Compensation Method of an Optimal-order Wilson Nonconforming Multigrid

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1 Introduction

It is well known that the nonconforming finite elements are widely used in solving elliptic boundary value problems, because they have fewer degrees of freedom, simpler basis functions and better convergence behavior. As usual, we apply them with two methods. One is the so called tolerance method, and the other is the penalty method. But they both have some disadvantages: The convergence order is lower than for conforming elements with the same degree of piecewise polynomial interpolation if the former method is used (cf.[1] and [2]). Using the second method, the convergence order is only half of that of conforming elements with the same degree of piecewise polynomial (see[3]). To increase the convergence order, the compensation method was introduced in [4]. Through applying this method, the same accuracy order as for conforming elements with the same degree of piecewise piecewise polynomial can be obtained.

Additionally, the nonconforming multigrid method is also a very efficient method for solving the elliptic boundary value problem. Its characteristic feature is its fast convergence. Moreover, one can obtain an acceptable approximation of the discrete problem at an expense of computational work proportional to the number of unknowns. It is not only the complexity which is optimal, also the constants of proportionality are so small that other methods can hardly surpass the multigrid efficiency.

In the present paper, we will give a new method by combining above two methods

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for the Wilson element, such that the accuracy order using Wilson nonconforming element is almost the same as for a quadratic conforming element.

The paper is organized as follows. We will begin with a discussion of the compensation method for the Wilson element. The intergrid transfer operator is defined and its properties are discussed in section 3. In section 4, the $k-$level iteration and nested iteration are given. The last section contains the error estimate for the new method.

2 Compensation for the Wilson Element

For simplicity, the Poisson's equation is considered:

$$\begin{cases} -\Delta u = f \text{ in } \Omega \\ u = 0 \text{ on } \partial\Omega, \end{cases} \tag{2.1}$$

where $f \in L^2(\Omega)$, and Ω is a rectangular domain. The variational form of the problem (2.1) is: Find $u \in H_0^1(\Omega)$, such that

$$a(u,v) = (f,v), \forall v \in H_0^1(\Omega), \tag{2.2}$$

where

$$a(u,v) = \int_\Omega \nabla u \nabla v dx,$$
$$(f,v) = \int_\Omega f v dx.$$

For $h_k (k \geq 1)$ in a null sequence, let τ_k be a subdivision of $\bar{\Omega}$ into rectangles P; τ_{k+1} can be obtained by connecting the midpoints of the edges of $P \in \tau_k$. Then $h_k = 2h_{k+1}$. In addition, we assume that $\tau_k (k \geq 1)$ satisfy the regular and uniformly conditions, namely, there exist two constants σ, γ, which are independent of P, such that

$$\bar{h}_{k.P} \leq \sigma h_{k,P}, \ h_k \leq \gamma \bar{h}_{k,P}, \ \ \forall P \in \tau_k,$$

where $\bar{h}_{k,P}$ and $h_{k,P}$ denote the diameters of P, i.e. the length of the longest sides of P, and of the smallest sides of P, respectively. $h_k = \max_{P \in \tau_k} \bar{h}_{k.P}$.

Let V_k be the Wilson finite element space associated with τ_k. For every $v \in V_k$, it has the following properties:
(i) $v|_P$ is quadratic polynomial;
(ii) v is continuous at the vertices and vanishes at the vertices along $\partial\Omega$;
(iii) The remainning two degrees of freedom are the mean values of second derivatives $\frac{\partial^2 v}{\partial x_i^2} (i = 1, 2)$ for each $P \in \tau_k$.

If $u, v \in V_k$, set

$$a_k(u,v) = \sum_{P \in \tau_k} \int_P \nabla u \bigtriangledown v dx,$$
$$b_k(u,v) = \sum_{P \in \tau_k} \int_{\partial P} \{\bar{u}_n[v] + \bar{v}_n[u] + \frac{\theta}{h}[u][v]\} ds,$$

and

$$c_k(u,v) = a_k(u,v) + b_k(u,v),$$

where

$$[u] = u^+ - u^-,$$
$$u_n = \frac{\partial u}{\partial n},$$
$$\bar{u}_n = \frac{1}{2}(u_n^+ + u_n^-),$$

and n is the normal direction of ∂P pointing in the "+" direction. The positive constant θ, which is independent of h_k, will be determined later.

Now, a new discrete variational problem is given as follows.

Find $u_k \in V_k$, such that

$$c_k(u_k, v) = (f, v), \ \forall v \in V_k. \tag{2.3}$$

Obviously, if V_k is conforming, then $b_k(u,v) = 0$, i.e. the problem (2.3) becomes a standard conforming finite problem.

For the tolerance method, the approximation solution u_k^* is obtained by solving the problem: Find $u_k^* \in V_k$, such that

$$a_k(u_k^*, v) = (f, v), \ \forall v \in V_k. \tag{2.4}$$

From [1], the following error estimates hold.

$$\|u - u_k^*\|_k \leq C h_k |u|_{H^2(\Omega)}, \tag{2.5}$$

$$\|u - u_k^*\|_{L^2(\Omega)} \leq C h_k^2 |u|_{H^2(\Omega)}. \tag{2.6}$$

We can easily get the following lemmas.

Lemma 2.1 *If $v \in V_k$, then*

$$\int_{\partial p} u^2 ds \leq C h_k^{-1} |u|_{o.p}^2$$

holds (see [4]).

Lemma 2.2 *If $|v|_{1.h_k}^2 = \sum_{P \in \tau_k} |v|_{1.P}^2$, then $|v|_{1.h_k}$ is a norm on V_k.*

Theorem 2.1 *The bilinear form $c_k(u,v)$ is not only uniformly V_h elliptic, but also bounded. So the problem (2.3) has a unique solution.*

We omit the proof.

Theorem 2.2 *If u_k is the solution of problem (2.3) and $u \in H^3(\Omega)$, then there exists a constant C independent of h_k, such that*

$$|u - u_k|_{1.h_k} \leq C h_k^2 |u|_{3.\Omega},$$

$$|u - u_k|_{L^2(\Omega)} \leq C h_k^3 |u|_{3.\Omega}.$$

Proof. We can prove this theorem by using Green's formula on each element P in τ_k and a duality argument.

3 The Intergrid Transfer Operator and its Properties

The intergrid transfer operator I_{k-1}^k is defined as follows. For any $v \in V_{k-1}$, I_{k-1}^k satisfies following conditions:
(i) If A is a vertex of element P $\in \tau_k$ inside Ω and a vertex of a element in τ_{k-1}, then

$$(I_{k-1}^k v)(A) := v(A).$$

(ii) If A is a common vertex of two elements P_1 and P_2 in τ_{k-1}, then

$$(I_{k-1}^k v)(A) := \frac{1}{2}[v|_{P_1}(A) + v|_{P_2}(A)].$$

If A is in the interior of a rectangle in τ, then

$$(I_{k-1}^k v)(A) := v(A).$$

(iii) $I_{k-1}^k v = 0$ at the vertices along $\partial\Omega$.
(iv) The remaining rest degrees of freedom are the mean values of the second derivatives of v on $P \in \tau_k$.

We define a mesh-dependent energy norm by

$$\|v\|_k := \sqrt{c_k(v,v)}.$$

From Theorem 2.1, we have the following lemma.

Lemma 3.1 *The norm $|v|_{1.h_k}$ is equivalent to the energy norm $\|v\|_k$.*

Now, we give two properties of the operator I_{k-1}^k; proofs can be found in [7].

Property A. *There exists a constant C, independent of h_k, such that*

$$\|I_{k-1}^k v\|_{L^2(\Omega)} \leq C\|v\|_{L^2(\Omega)} \ , \forall v \in V_{k-1}.$$

Property B. *There exists a constant C, independent of h_k, such that*

$$\|I_{k-1}^k v\|_k \leq C\|v\|_{k-1}, \forall v \in V_{k-1}.$$

Assume that $0 \leq \lambda_1 \leq \lambda_2 \cdots \leq \lambda_{N_k}$ and $\phi_1, \phi_2, \ldots, \phi_{N_k}$ are the eigenvalues and eigenfunctions of $c_k(\cdot,\cdot)$ with respect to $(\cdot,\cdot)$, respectively. From the inverse inequality, there exists a constant C^*, such that

$$\lambda_{n_k} \leq C^* h_k^{-2}. \tag{3.1}$$

If $v = \sum_{i=1}^{i=N_k} c_i\phi_i$, then the discrete norm $|||v|||_{s.k}$ can be defined as follows.

$$|||v|||_{s,k}^2 := \sum_{i=1}^{i=N_k} \lambda_i^s c_i^2, s \in R.$$

Obviously, $|||v|||_{0,k} = \|v\|_{L^2(\Omega)}$, $|||v|||_{1.k} = \|v\|_k$.

4 The k-Level Iteration and the Nested Iteration

If z_0 is an initial value of the solution; then the approximation $MG(k, z_0, G)$ can be obtained by solving following problem: Find $z \in V_k$, such that

$$c_k(z, v) = G(v), \ \forall v \in V_k, G \in V_k',$$

where V_k' denote the conjugate space of V_k and $c_k(u, v)$ is defined as in section 2.

If $k = 1$, then $z_1 := MG(1, z_0, G)$ is the solution using a direct method. For $k \geq 1$, z_m are obtained by solving the equation:

$$(z_i - z_{i-1}, v) = \Lambda_k^{-1}(G(v) - c_k(z_{i-1}, v)), \ \forall v \in V_k,$$

where $1 \leq i \leq m$, $\Lambda_k \leq C^* h_k^{-2}$ (see (3.1)), and m is an integer to be determined later. In addition, q_p is obtained by a $(k-1)$-level iteration p times ($p = 2, 3$), namely,

$$q_0 = 0,$$
$$q_i = MG(k-1, q_{i-1}, \bar{G}), \ 1 \leq i \leq p,$$

where

$$\begin{aligned} \bar{G}(v) &= G(I_{k-1}^k v) - c_k(z_m, I_{k-1}^k v) \\ &= c_k(z - z_m, I_{k-1}^k v), \ \forall v \in V_{k-1}. \end{aligned}$$

The approximate solution is $MG(k, z_0, G) := z_m + I_{k-1}^k q_p$.

The full multigrid method is defined as follows. Let $\hat{u}_1$ is the solution obtained by using a direct method. The approximation $\hat{u}_k (k \geq 2)$ is obtained, recursively, by

$$\begin{aligned} &u_0^j = I_{j-1}^j \hat{u}_{j-1}, \\ &u_l^j = MG(j, u_{l-1}^j, G), \ 1 \leq l \leq r, \ G(v) = \int_G f v dx_1 dx_2, \\ &\hat{u}_j = u_r^j. \end{aligned}$$

Here r is a positive integer to be determined later.

5 The Error Estimate

In this section, we will prove that the convergence order of the multigrid method with compensation is almost the same as that of the conforming quadratic element.

Theorem 5.1 *If the number of smoothing steps m is large enough, then the k-level iteration is a contraction for the energy norm.*

The proof is given in [7].

Theorem 5.2 *If $\hat{u}_k$ is the approximation using the full multigrid algorithm and r is large enough, then exists a constant C, independent of k, such that*

$$\|u - \hat{u}_k\|_{L^2} \leq C h_k^2 (h_k |u|_{3.\Omega} + |u|_{2.\Omega}),$$

$$\|u - \hat{u}_k\|_k \leq Ch_k(h_k|u|_{3.\Omega} + |u|_{2.\Omega}),$$

where we assume that $u \in H^3(\Omega) \cap H_0^1(\Omega)$ is the solution of problem (2.2).
Proof: We only prove the first inequality because the proof of the second inequality is simpler.

Let $\bar{\pi}_{k-1}$ be the Lagrange interpolant operator. Then,

$$I_{k-1}^k(\bar{\pi}_{k-1}u) = \bar{\pi}_{k-1}u,$$
$$\|u - \bar{\pi}_{k-1}u\|_{L^2(\Omega)} \leq Ch_k^2|u|_{2.\Omega}.$$

Thus,

$$\begin{aligned}
&\|u_k - \hat{u}_k\|_{L^2(\Omega)} \\
&\leq \gamma^r[\|u_k - u\|_{L^2(\Omega)} + \|u - \bar{\pi_{k-1}}u\|_{L^2(\Omega)} + \|I_{k-1}^k(\bar{\pi_{k-1}}u - \hat{u}_{k-1}\|_{L^2(\Omega)}] \\
&\leq \frac{C\gamma^r}{1 - 2C\gamma^r}(h_k^3|u|_{3.\Omega} + h_k^2|u|_{2.\Omega}) \ .
\end{aligned}$$

Choosing γ such that $1 - 2C\gamma > 0$, we find

$$\|u_k - \hat{u}_k\|_{L^2(\Omega)} \leq Ch_k^2(|u|_{2.\Omega} + h_k|u|_{3.\Omega}).$$

The first inequality follows by using a triangle inequality.

REFERENCES

[1] Ciarlet P. G. (1978) *The Finite Element Method For Elliptic Problem,* North-Holland, Amsterdam.

[2] Stummel F. (1979) The generalized patch test. *SIAM J. Numer. Anal.,* 16:449-471.

[3] Babuska I. and Zlamal M. (1973) Nonconforming element in the finite element method with penalty. *SIAM J. NUmer. Anal.,*10: 863-875.

[4] Zhan Chongxi (1987) Nonconforming finite elements with compensation for plate bending problems. *J. Compute. Math.,* 5: 238-248.

[5] Shi Zhongci (1986) A remark on the optimal order of convergence of Wilson nonconforming element. *J.Comput.Math.,* 2:159-163.

[6] Agmon S.(1965) *Lectures on Elliptic Boundary Value Problems,* D.Van Nostrand Co, Int.

[7] Yu Xijun (1993) A multigrid method of nonconforming Wilson finite element. *Math. Numer. Sinica* 2: 346-351(in Chinese).