Schwarz Domain Decomposition Method with Time Stepping along Characteristic for Convection Diffusion Equations

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1 Introduction

The multiplicative Schwarz domain decomposition method is a powerful iteration method for solving elliptic equations and other stationary problems. A systematic theory has been developed for elliptic finite element problems in the past few years, see [1, 2, 7, 8, 9]. In this paper, we are interested in solving the parabolic convection diffusion problems. We use time-stepping along characteristic method mentioned by Douglas, Russell [6], which is powerful especially for convection-dominated equations, and Galerkin approximation in the space variables. At a fixed time level, the resulting equation is equivalent to an elliptic problem which depends on a time-step parameter Δt. This suggests that we might apply the multiplicative Schwarz domain decomposition method, originally proposed for elliptic equations to the parabolic cases at every time level. The crucial mathematical questions is then to know how the convergence rate depends on the space mesh and the time step parameters. In the present paper, we introduce two kinds of domain decomposition algorithm, give the convergence rate and error estimates which tell us that after iterating only one cycle at every time-level, the global approximate solutions converge to the exact solution.

Other domain decomposition methods for parabolic problem can be found in [2,

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5, 8]. In [8], Lions has given a kind of Schwarz alternating algorithm in the case of two subdomains for the heat equation and a convergence result but without an error estimate. In [5] Dawson, Dupont have given a kind of nonoverlapping domain decomposition method for parabolic equations, but since they have used explicit schemes at the intersection points, a stability condition is needed in the convergence analysis. In [2], Cai has given a kind of additive Schwarz algorithms for parabolic convection diffusion equations.

The outline of the paper is as following: In the next section, we will introduce two kinds of multiplicative Schwarz algorithms with time-stepping along characteristic. In Sect. 3, we will give the convergence rate of this algorithm, we also give L^2 error estimates with a fixed number of iterations at every time level, which tell us that when $h, \Delta t$ are sufficiently small, the approximate solution converges after a fixed number of iterations at every time level. Throughout this paper, c and C, with or without subscripts, denote generic, strictly positive constants. Their values may be different at different occurrences, but they are independent of the mesh parameters h and the time step Δt, which will be introduced later.

2 Multiplicative Schwarz Algorithms

Without loss of generality, we consider the following model problem in a bounded polygonal domain $\Omega \subset R^2$

$$\begin{cases} \frac{\partial u}{\partial t} + b \cdot \nabla u - \sum\limits_{i,j=1}^{2} \frac{\partial}{\partial x_j}(a_{ij}\frac{\partial u}{\partial x_i}) = f, & \text{in} \quad \Omega, \\ u = 0, & \text{on} \quad \partial\Omega, \\ u(x,0) = u^0(x), & \text{in} \quad \Omega. \end{cases} \tag{1}$$

Here $b = (b_1, b_2), b \cdot \nabla u = b_1 \frac{\partial u}{\partial x_1} + b_2 \frac{\partial u}{\partial x_2} andJ = (0, T]$ denotes the time interval. The coefficient satisfy $a_{ij} = a_{ji}$ and there exists a positive constant γ such that

$$\sum_{ij=1}^{2} a_{ij}\xi_i\xi_j \geq \gamma |\xi|^2, \quad \forall \xi = (\xi_1, \xi_2)^T \in R^2. \tag{2}$$

The variational formulation of problem (1) is: For $t \in J$, find $u(t) \in H_0^1(\Omega)$ such that

$$\begin{cases} (\frac{\partial u}{\partial t}, v) + a(u,v) + (b \cdot \nabla u, v) = (f, v), & v \in H_0^1(\Omega), \\ (u(0), v) = (u^0, v) \end{cases} \tag{3}$$

where

$$a(u,v) = \int_\Omega \sum_{ij=1}^{2} a_{ij} \frac{\partial u}{\partial x_i} \frac{\partial v}{\partial x_j} dx. \tag{4}$$

Let Δt denote the time-step, and let $t^n = n\Delta t$ and $u^n = u(t^n)$. For any point $x = (x_1, x_2)$, let

$$\bar{x} = \begin{cases} x - b\Delta t = (x_1 - b_1\Delta t, x_2 - b_2\Delta t), & \text{when} \quad x - b\Delta t \in \Omega \\ 2Y(x - b\Delta t) - X(x - b\Delta t), & \text{when} \quad x - b\Delta t \notin \Omega \end{cases} \tag{5}$$

where $Y(x) \in \partial\Omega$ denotes the point of projection of x, $X(x) \in \Omega$ denotes the symmetric point of x, we also let $\overline{u}^{n-1} = u(\overline{x}, t^{n-1})$. Then

$$\frac{u^n - \bar{u}^{n-1}}{\Delta t} = \frac{\partial u}{\partial t} + b \cdot \nabla u + O(\frac{\partial^2 u}{\partial \tau^2}\Delta t) \tag{6}$$

where τ denotes the unit vector in the characteristic direction of the transport term $(\frac{\partial u}{\partial t} + b \cdot \nabla u)$. Then the form (3) can be changed to

$$(\frac{u^n - \bar{u}^{n-1}}{\Delta t}, v) + a(u, v) = (f^{n-1}, v) + (\rho^n, v), \tag{7}$$

where

$$\rho^n = \frac{u^n - \bar{u}^{n-1}}{\Delta t} - (\frac{\partial u}{\partial t} + b \cdot \nabla u)^n = O(\frac{\partial^2 u}{\partial \tau^2}\Delta t).$$

We now divide Ω into overlapping subdomains $\Omega_1, \Omega_2, \cdots, \Omega_p$ satisfying Condition(A):

Condition(A): For any $x \in \bar{\Omega}$ there exist an open domain D_x and an $i_0 \in \{1, 2, \cdots, p\}$ such that $x \in D_x$ and $D_x \cap \Omega \subset \Omega_{i_0}$.

Extending the elements in $H_0^1(\Omega_j)$ to Ω by zero, we give the semi-discreted multiplicative Schwarz algorithms:

Scheme I: Let $U^0 = u^0$, for $n \geq 1$, we find $U^n \in H_0^1(\Omega)$ by three steps:

1) Set $U_0^n = U^{n-1}$.

2) Find $U_{jp+i}^n (j = 0, 1, \cdots, m-1, i = 1, 2, \cdots, p)$ such that

$$\begin{cases} (\frac{U_{jp+i}^n - \bar{U}^{n-1}}{\Delta t}, v) + a(U_{jp+i}^n, v) = (f^n, v), & v \in H_0^1(\Omega_i), \\ U_{jp+i}^n = U_{jp+i-1}^n, & x \in \Omega \setminus \Omega_i. \end{cases} \tag{8}$$

3) Let $U^n = U_{mp}^n, x \in \Omega$. Here m denotes the iteration time at the time-level in question.

Problem (8) are continuous in the spatial variables; in practical computation we can use an appropriate numerical method to solve it. Next we shall give a kind of multiplicative Schwarz algorithm combined with Galerkin finite element method. Let T_h denote a quasi-uniform triangulation of Ω which is aligned with the above domain decomposition, let h be the mesh parameter. $M_h \subset H_0^1(\Omega)$ denotes a standard finite element space such that

$$\inf_{\varphi \in M_h} (\|u - \varphi\| + h\|u - \varphi\|_1) \leq C\|u\|_{r+1} h^{r+1}, \; u \in H_0^1(\Omega) \cap H^{r+1}(\Omega). \tag{9}$$

Let $T_{i,h}$ denote the restriction of T_h to Ω_i, let $M_h(\Omega_i)$ denote the restriction of M_h to Ω_i, and let $M_h^0(\Omega_i) = M^h(\Omega_i) \cap H_0^1(\Omega_i)$. Set the initial approximation $W_h^0 \in M_h$ satisfying

$$\|W_h^0 - u^0\| \leq Ch^{r+1}. \tag{10}$$

We give the following multiplicative Schwarz algorithm :

Scheme II: For $n \geq 1$ find $W_h^n \in M_h$ in three steps:

1). Let $W_0^n = W_h^{n-1}$

2). Find $W_{jp+i}^n (j = 0, 1, \cdots, m-1, i = 1, 2, \cdots, p)$ satisfying

$$\begin{cases} (\frac{W_{jp+i}^n - \bar{W}^{n-1}}{\Delta t}, v) + a(W_{jp+i}^n, v) = (f^n, v), & v \in M_h^0(\Omega_i), \\ W_{jp+i}^n = W_{jp+i-1}^n, & x \in \Omega \setminus \Omega_i. \end{cases} \tag{11}$$

3) Let $W^n = W^n_{mp}, x \in \Omega$.

It is clear that the solutions of Scheme I and Scheme II are unique.

3 Convergence Analysis and Error Estimates

First we give two lemmas given Condition (A). We use the notations

$$\begin{cases} A(u,v) = (u,v) + \Delta t a(u,v), \forall v \in H_0^1(\Omega), \\ \|u\|_a = (a(u,u))^{\frac{1}{2}}, \\ \|u\|_A = (A(u,u))^{\frac{1}{2}} = (\|u\|_0^2 + \Delta t \|u\|_a^2)^{\frac{1}{2}}. \end{cases} \tag{12}$$

Lemma 3.1 *Suppose that the domain decomposition satisfies Condition (A). Then for $u \in H_0^1(\Omega)$ there exist a decomposition $u = \sum_{i=1}^{p} u_l, u_l \in H_0^1(\Omega_l)$ such that*

$$\sum_{l=1}^{p} \|u_l\|_A^2 \le (1 + C_1 \Delta t)\|u\|_A^2, \ \forall u \in H_0^1(\Omega), \tag{13}$$

where C_1 denotes a constant independent of Δt and u.

Lemma 3.2 *Under the condition of Lemma 3.1, there exists a constant C_2 independent of $h, \Delta t$. For $u \in M_h$, there exists a decomposition $u = \sum_{l=1}^{p} u_l, u_l \in M_h^0(\Omega_l)$ such that*

$$\sum_{l=1}^{p} \|u_l\|_A^2 \le (1 + C_2(\Delta t + h))\|u\|_A^2. \tag{14}$$

In order to estimate the convergence rate, we need one of the following stronger but reasonable conditions, which can be easily satisfied.

Condition (B): The subregion $\Omega_i (1 \le i \le p)$ can be divided into four parts:

$$D_j = \sum_{r_{j-1} \le i \le r_j} \Omega_i, \ j = 1,2,3,4, \ r_0 = 0, \ r_4 = p \tag{15}$$

Subdomains in D_j are disjoint and $\{D_1, D_2\}, \{D_3, D_4\}, \{D_1 \cup D_2, D_3 \cup D_4\}$ are domain decompositions of $D_1 \cup D_2, D_3 \cup D_4, \Omega$ respectively, which satisfy Condition (A) for $p = 2$.

Condition (C): The subregion $\Omega_i (1 \le i \le p)$ can be divided into k parts:

$$D_j = \sum_{r_{j-1} \le i \le r_j} \Omega_i, j = 1,2,...,k, r_0 = 0, r_k = p \tag{16}$$

such that: (1) $\{\Omega_j, r_{j-1} \le i \le r_j\}$ is a domain decompositions of D_j satisfying condition (A) and for $r_{j-1}+1 \le i, l \le r_j$, $\Omega_i \cap \Omega_l = \emptyset$, if $l \ne i-1, i+1$; (2) $\{D_j, 1 \le j \le k\}$ is a domain decomposition of Ω satisfying Condition(A) and for $1 \le j, l \le k$, $\Omega_j \cap \Omega_l = \emptyset$, if $l \ne j-1, j+1$.

Remark 1. When Condition (B) holds, the above method can be parallelized by coloring the subdomains and solving in parallel on disjoint subdomains of the same color.

For error estimates of Scheme I, defining the operator $R_i : H_0^1(\Omega) \to H_0^1(\Omega_i)$ such that

$$A(R_i u - u, v) = 0, \forall v \in H_0^1(\Omega_i). \tag{17}$$

Theorem 3.1 *Suppose that Condition (B) is satisfied for the domain decomposition. Then there exists a constant C_3, independent of $u, \Delta t$, such that*

$$\|(I - R_p)\cdots(I - R_2)(I - R_1)u\|_A \le C_3 \Delta t^{\frac{1}{2}} \|u\|_A, \forall u \in H_0^1(\Omega). \tag{18}$$

Theorem 3.2 *Suppose that Condition (C) is satisfied for the domain decomposition. Then there exists a constant C_4 independent of $\Delta t, u$ such that*

$$\|(I - R_p)\cdots(I - R_2)(I - R_1)u\|_A \le C_4 \Delta t^{\frac{1}{2}} \|u\|_A, \forall u \in H_0^1(\Omega). \tag{19}$$

Let $\partial_t u^n = (u^n - u^{n-1})/\Delta t$, $e^n = U^n - u^n$, and $e_i^n = U_i^n - u^n$. Then $e^n = e_{mp}^n$ and

$$\begin{cases} (\frac{e_{jp+i}^n - e^{n-1}}{\Delta t}, v) + a(e_{jp+i}^n, v) = -(\rho^n, v), & \forall v \in H_0^1(\Omega_i), \\ (e_{jp+i}^n - e_{jp+i-1}^n) = 0, & x \in (\Omega - \Omega_i). \end{cases} \tag{20}$$

Theorem 3.3 *Suppose that Condition (B) or Condition (C) hold. Then there exists a constant C_5, independent of Δt, such that*

$$\|e^n\|_A \le (1 + C_5 \Delta t^m)\|e^{n-1}\|_A + C\Delta t(\Delta t^{\frac{m}{2}} + \Delta t) \tag{21}$$

where the constant C is independent of Δt.

Theorem 3.4 *Suppose that the solution of (1) is sufficiently smooth. Suppose also that Condition (B) or Condition(C) holds for the domain decomposition. For $m \ge 1$ and the solution of Scheme I, we have*

$$\|u^n - U^n\| \le C(\Delta t + \Delta t\|\frac{\partial^2 u}{\partial \tau^2}\|_{L^\infty(J;L^2(\Omega))} + \Delta t^{\frac{m}{2}}),$$

where C denotes a generic constant which is independent of Δt.

For error estimates of Algorithm II, define $R_{h,i} : M_h \to M_h^0(\Omega_i)$ such that

$$A(R_{h,i}u - u, v) = 0, \quad \forall v \in M_h^0(\Omega_i). \tag{22}$$

Theorem 3.5 *Suppose that Condition (B) or condition (C) are satisfied for the domain decomposition. Then, there exists a constant C_6, independent of $u, h, \Delta t$, such that*

$$\|(I - R_{h,p})\cdots(I - R_{h,2})(I - R_{h,1})u\|_A \le C_6(h + \Delta t)^{\frac{1}{2}}\|u\|_A, \forall u \in M_h \tag{23}$$

Define an auxiliary function $\widetilde{u}_h^n \in M_h$ such that

$$a(\widetilde{u}_h^n - u^n, v) = 0, \forall v \in M_h, \tag{24}$$

and let $\eta = u^n - \widetilde{u}_h^n$, $E^n = W^n - \widetilde{u}_h^n, E_i^n = W_i^n - \widetilde{u}_h^n$, as in (18), (19) and (21). If Condition (B) or Condition (C) is satisfied, we can prove that

$$\|(I - R_{h,p}) \cdots (I - R_{h,2})(I - R_{h,1})\| \leq C(\Delta t + h)^{\frac{1}{2}}, \tag{25}$$

$$\|E^n\|_A \leq (1 + C_7(\Delta t + h)^m)\|E^{n-1}\|_A + C\Delta t(\Delta t + h^{r+1} + (\Delta t + h)^{\frac{m}{2}}), \tag{26}$$

where C_7 denotes a constant which satisfies $1 + C^2(\Delta t + h)^m \leq 1 + 2C_7(\Delta t + h)^m$.

Theorem 3.6 *Under the assumptions of Theorem 3.4, and for $h^m = O(\Delta t)$,there exists a constant C, independent of $h, \Delta t$, such that for the solution of Scheme II*

$$\|u^n - W^n\| \leq C(\Delta t + \Delta t\|\frac{\partial^2 u}{\partial \tau^2}\|_{L^\infty(J;L^2(\Omega))} + \Delta t^{-\frac{1}{2}} h^{m+1} + (\Delta t + h)^{\frac{m}{2}}). \tag{27}$$

Here C denote a generic constant independent of Δt and h.

Remark 2 If $b_1 = b_2 = 0$, the term $\Delta t^{-\frac{1}{2}} h^{m+1}$ can be removed and the error estimates are of optimal order.

Remark 3 Theorem 3.1, Theorem 3.2 and (25) tell us that the convergence rate for Algorithm I, Algorithm II is $\rho = C\Delta t^{\frac{1}{2}}$, $\rho = C(\Delta t + h)^{\frac{1}{2}}$ respectively. Since we do not use a coarse mesh triangulation, the constant C depends on the overlapping parts of subdomains.

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