Domain Decomposition Methods with Strip Substructures

Monika Mróz [1]

Abstract: Domain Decomposition Method for finite element discretization of planar elliptic variational problems with regular and discontinuous coefficients is analyzed. The domain is divided into strip shaped subdomains. We construct Neumann – Dirichlet substructuring algorithms. The approximate solution is obtained iteratively by solving local problems associated with each strip and the global problem associated with the coarse triangulation. Convergence of algorithms is almost optimal with respect to the parameters of triangulations and independent of the jumps of coefficients.

1 Introduction

Domain Decomposition Method for finite element discretization of elliptic problems with discontinuous coefficients is analyzed. We introduce two level nested triangulation. The domain is divided into strip shaped subdomains. We construct Neumann – Dirichlet substructuring algorithm with coarse space.

The partition of the domain into such subdomains has several advantages. The bandwidth of local matrices is narrow, which minimizes computations and memory requirements. Also the structure of local problems is useful for vectorization of an algorithm.

The substructuring preconditioners are usually constructed on subdomains (boxes) defined by the coarse triangulation, cf. Bramble, Pasciak, Schatz (1986). The convergence rate of iterative method which is depends on the condition number of preconditioned system, is bounded polylogarithimcally in H/h. Here H and h denote parameters of coarse and fine triangulations. Our algorithms have similar convergence properties. The condition number of preconditioned system for strips is proportional to $(1 + \ln(H/h))$. The related numerical experiments are included in Mróz (1995).

[1] Institute of Applied Mathematics and Mechanics, Warsaw University,Banacha 2, 02-097 Warsaw, Poland, Email monika@appli.mimuw.edu.pl. This work has been supported by Polish Scientific Grant 211669101

Domain Decomposition Methods in Sciences and Engineering, edited by R. Glowinski *et al.*

The algorithms presented in this paper can be extended to 3–D case. Also they are applicable to parabolic and nonlinear problems. The Neumann–Dirichlet preconditioner can be used as inexact solver in Additive Schwarz Method. In such situation we obtain optimal convergence and much simpler implementation. We can construct substructuring preconditioner for boxes. Each strip is divided into boxes and for each strip we apply our Neumann–Dirichlet preconditioner. The convergence of such method is the same as for strips. All these extensions are analyzed in Mróz (1995).

2 Model problem

We consider the problem of finding an approximate solution of the following elliptic, boundary value problem.

For given a bilinear form $a(\cdot,\cdot)$ and linear functional $l(\cdot)$ on $H_0^1(\Omega)$ we want to find $u \in H_0^1(\Omega)$ such that

$$a(u,v) = l(v) \quad \forall v \in H_0^1(\Omega) , \tag{1}$$

where Ω is a Lipschitz bounded domain in R^2. For simplicity of presentation we assume that Ω is a polygon.

We will distinguish two cases for $a(\cdot,\cdot)$.

bilinear form with regular coefficients

The bilinear form in this case is as follows:

$$a(u,v) = \sum_{i=1}^{2} \int_\Omega \frac{\partial u}{\partial x_i} \frac{\partial v}{\partial x_i})dx . \tag{2}$$

bilinear form with discontinuous coefficients

We consider the variational problem of the form (1) up to replacing $a(\cdot,\cdot)$ by the form

$$a^\rho(u,v) = \int_\Omega \rho(x) \sum_{i=1}^{2} \frac{\partial u}{\partial x_i} \frac{\partial v}{\partial x_i} dx . \tag{3}$$

The function $\rho(x)$ is piecewise constant i.e

$$\rho(x) = \rho_i > 0 , \quad x \in \Omega_i , \tag{4}$$

where Ω_i denote the strip shape subdomains consisting of elements Ω^j, defined in Section 2. The jumps of coefficients between subdomains may be large. This model problem can be applied to the case when the function $\rho(x)$ varies moderately on each subdomain and is discontinuous between subdomains. In this case the coefficient is merely equal to the mean value of $\rho(x)$ on the subdomain.

Let $l(v)$ denote the linear form defined by

$$l(v) = (f,v)_{L^2(\Omega)} = \int_\Omega fv dx . \tag{5}$$

3 Finite Element approximation

A two level triangulation is defined on the domain Ω. First, we construct a coarse triangulation Ω_H that consists of shape regular (cf. Ciarlet (1978)), nonoverlaping triangles Ω^j of diameter of order H. In second step, we further divide each element of the triangulation Ω_H into smaller, shape regular triangles of diameter $O(h)$. They form the fine triangulation Ω_h.

Spaces of piecewise linear, continuous functions on Ω_H and Ω_h are denoted by $V^H(\Omega)$ and $V^h(\Omega)$. The restriction to subspaces of functions vanishing on $\partial\Omega$ is denoted by $V_0^h(\Omega)$ and $V_0^H(\Omega)$ respectively. The corresponding approximate problem for (1) is then:

Find $u^* \in V_0^h(\Omega)$ such that

$$a(u^*, v) = l(v) \quad \forall v \in V_0^h(\Omega) . \tag{6}$$

Let $\{\phi_j^h\}$ be the set of standard, piecewise linear, nodal basis functions, thus $V_0^h(\Omega) = span\{\phi_j^h\}$. In this basis, the discrete variational problem (6) can be rewritten as a system of linear equations

$$A\underline{u} = \underline{f} , \tag{7}$$

where coefficients $A_{jl} = a(\phi_j^h, \phi_l^h)$ and $f_j = l(\phi_j^h)$ The matrix A is positive definite and symmetric. The condition number of A is proportional to h^{-2}.

In the same manner, we can formulate the discrete variational problem for the bilinear form $a^\rho(\cdot,\cdot)$ from (3). The condition number of A^ρ is proportional to $\frac{\max_j \rho^j}{\min_j \rho^j} h^{-2}$.

4 Neumann–Dirichlet preconditioner – regular coefficients

In this section we construct a Neumann–Dirichlet preconditioner for the problem (2) with regular coefficients.

The domain Ω is divided into N strips Ω_i, $i = 1, \ldots, N$ called strips. We assume that the boundary of each strip consists only of boundaries of elements from the coarse triangulation Ω_H and there are no nodes of Ω_H inside the strip. The strip Ω_i has common boundary only with at most two neighboring strips. This common interface between two strips is called Γ_i,

$$\Gamma_i = \partial\Omega_i \cap \partial\Omega_{i+1} .$$

Every point of Γ_i belongs to exactly two strips. The boundary of each strip consists of the two interface lines Γ_{i-1} and Γ_i and parts of $\partial\Omega$.

Let us denote the odd strips by Dirichlet superscript Ω_i^D and even strips by Neumann superscript Ω_i^N. The bilinear forms $a_i(\cdot,\cdot)$ represents restrictions of $a(\cdot,\cdot)$ to Ω_i. In order to distinguish the form $a_i(\cdot,\cdot)$ defined on Dirichlet or Neumann type strips we will add suitable superscript to this notation. Thus bilinear forms $a_i^D(\cdot,\cdot)$ and $a_i^N(\cdot,\cdot)$ are defined on Dirichlet or Neumann type strips respectively.

We define the local orthogonal projection P_i of the space $V^h(\Omega_i) \cap V_0^h(\Omega)$ onto $V_0^h(\Omega_i)$ by

$$a_i(P_i u, v) = a_i(u, v) \quad \forall v \in V_0^h(\Omega_i) , \tag{8}$$

and the local discrete harmonic function $H_i u$ with respect to bilinear form $a_i(\cdot,\cdot)$ by

$$\begin{aligned} a_i(H_i u, v) &= 0 \qquad \forall v \in V_0^h(\Omega_i)\,, \\ H_i u &= u \qquad \text{on } \partial\Omega_i\,. \end{aligned} \tag{9}$$

For any function $V^h(\Omega_i) \cap V_0^h(\Omega)$ we have

$$u = P_i u + H_i u\,.$$

Thus

$$a(u,u) = \sum_{even\ i} a_i^N(u,u) + \sum_{odd\ i} \left(a_i^D(P_i u, P_i u) + a_i^D(H_i u, H_i u)\right)\,. \tag{10}$$

Dryja and Proskurowski (1985) constructed the Neumann–Dirichlet preconditioner by omitting the last term of representation (10). Then the upper estimate of $a(u,u)$ by this preconditioner depends on H^{-2} since the Trace and Extension Lemmas were used for strip shape subdomains. In order to avoid such dependence, the mechanism of global transportation of information should be introduced into the definition of the preconditioner (cf. Widlund (1988). To meet this requirement we include the term $a(I_H u, I_H u)$ to the definition of the preconditioner, where I_H denotes the nodal value interpolation operator from $V_0^h(\Omega)$ onto $V_0^H(\Omega)$, defined by

$$(I_H u)(x) = u(x)\,, \tag{11}$$

if x is a node of the triangulation Ω_H.

We are ready to define a bilinear form $b(\cdot,\cdot)$ that corresponds to the Neumann–Dirichlet preconditioner,

$$\begin{aligned} b(u,v) &= \sum_{odd\ i} a_i^D(\mathcal{P}_i(u - I_H u), \mathcal{P}_i(v - I_H v)) + \\ &+ \sum_{even\ i} \tilde{a}_i^N(u - I_H u, v - I_H v) + a(I_H u, I_H v)\,. \end{aligned} \tag{12}$$

where

$$\tilde{a}_i^N(u,v) = a_i^N(u,v) + H^{-2}(u,v)_{L^2(\Omega_i^N)}\,. \tag{13}$$

Theorem 1 *For any function $u \in V_0^h(\Omega)$, the following inequalities holds*

$$m\,(1 + \ln\frac{H}{h})^{-1} b(u,u) \le a(u,u) \le M\, b(u,u)\,, \tag{14}$$

where the positive m and M are independent of H and h

The proof of this theorem is given in Mróz (1995).

To solve the linear system (7), we use a preconditioned gradient method (PCG) (cf. Concus, Golub, O'Leary (1976)). In each step of the PCG method the system corresponding to the preconditioner is to be solved:

Find $u \in V_0^h(\Omega)$ such that

$$b(u,v) = g(v) \quad \forall v \in V_0^h(\Omega)\,. \tag{15}$$

The finite element space $V_0^h(\Omega)$ is decomposed,

$$V_0^h(\Omega) = \tilde{V}_0^h(\Omega) \oplus V_0^H(\Omega) ,$$

where $\tilde{V}_0^h(\Omega)$ is a subspace of $V_0^h(\Omega)$ consisting of functions vanishing at the nodes of the coarse triangulation Ω_H.

The algorithm of solving the problem (15) is as follows.

Algorithm 2

1. *Construct the system for the nodal basis functions ϕ_j^h connected with the nodes of the interiors of Ω_i^D. For such functions $I_H\phi_j^h = 0$, hence (15) is reduced to separate subproblems of finding local projections $\mathcal{P}_i w$, $i = 1, 3, \ldots$ by solving*

$$a_i^D(\mathcal{P}_i w, \phi_j^h) = g(\phi_j^h) \quad \forall \phi_j^h \in V_0^h(\Omega_i^D) .$$

   These systems correspond to solving the subproblems individually for each strip Ω_i with homogeneous Dirichlet boundary conditions on interface lines Γ_i.

2. *We build the system associated with $\overline{\Omega}_i^N$ excluding nodes of the coarse triangulation Ω_H. For such basis functions $I_H\phi_j^h = 0$, thus we search for w on Ω_i^N $i = 2, 4, \ldots$ by solving*

$$\tilde{a}_i^N(w, \phi_j^h) = g(\phi_j^h) - \sum_{odd\ i} a_i^D(\mathcal{P}_i w, \phi_j^h) \quad \forall \phi_j^h \in \tilde{V}^h(\Omega_i^N)$$

   These systems correspond to solving the subproblems individually for each strip Ω_i^N with Neumann boundary conditions on interface lines Γ_i, excluding the nodes of the triangulation Ω_H, where we impose homogeneous Dirichlet boundary conditions.

3. *Construct the system for basis function ϕ_j^H from $V_0^H(\Omega)$. Note that $\phi_j^H - I_H\phi_j^H = 0$, thus in order to find the interpolation $I_H u$ we solve the global system*

$$a(I_H u, \phi_j^H) = g(\phi_j^H) \quad \forall \phi_j^H \in V_0^H(\Omega)$$

4. *In Step 1 the projections $\mathcal{P}_i w$ have been computed, so now we find the solution u on Ω_i^D. The discrete harmonic function $\mathcal{H}_i w$ is obtained from the system.*

$$a_i^D(\mathcal{H}_i w, \phi_j^h) = 0 \quad \forall \phi_j^h \in V_0^h(\Omega_i^D) ,$$

   and then $w = \mathcal{H}_i w + \mathcal{P}_i w$ or we can solve the system

$$a_i^D(w, \phi_j^h) = g(\phi_j^h) \quad \forall \phi_j^h \in V_0^h(\Omega_i^D) .$$

   The boundary conditions are to be imposed so as to fix the values of w on the interface lines Γ_i as calculated in the previous Step. In the second case we do not need to keep the values of $\mathcal{P}_i w$ in all nodes of Ω_i after Step 1 is completed.

   The solution u on Ω is obtained from the formula

$$u = w + I_H u .$$

This algorithm admit quite high level of coarse-grained parallelism. Steps 1, 2 and 4 consist of solving a number of small separate systems. Furthermore Steps 2 and 3 may be performed concurrently.

5 Neumann–Dirichlet preconditioner – discontinuous coefficients

In this section we consider the differential problem with coefficients constant on each strip Ω_i

We will use the notation and definitions introduced in previous Section. The restriction of the bilinear form $a^\rho(\cdot,\cdot)$ to Ω_i is denoted by $a_i^\rho(\cdot,\cdot)$. The definition of projection $P_i^\rho u$ and discrete harmonic function $H_i^\rho u$ is the same as in (8) and (9) with respect to the bilinear form $a_i^\rho(\cdot,\cdot)$. The idea of construction of the Neumann–Dirichlet preconditioner is similar to that for regular coefficients. In order to avoid the dependence on jumps of coefficients, the weights are introduced into Neumann problems on even (Neumann type) strips. We first define the function u^ρ on $\partial\Omega$ and on Γ_i, $i = 1, \ldots, n-1$

$$\begin{array}{ll} u^\rho(x) = (\rho_i + \rho_{i+1})^{1/2}(u(x) - I_H u(x)) & x \in \Gamma_i \,, \\ u^\rho(x) = 0 & x \in \partial\Omega \,. \end{array} \tag{16}$$

The function $\tilde{\mathcal{H}}_i u^\rho \in V^h(\Omega_i^N)$ is a local discrete harmonic function defined on Ω_i^N with respect to the bilinear form

$$\tilde{a}_i^N(u,v) = (\nabla u, \nabla v)_{L^2(\Omega_i^N)} + H^{-2}(u,v)_{L^2(\Omega_i^N)} \,, \tag{17}$$

i.e.

$$\begin{array}{rcl} \tilde{a}_i^N(\tilde{\mathcal{H}}_i u^\rho, v) & = & 0 \qquad \forall v \in V_0^h(\Omega_i^N) \,, \\ \tilde{\mathcal{H}}_i u^\rho(x) & = & u^\rho(x) \,, \quad x \in \partial\Omega_i^N . \end{array}$$

The bilinear form $b^\rho(\cdot,\cdot)$ that corresponds to the Neumann–Dirichlet preconditioner for the case with piecewise constant coefficients is defined by

$$\begin{array}{rcl} b^\rho(u,v) & = & \sum_i \rho_i (\nabla \mathcal{P}_i(u - I_H u), \nabla(v - I_H v))_{L^2(\Omega_i)} + \\ & + & \sum_{even\ i} \tilde{a}_i^N(\tilde{\mathcal{H}}_i u^\rho, \tilde{\mathcal{H}}_i v^\rho) + a^\rho(I_H u, I_H v) \,. \end{array} \tag{18}$$

Theorem 2 *For any function $u \in V_0^h(\Omega)$, the following inequalities hold*

$$m\,(1 + \ln\frac{H}{h})^{-1} b^\rho(u,u) \le a^\rho(u,u) \le M\, b^\rho(u,u), \tag{19}$$

provided coefficients of the bilinear form $a^\rho(\cdot,\cdot)$ are constant on strip Ω_i, (see (4)), the positive constants m and M are independent of H, h and the jumps of ρ_i.

This theorem is proved in Mróz (1995).

6 Neumann-Dirichlet preconditioner as inexact solver in ASM

In this section we apply the Neumann–Dirichlet preconditioner to Additive Schwarz Method as inexact solver for local problems. Such approach allow us to construct a structural algorithm with optimal estimates on convergence. We use the framework of ASM developed by Dryja and Widlund (1990). The algorithm is presented for the case of regular coefficients.

The space $V_0^h(\Omega)$ is represented as a sum of two spaces

$$V_0^h(\Omega) = V_0 + V_1 = V_0^H(\Omega) + V_0^h(\Omega) . \tag{20}$$

Let us define inner local products $b_i(\cdot,\cdot)$, $i = 0, 1$

$$\begin{aligned} b_0(u,v) = &\ a(u,v), & u,v \in V_0^H(\Omega) , \\ b_1(u,v) = &\ \sum_{odd\ i} a_i^D(\mathcal{P}_i u, \mathcal{P}_i v) + \sum_{even\ i} \tilde{a}_i^N(u,v) & u,v \in V_0^h(\Omega) , \end{aligned} \tag{21}$$

where $\tilde{a}_i^N(u,v)$ was introduced in (13).

Let $\mathcal{T}_i$ denote the approximate projections from $V_0^h(\Omega)$ to V_i with respect to bilinear form $b_i(\cdot,\cdot)$

$$b_i(\mathcal{T}_i u, v) = a(u,v) \quad \forall v \in V_i \tag{22}$$

If the operator $\mathcal{T} = \mathcal{T}_0 + \mathcal{T}_1$ is invertible then (6) is equivalent the following auxiliary problem:

Find $u \in V_0^h(\Omega)$ which satisfies

$$\mathcal{T} u = g \tag{23}$$

where the right hand side g has to be chosen so that the auxiliary equation (23) has the same solution as (6).

Theorem 3 *The operator $\mathcal{T} : V_0^h(\Omega) \to V_0^h(\Omega)$ is symmetric and the following estimates hold*

$$m\ a(u,u) \le a(\mathcal{T}u,u) \le M\ a(u,u) \quad \forall u \in V_0^h(\Omega) , \tag{24}$$

where constants m and M are independent of H, h.

The proof of this theorem can be found in Mróz (1995).

In each step of PCG method (cf. Concus, Golub O'Leary (1976)) we have to calculate the function $w \in V_0^h(\Omega)$

$$w = \mathcal{T}u = w_0 + w_1 , \quad w_i = \mathcal{T}_i u ,$$

where $u \in V_0^h(\Omega)$ is a given function. The algorithm of finding $w_0 = \mathcal{T}_0 u$ involves solving the global problem defined on the coarse space $V_0^H(\Omega)$, see Step 3 of Algorithm 1. Let us now outline the algorithm of finding $w_1 = \mathcal{T}_1 u$.

Algorithm 2

1. *Construct the system for the nodal basis functions ϕ_j^h associated with the nodes of interiors of Ω_i^D*

$$a_i^D(\mathcal{P}_i w_1, \phi_j^h) = a(u, \phi_j^h) \quad \forall \phi_j^h \in V_0^h(\Omega_i^D) .$$

2. *Build the system for the nodal basis functions* ϕ_j^h *associated with the nodes of* $\overline{\Omega}_i^N$. *Thus we compute* w_1 *on* $\overline{\Omega}_i^N$ $i = 2, 4, \ldots$ *by solving*

$$\tilde{a}_i^N(w_1, \phi_j^h) = a(u, \phi_j^h) - \sum_{odd\ i} a_i^D(\mathcal{P}_i w_1, \phi_j^h) \quad \forall \phi_j^h \in V^h(\Omega_i^N) \cap V_0^h(\Omega)$$

   It reduces to solving the subproblems in each subdomain Ω_i^N *with Neumann boundary conditions on interface lines* Γ_i, *and homogeneous Dirichlet boundary conditions on* $\partial\Omega$.
3. *In Step 1 the projections* $\mathcal{P}_i w_1$ *have been computed, so now we find the function* w_1 *on* Ω_i^D. *This is done in the same way as in Step 4 of Algorithm 1 .*

References

Bramble J. H., Pasciak J. E., Schatz A. H. (1986) An iterative method for elliptic problems on regions partitioned into substructures. *Math. Comp.* 46, 361–369.

Ciarlet P. G. (1978) *The Finite Element Method for Elliptic Problems.* North–Holland.

Concus P., Golub G. H., O'Leary D. P.(1976) A generalized conjugate gradient method for the numerical solution of elliptic PDE. In *Sparse Matrix Computations* (Ed. J. R. Bunch and D. J. Rose). Academic Press, N.J., pp. 309–332.

Dryja M., Proskurowski W. (1985) Capacitance matrix method using strips with alternating Neumann and Dirichlet boundary conditions. *Applied Numer. Math.* 1.

Dryja, M., Widlund, O. B. (1990) Towards a Unified Theory of Domain Decomposition Algorithms for Elliptic Problems, in *Third International Symposium on Domain Decomposition Methods for Partial Differential Equations, held in Houston, Texas, March 20-22, 1989* (ed. T. Chan and others) SIAM, Philadelphia, PA.

Mróz M. (1995) Domain Decomposition Methods with Strip Substructures for Finite Element Problems, PHD Thesis, Institute of Applied Mathematics and Mechanics, Warsaw University.

Widlund O. B. (1988) Iterative substructuring methods: algorithms and theory for problems in the plane. *Domain Decomposition Methods for PDEs* (Ed. R. Glowinski and others) SIAM, Philadelphia.