & \\ \frac{4096}{2835} & \cdots & & & \\ \cdots & & & & \end{array}$$

As a special case, when the number of split $m = 1$, the extrapolation coefficients for a s-dimensional problem are:

$$a_{(0,\cdots,0)} = -\frac{(4s-3)}{3}, a_{(1,0,\cdots,0)} = \cdots = a_{(0,\cdots,0,1)} = \frac{4}{3},$$

i.e., there are only two different values and $\tilde{a}_\beta = a_\beta$.

Some of the coefficients a_β and $\tilde{a}_\beta$, $0 \le |\beta| \le m$, can be found in [4] and [5].

MULTIVARIATE ASYMPTOTIC ERROR EXPANSION OF THE NUMERICAL SOLUTION TO PDE

Finite Difference Methods

Consider the following semilinear elliptic equation:

$$\left\{ \begin{array}{l} \Delta u = f(x,u), \text{ in } \Omega = (0,1)^s, \\ u = 0, \text{ on } \partial\Omega. \end{array} \right. \tag{5}$$

where $f_u^{'}(x,u) \ge 0, h = (h_1, h_2, \cdots, h_s)$, and $h_i = \frac{1}{N_i}$, for $i = 1, 2, \cdots, s$.
Using the central difference scheme, we have the following difference equation:

$$\Delta^h u^h = \sum_{i=1}^{s} h_i^{-2} \{u^h(x_1, x_2, \cdots, x_i - h_i, \cdots, x_s) - 2u^h(x_1, \cdots, x_i, \cdots, x_s)$$

$$+ u^h(x_1, \cdots, x_i + h_i, \cdots, x_s)\} = f(x, u^h), \ x = (x_1, \cdots, x_s) \in \Omega^h, \tag{6}$$

$$u^h = 0, x \in \partial\Omega^h,$$

where $\Omega^h = \{x = (x_1, \cdots, x_s) : x_i = jh_i, 1 \le j \le N_i, 1 \le i \le s\}$.

Theorem 1. If $u \in C^{7+\sigma}(\Omega), 0 < \sigma \le 1$, then $\exists w_\beta \in C^{5+\sigma-2|\beta|}(\Omega)$, $1 \le |\beta| \le 2$, such that

$$u^h - u + \sum_{1 \le |\beta| \le 2} w_\beta h^{2\beta} = O\left(h_0^{5+\sigma}\right),$$

where $h_0 = \max_{1 \le i \le s} h_i$.
Remark: If Ω is a smooth domain, the above theorem will still hold if quadratic interpolation polynomials are applied at irregular points.

Finite Element Methods

One of the earliest work on splitting extrapolation of finite element methods was published by Q. Lin and T. Lu in 1983. A recent development is the monograph by Q. Lin and Q. Zhu (1994). If $s = 2$, using bilinear elements on rectangular grids, one can prove the following theorem for equation (5):

Theorem 2. If $\Im^h$ is a regular rectangular subdivision of Ω, and $u \in W_q^3(\Omega) \cap H_0^1(\Omega) \cap \left(\Pi_{e \in \Im^h} W_q^4(e)\right)$, where $1 \le q \le \infty$, then there exist functions w_1 and w_2, independent of $h = (h_1, h_2)$, such that

$$\left\| u^h - u^I - h_1^2 w_1^I - h_2^2 w_2^I \right\|_0 \le c h_0^4 \|u\|'_{4,2}, \tag{7}$$

and

$$\left\| u^h - u^I - h_1^2 w_1^I - h_2^2 w_2^I \right\|_\infty \le c h_0^4 \left|\ln h_0\right| \|u\|'_{4,\infty} \tag{8}$$

where $h_0 = \max\{h_1, h_2\}$, $\|\cdot\|_0$ denotes the norm in $L^2(\Omega)$, $\|\cdot\|_\infty$ denotes the norm in $L^\infty(\Omega)$, u^I denotes the interpolation of u, and

$$\|u\|'_{n,p} = \left(\sum_{e \in \Im^h} \|u\|^p_{n,p,e} \right)^{\frac{1}{p}}, \text{ for } p = 2, \infty.$$

The above theorem shows that one split can be used while under stronger conditions, more splits are allowed.

Similar result remains true for three dimensional case. But a more important case is that under certain conditions, one can choose independent grid parameters according to the size and the geometry of the problem. In general, the more the independent grid parameters, the higher the parallelism and more computer CPU and storage can be saved. In order to explain the idea, consider the following two figures:

In figure 1, there are three independent grid parameters for a two-dimensional problem.

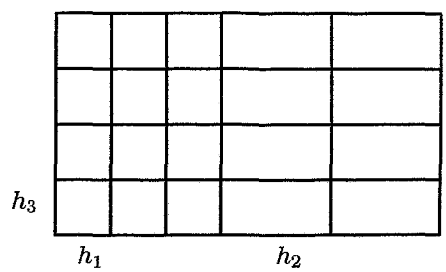

Fig.1

In figure 2, higher accuracy is needed near the point A. Therefore we can use six independent grid sizes where h_2 and h_5 can be chosen to be smaller than others.

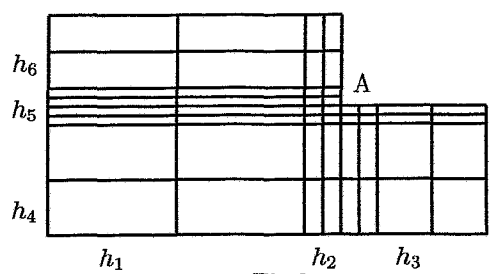

Fig.2

TABLE 1. Comparison of numerical results of three extrapolation methods

No. of split	Richardson extrapolation		Type 1 SEM		Type 2 SEM	
	max error	CPU	max error	CPU	max error	CPU
0	3.91E-4	0.04	3.91E-4	0.04	3.91E-4	0.04
1	2.44E-6	0.71	6.79E-6	0.52	6.79E-6	0.49
2	7.61E-9	13.60	1.90E-7	2.53	1.90E-7	2.09
3	3.23E-11	285.67	5.06E-9	12.98	5.32E-9	7.41
4					1.41E-10	21.94
5					7.89E-12	57.11

TABLE 2. Comparison of computer storage needed by three extrapolation methods

No. of split	Richardson extrapolation	Type 1 SEM	Type 2 SEM
0	162	162	162
1	1,769	339	339
2	16,956	945	945
3	149,063	2,135	2,135
4	1,250,370	4,410	3,430
5	2,048,543	9,051	5,411

NUMERICAL EXPERIMENTS

Example 1: Consider the three dimensional Poisson equation:

$$\begin{cases} \Delta u &= f \quad , \quad \text{in} \quad \Omega = (0,1)^3, \\ u &= 0 \quad , \quad \text{on} \quad \partial\Omega. \end{cases}$$

The exact solution is $u = \prod_{i=1}^{3} \left[x_i (1 - x_i) \cos\left(\frac{\pi x_i}{2}\right)\right]$. Using the seven-point finite difference scheme with initial grid sizes $h_1 = h_2 = h_3 = \frac{1}{4}$, the results are recorded in Tables 1—3.

TABLE 3. Comparision of the degree of parallelism for three extrapolation methods, N is the processor used

No. of split	Richardson extrapolation		Type 1 SEM		Type 2 SEM	
	N	max CPU	N	max CPU	N	max CPU
1	2	0.33	4	0.13	4	0.13
2	3	12.94	10	0.38	10	0.38
3	4	272.01	20	1.42	20	0.67
4	5	⋯	35	⋯	35	1.44
5	6	⋯	56	⋯	56	2.54

TABLE 4. Maximum relative errors

err0	err1	err2	err3	err4
4.120E-2	4.125E-2	4.125E-2	4.125E-2	4.119E-2
err5	err6	errspl	errRich	
2.518E-2	2.498E-2	3.189E-3	7.535E-3	

TABLE 5. Relative errors at some points

Coordinates	err0	errspl	errRich
(1/2, 1/2)	8.55E-3	2.43E-5	3.93E-5
(3/2, 1/2)	9.63E-3	1.03E-5	3.05E-5
(5/2, 1/2)	5.09E-3	8.80E-5	4.36E-5
(7/2, 1/2)	3.84E-3	1.37E-4	1.88E-5
(8/2, 1/2)	1.18E-2	6.80E-5	7.23E-5
(9/2, 1/2)	2.44E-2	7.85E-4	3.56E-4

Example 2: Consider the two dimensional Poisson equation:

$$\begin{cases} \Delta u = f, & \text{in} \quad \Omega = (0,5) \times (0,1), \\ u = 0, & \text{on} \quad \partial\Omega. \end{cases}$$

The exact solution is $u(x,y) = xy(1-y)\left(1-\frac{x}{5}\right)e^{xy}$. Using rectangular elements and choose six independent grid sizes $h_i\ (i = 1, 2, \cdots, 6)$ in five subdomains $\Omega_i\ (i = 1, 2, \cdots, 5)$.(Fig.3). Let $u_0 = u(h_1, h_2, \cdots, h_6)$, $u_i = u\left(h_1, \cdots, \frac{h_i}{2}, \cdots, h_6\right)$, denote the maximum relative error of $u_i\ (i = 0, 1, \cdots, 6)$, of the splitting extrapolation solution and of Richardson extrapolation by erri, errspl and errRich respectively. The results are shown in Table 4 and Table 5.

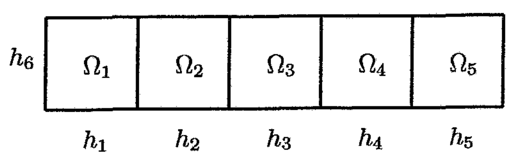

Fig.3

CONCLUSION

If there exists the asymptotic expansion (1), the numerical accuracy of the splitting extrapolation and the Richardson extrapolation methods are comparable, while the former is superior in the sense of parallelism, computational complexity and computer storage needed.

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