Domain Decomposition Methods with Strip Substructures

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Abstract: Domain Decomposition Method for finite element discretization of planar elliptic variational problems with regular and discontinuous coefficients is analyzed. The domain is divided into strip shaped subdomains. We construct Neumann–Dirichlet substructuring algorithms. The approximate solution is obtained iteratively by solving local problems associated with each strip and the global problem associated with the coarse triangulation. Convergence of algorithms is almost optimal with respect to the parameters of triangulations and independent of the jumps of coefficients.

1 Introduction

Domain Decomposition Method for finite element discretization of elliptic problems with discontinuous coefficients is analyzed. We introduce two level nested triangulation. The domain is divided into strip shaped subdomains. We construct Neumann–Dirichlet substructuring algorithm with coarse space.

The partition of the domain into such subdomains has several advantages. The bandwidth of local matrices is narrow, which minimizes computations and memory requirements. Also the structure of local problems is useful for vectorization of an algorithm.

The substructuring preconditioners are usually constructed on subdomains (boxes) defined by the coarse triangulation, cf. Bramble, Pasciak, Schatz (1986). The convergence rate of iterative method which is depends on the condition number of preconditioned system, is bounded polylogarithmically in $H/h$. Here $H$ and $h$ denote parameters of coarse and fine triangulations. Our algorithms have similar convergence properties. The condition number of preconditioned system for strips is proportional to $(1 + \ln(H/h))$. The related numerical experiments are included in Mróz (1995).

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The algorithms presented in this paper can be extended to 3-D case. Also they are applicable to parabolic and nonlinear problems. The Neumann–Dirichlet preconditioner can be used as inexact solver in Additive Schwarz Method. In such situation we obtain optimal convergence and much simpler implementation. We can construct substructuring preconditioner for boxes. Each strip is divided into boxes and for each strip we apply our Neumann–Dirichlet preconditioner. The convergence of such method is the same as for strips. All these extensions are analyzed in Mróz (1995).

2 Model problem

We consider the problem of finding an approximate solution of the following elliptic, boundary value problem.

For given a bilinear form \( a(\cdot, \cdot) \) and linear functional \( l(\cdot) \) on \( H^1_0(\Omega) \) we want to find \( u \in H^1_0(\Omega) \) such that

\[
 a(u, v) = l(v) \quad \forall v \in H^1_0(\Omega),
\]

where \( \Omega \) is a Lipschitz bounded domain in \( \mathbb{R}^2 \). For simplicity of presentation we assume that \( \Omega \) is a polygon.

We will distinguish two cases for \( a(\cdot, \cdot) \).

**bilinear form with regular coefficients**

The bilinear form in this case is as follows:

\[
 a(u, v) = \sum_{i=1}^{2} \int_{\Omega} \frac{\partial u}{\partial x_i} \frac{\partial v}{\partial x_i} dx .
\]

**bilinear form with discontinuous coefficients**

We consider the variational problem of the form (1) up to replacing \( a(\cdot, \cdot) \) by the form

\[
 a^\rho(u, v) = \int_{\Omega} \rho(x) \sum_{i=1}^{2} \frac{\partial u}{\partial x_i} \frac{\partial v}{\partial x_i} dx .
\]

The function \( \rho(x) \) is piecewise constant i.e

\[
 \rho(x) = \rho_i > 0, \quad x \in \Omega_i,
\]

where \( \Omega_i \) denote the strip shape subdomains consisting of elements \( \Omega_i^j \), defined in Section 2. The jumps of coefficients between subdomains may be large. This model problem can be applied to the case when the function \( \rho(x) \) varies moderately on each subdomain and is discontinuous between subdomains. In this case the coefficient is merely equal to the mean value of \( \rho(x) \) on the subdomain.

Let \( l(v) \) denote the linear form defined by

\[
 l(v) = (f, v)_{L^2(\Omega)} = \int_{\Omega} fv dx .
\]
3 Finite Element approximation

A two level triangulation is defined on the domain $\Omega$. First, we construct a coarse
triangulation $\Omega_H$ that consists of shape regular (cf. Ciarlet (1978)), nonoverlapping
triangles $\Omega^H$ of diameter of order $H$. In second step, we further divide each element
of the triangulation $\Omega_H$ into smaller, shape regular triangles of diameter $O(h)$. They
form the fine triangulation $\Omega_h$.

Spaces of piecewise linear, continuous functions on $\Omega_H$ and $\Omega_h$ are denoted by
$V^H(\Omega)$ and $V^h(\Omega)$. The restriction to subspaces of functions vanishing on $\partial\Omega$ is
denoted by $V^H_0(\Omega)$ and $V^h_0(\Omega)$ respectively. The corresponding approximate problem
for (1) is then:

Find $u^* \in V^h_0(\Omega)$ such that
\begin{equation}
    a(u^*, v) = l(v) \quad \forall v \in V^h_0(\Omega) \, .
\end{equation}

Let $\{\phi_j^h\}$ be the set of standard, piecewise linear, nodal basis functions, thus $V^h_0(\Omega) = \text{span}\{\phi_j^h\}$. In this basis, the discrete variational problem (6) can be rewritten as a
system of linear equations
\begin{equation}
    Au = f_h ,
\end{equation}

where coefficients $A_{ij} = a(\phi_j^h, \phi_i^h)$ and $f_j = l(\phi_j^h)$ The matrix $A$ is positive definite
and symmetric. The condition number of $A$ is proportional to $h^{-2}$.

In the same manner, we can formulate the discrete variational problem for the
bilinear form $a^p(\cdot, \cdot)$ from (3). The condition number of $A^p$ is proportional to
$\frac{\max_{i,j} \rho^p_i}{\min_{i,j} \rho^p_j} h^{-2}$.

4 Neumann–Dirichlet preconditioner – regular coefficients

In this section we construct a Neumann–Dirichlet preconditioner for the problem (2)
with regular coefficients.

The domain $\Omega$ is divided into $N$ strips $\Omega_i$, $i = 1, \ldots, N$ called strips. We assume
that the boundary of each strip consists only of boundaries of elements from the
coarse triangulation $\Omega_H$ and there are no nodes of $\Omega_H$ inside the strip. The strip
$\Omega_i$ has common boundary only with at most two neighboring strips. This common
interface between two strips is called $\Gamma_i$,

$$
\Gamma_i = \partial \Omega_i \cap \partial \Omega_{i+1} .
$$

Every point of $\Gamma_i$ belongs to exactly two strips. The boundary of each strip consists of
the two interface lines $\Gamma_{i-1}$ and $\Gamma_i$ and parts of $\partial \Omega$.

Let us denote the odd strips by Dirichlet superscript $\Omega_i^D$ and even strips by Neumann
superscript $\Omega_i^N$. The bilinear forms $a_i(\cdot, \cdot)$ represents restrictions of $a(\cdot, \cdot)$ to $\Omega_i$. In
order to distinguish the form $a_i(\cdot, \cdot)$ defined on Dirichlet or Neumann type strips we
will add suitable superscript to this notation. Thus bilinear forms $a_i^D(\cdot, \cdot)$ and $a_i^N(\cdot, \cdot)$
are defined on Dirichlet or Neumann type strips respectively.

We define the local orthogonal projection $P_i$ of the space $V^h(\Omega_i) \cap V^h_0(\Omega)$ onto
$V^h_0(\Omega_i)$ by
\begin{equation}
    a_i(P_iu, v) = a_i(u, v) \quad \forall v \in V^h_0(\Omega_i) ,
\end{equation}

and the local discrete harmonic function $H_i u$ with respect to bilinear form $a_i(\cdot, \cdot)$ by

$$ a_i(H_i u, v) = 0 \quad \forall v \in V_0^h(\Omega_i), $$

$$ H_i u = u \quad \text{on } \partial\Omega_i. \tag{9} $$

For any function $V^h(\Omega_i) \cap V_0^h(\Omega)$ we have

$$ u = P_i u + H_i u. $$

Thus

$$ a(u, u) = \sum_{even \ i} a_i^N(u, u) + \sum_{odd \ i} (a_i^D(u, P_i u) + a_i^D(H_i u, H_i u)) \tag{10} . $$

Dryja and Proskurowski (1985) constructed the Neumann–Dirichlet preconditioner by omitting the last term of representation (10). Then the upper estimate of $a(u, u)$ by this preconditioner depends on $H^{-2}$ since the Trace and Extension Lemmas were used for strip shape subdomains. In order to avoid such dependence, the mechanism of global transportation of information should be introduced into the definition of the preconditioner (cf. Widlund (1988)). To meet this requirement we include the term $a(I_H u, I_H u)$ to the definition of the preconditioner, where $I_H$ denotes the nodal value interpolation operator from $V_0^h(\Omega)$ onto $V_0^H(\Omega)$, defined by

$$ (I_H u)(x) = u(x), \tag{11} $$

if $x$ is a node of the triangulation $\Omega_H$.

We are ready to define a bilinear form $b(\cdot, \cdot)$ that corresponds to the Neumann–Dirichlet preconditioner,

$$ b(u, v) = \sum_{odd \ i} a_i^D(P_i(u - I_H u), P_i(v - I_H v)) + $$

$$ + \sum_{even \ i} \tilde{a}_i^N(u - I_H u, v - I_H v) + a(I_H u, I_H v) \tag{12} . $$

where

$$ \tilde{a}_i^N(u, v) = a_i^N(u, v) + H^{-2}(u, v)_{L^2(\Omega_i^N)}. \tag{13} $$

**Theorem 1** For any function $u \in V_0^h(\Omega)$, the following inequalities holds

$$ m \left(1 + \ln \frac{H}{h}\right)^{-1} b(u, u) \leq a(u, u) \leq M b(u, u), \tag{14} $$

where the positive $m$ and $M$ are independent of $H$ and $h$.

The proof of this theorem is given in Mróz (1995).

To solve the linear system (7), we use a preconditioned gradient method (PCG) (cf. Concus, Golub, O'Leary (1976)). In each step of the PCG method the system corresponding to the preconditioner is to be solved:

Find $u \in V_0^h(\Omega)$ such that

$$ b(u, v) = g(v) \quad \forall v \in V_0^h(\Omega). \tag{15} $$
The finite element space $V_0^h(\Omega)$ is decomposed,
\[ V_0^h(\Omega) = \tilde{V}_0^h(\Omega) \oplus V_0^H(\Omega), \]
where $\tilde{V}_0^h(\Omega)$ is a subspace of $V_0^h(\Omega)$ consisting of functions vanishing at the nodes of the coarse triangulation $\Omega_H$.

The algorithm of solving the problem (15) is as follows.

**Algorithm 2**

1. Construct the system for the nodal basis functions $\phi^h_j$ connected with the nodes of the interiors of $\Omega_i^D$. For such functions $I_H \phi^h_j = 0$, hence (15) is reduced to separate subproblems of finding local projections $P_i w$, $i = 1, 3, \ldots$ by solving
\[ a^D_i (P_i w, \phi^h_j) = g(\phi^h_j), \quad \forall \phi^h_j \in V_0^h(\Omega_i^D). \]
These systems correspond to solving the subproblems individually for each strip $\Omega_i$ with homogeneous Dirichlet boundary conditions on interface lines $\Gamma_i$.

2. We build the system associated with $\Omega_i^N$ excluding nodes of the coarse triangulation $\Omega_H$. For such basis functions $I_H \phi^h_j = 0$, thus we search for $w$ on $\Omega_i^N$, $i = 2, 4, \ldots$ by solving
\[ a^N_i (w, \phi^h_j) = g(\phi^h_j) - \sum_{\text{odd } i} a^D_i (P_i w, \phi^h_j), \quad \forall \phi^h_j \in \tilde{V}^h(\Omega_i^N). \]
These systems correspond to solving the subproblems individually for each strip $\Omega_i^N$ with Neumann boundary conditions on interface lines $\Gamma_i$, excluding the nodes of the triangulation $\Omega_H$, where we impose homogeneous Dirichlet boundary conditions.

3. Construct the system for basis function $\phi^H_j$ from $V_0^H(\Omega)$. Note that $\phi^H_j - I_H \phi^H_j = 0$, thus in order to find the interpolation $I_H u$ we solve the global system
\[ a(I_H u, \phi^H_j) = g(\phi^H_j), \quad \forall \phi^H_j \in V_0^H(\Omega). \]

4. In Step 1 the projections $P_i w$ have been computed, so now we find the solution $u$ on $\Omega_i^D$. The discrete harmonic function $H_i w$ is obtained from the system.
\[ a^D_i (H_i w, \phi^h_j) = 0, \quad \forall \phi^h_j \in V_0^h(\Omega_i^D), \]
and then $w = H_i w + P_i w$ or we can solve the system
\[ a^D_i (w, \phi^h_j) = g(\phi^h_j), \quad \forall \phi^h_j \in V_0^h(\Omega_i^D). \]

The boundary conditions are to be imposed so as to fix the values of $w$ on the interface lines $\Gamma_i$ as calculated in the previous Step. In the second case we do not need to keep the values of $P_i w$ in all nodes of $\Omega_i$ after Step 1 is completed.

The solution $u$ on $\Omega$ is obtained from the formula
\[ u = w + I_H u. \]
This algorithm admit quite high level of coarse-grained parallelism. Steps 1, 2 and 4 consist of solving a number of small separate systems. Furthermore Steps 2 and 3 may be performed concurrently.

5 Neumann–Dirichlet preconditioner — discontinuous coefficients

In this section we consider the differential problem with coefficients constant on each strip \( \Omega_i \).

We will use the notation and definitions introduced in previous Section. The restriction of the bilinear form \( a^\rho(\cdot, \cdot) \) to \( \Omega_i \) is denoted by \( a_i^\rho(\cdot, \cdot) \). The definition of projection \( P_i^\rho u \) and discrete harmonic function \( H^\rho_i u \) is the same as in (8) and (9) with respect to the bilinear form \( a_i^\rho(\cdot, \cdot) \). The idea of construction of the Neumann–Dirichlet preconditioner is similar to that for regular coefficients. In order to avoid the dependence on jumps of coefficients, the weights are introduced into Neumann problems on even (Neumann type) strips. We first define the function \( u^\rho \) on \( \partial \Omega \) and on \( \Gamma_i, i = 1, \ldots, n - 1 \)

\[
\begin{align*}
w^\rho(x) &= (\rho_i + \rho_{i+1})^{1/2}(u(x) - I_H u(x)) \quad x \in \Gamma_i, \\
\tilde{w}^\rho(x) &= 0 \quad x \in \partial \Omega.
\end{align*}
\] (16)

The function \( \tilde{H}_i u^\rho \in V^h(\Omega_i^N) \) is a local discrete harmonic function defined on \( \Omega_i^N \) with respect to the bilinear form

\[
\tilde{a}_i^N(u, v) = (\nabla u, \nabla v)_{L^2(\Omega_i^N)} + H^{-2}(u,v)_{L^2(\Omega_i^N)},
\] (17)

i.e.

\[
\tilde{a}_i^N(\tilde{H}_i u^\rho, v) = 0 \quad \forall v \in V^h_0(\Omega_i^N),
\]

\[
\tilde{H}_i u^\rho(x) = u^\rho(x), \quad x \in \partial \Omega_i^N.
\]

The bilinear form \( b^\rho(\cdot, \cdot) \) that corresponds to the Neumann–Dirichlet preconditioner for the case with piecewise constant coefficients is defined by

\[
b^\rho(u, v) = \sum_i \rho_i (\nabla P_i (u - I_H u), \nabla (v - I_H v))_{L^2(\Omega_i)} + \sum_{even \ i} \tilde{a}_i^N(\tilde{H}_i u^\rho, \tilde{H}_i v^\rho) + a^\rho(I_H u, I_H v).
\] (18)

**Theorem 2** For any function \( u \in V^h_0(\Omega) \), the following inequalities hold

\[
m (1 + \ln \frac{H}{h})^{-1} b^\rho(u, u) \leq a^\rho(u, u) \leq M b^\rho(u, u),
\] (19)

provided coefficients of the bilinear form \( a^\rho(\cdot, \cdot) \) are constant on strip \( \Omega_i \), (see (4)), the positive constants \( m \) and \( M \) are independent of \( H, h \) and the jumps of \( \rho_i \).

This theorem is proved in Mróz (1995).
6 Neumann-Dirichlet preconditioner as inexact solver in ASM

In this section we apply the Neumann–Dirichlet preconditioner to Additive Schwarz Method as inexact solver for local problems. Such approach allow us to construct a structural algorithm with optimal estimates on convergence. We use the framework of ASM developed by Dryja and Widlund (1990). The algorithm is presented for the case of regular coefficients.

The space $V_0^h(\Omega)$ is represented as a sum of two spaces

$$V_0^h(\Omega) = V_0 + V_1 = V_0^H(\Omega) + V_0^D(\Omega).$$ (20)

Let us define inner local products $b_i(\cdot, \cdot)$, $i = 0, 1$

$$b_0(u, v) = a(u, v), \quad b_1(u, v) = \sum_{\text{odd } i} a_i^D(\mathcal{P}_i u, \mathcal{P}_i v) + \sum_{\text{even } i} \tilde{a}_i^N(u, v) \quad u, v \in V_0^H(\Omega),$$ (21)

where $\tilde{a}_i^N(u, v)$ was introduced in (13).

Let $T_i$ denote the approximate projections from $V_0^h(\Omega)$ to $V_i$ with respect to bilinear form $b_i(\cdot, \cdot)$

$$b_i(T_i u, v) = a(u, v) \quad \forall v \in V_i$$ (22)

If the operator $T = T_0 + T_1$ is invertible then (6) is equivalent the following auxiliary problem:

Find $u \in V_0^h(\Omega)$ which satisfies

$$Tu = g$$ (23)

where the right hand side $g$ has to be chosen so that the auxiliary equation (23) has the same solution as (6).

**Theorem 3** The operator $T : V_0^h(\Omega) \rightarrow V_0^h(\Omega)$ is symmetric and the following estimates hold

$$m a(u, u) \leq a(Tu, u) \leq M a(u, u) \quad \forall u \in V_0^h(\Omega),$$ (24)

where constants $m$ and $M$ are independent of $H, h$.

The proof of this theorem can be found in Mróz (1995).

In each step of PCG method (cf. Concus, Golub O’Leary (1976)) we have to calculate the function $w \in V_0^h(\Omega)$

$$w = Tu = w_0 + w_1, \quad w_i = T_i u,$$

where $u \in V_0^h(\Omega)$ is a given function. The algorithm of finding $w_0 = T_0 u$ involves solving the global problem defined on the coarse space $V_0^H(\Omega)$, see Step 3 of Algorithm 1. Let us now outline the algorithm of finding $w_1 = T_1 u$.

**Algorithm 2**

1. Construct the system for the nodal basis functions $\phi_j^h$ associated with the nodes of interiors of $\Omega^D_i$

$$a_i^D(\mathcal{P}_i w_1, \phi_j^h) = a(u, \phi_j^h) \quad \forall \phi_j^h \in V_0^h(\Omega^D_i).$$
2. Build the system for the nodal basis functions $\phi^h_j$ associated with the nodes of $\Omega_i^N$. Thus we compute $w_1$ on $\Omega_i^N$ $i = 2, 4, \ldots$ by solving

$$\tilde{a}_i^N(w_1, \phi^h_j) = a(u, \phi^h_j) - \sum_{\text{odd } i} a_i^D(\mathcal{P}_i w_1, \phi^h_j) \quad \forall \phi^h_j \in V^h(\Omega_i^N) \cap V_0^h(\Omega)$$

It reduces to solving the subproblems in each subdomain $\Omega_i^N$ with Neumann boundary conditions on interface lines $\Gamma_i$, and homogeneous Dirichlet boundary conditions on $\partial \Omega$.

3. In Step 1 the projections $\mathcal{P}_i w_1$ have been computed, so now we find the function $w_1$ on $\Omega_i^D$. This is done in the same way as in Step 4 of Algorithm 1.

References


