Substructure Preconditioners for Nonconforming Plate Elements

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1 Introduction

In this paper, we generalize the BPS algorithm [1] to nonconforming element approximations of the biharmonic equation. We construct a preconditioner for the Morley element by substructuring on the basis of a space decomposition. The space decomposition is introduced by partitioning discrete biharmonic functions into low and high frequency components through intergrid transfer operators between coarse and fine meshes and a conforming interpolation operator. The method leads to a preconditioned system with the condition number bounded by $C(1 + \log^2 H/h)$ in the case with interior cross points, and by C in the case without interior cross points, where H is the subdomain size and h is the mesh size. These techniques are applicable to other nonconforming plate elements and are well suited to parallel computation.

For conforming element discrete problems of a second order elliptic equation, Bramble et al [1] and Widlund [7] have obtained certain preconditioners which are easily inverted in parallel and can reduce the condition number of a discrete system from $O(h^{-2})$ to $O(1 + \log^2 H/h)$. The main idea is a decomposition given by $v = \Pi_H v + (v - \Pi_H v)$, where Π_H is the interpolation operator on coarse meshes and the nodal parameters of $v - \Pi_H v$ vanishes on the coarse mesh nodes, and an extension theorem. Gu and Hu [3] have obtained a similar result for Wilson nonconforming element. Zhang [9] has constructed preconditioners for certain conforming plate elements on the basis of a space decomposition by adding certain vertex spaces. However, for Morley element, since the finite element spaces are not nested, and the functions have bad discontinuities, a space decomposition similar to those mentioned above does not hold.

We introduce a conforming interpolation operator for the Morley element and related intergrid transfer operators, and then construct a space decomposition to overcome these difficulties. Brenner [2] has introduced a conforming interpolation operator E_h by taking averages of the nodal parameters associated with the function and its first derivatives among the relevant elements, and taking zero as the nodal

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Domain Decomposition Methods in Sciences and Engineering, edited by R. Glowinski *et al.*

parameters associated with its second-order derivatives, in order to deal with an overlapping domain decomposition method. To be suited to a parallel computation in the substructure preconditioning, we modify Brenner's approach so that the nodal parameters of $E_h v_h$ depend only on those of v_h on the boundaries of substructures. Zhang [9] on the other hand, has defined an interpolation operator for certain conforming plate elements by setting the nodal parameters for second-order derivatives to zero. We use it to define the intergrid transfer operators I_h from coarse to fine meshes and I_H from fine to coarse meshes. Then we generalize the BPS algorithms and Widlund theory of substructure preconditioning to nonconforming plate elements.

2 A Preconditioning Algorithm

Let Ω be a bounded polygonal domain in R^2. Let J_h and J_H be quasi-uniform triangulations of Ω with h and H as mesh parameters respectively. Assume that J_h can be obtained by refining J_H, so that J_H and J_h form a two-level triangulations on Ω and the nodes of J_H are those of J_h. Let $S^h(\Omega)$ be the Morley element space [5] and let $S_0^h(\Omega)$ be a subspace of $S^h(\Omega)$ with nodal parameters vanishing at the boundary nodes. The Morley element discrete problem is : Find $u_h \in S_0^h(\Omega)$ such that

$$a_h(u_h, v_h) = (f, v_h), \quad \forall v \in S_0^h(\Omega),$$

where

$$a_h(u,v) = \sum_{T\in J_h} \sum_{|\alpha|=2} \int_T D^\alpha u D^\alpha v dx, \quad (f,v) = \int_\Omega f v dx.$$

Let $J_H = \{\Omega_k\}_{k=1}^N$. The vertices of J_H will be labeled by v_j (ordered in some way) and Γ_{ij} will denote the edge with endpoints v_i and v_j. $S_0^h(\Omega_j)$ will denote the subspace of $S_0^h(\Omega)$ consisting of functions with nodal parameters vanishing on $\bar{\Omega}\backslash\Omega_j$. In addition, $S^h(\Omega_j)$ will be the set of functions which are restrictions, of those in $S_0^h(\Omega)$, to $\bar{\Omega}_j$. In what follows, c and C (with or without subscript) will denote generic positive constants which are independent of H, h and Ω_k.

We construct our preconditioner B through its corresponding bilinear form $B(\cdot,\cdot)$ defined on $S_0^h(\Omega) \times S_0^h(\Omega)$.

We decompose functions in $S_0^h(\Omega)$ as follows:

Write $w = w_P + w_H$, where $w_P \in S_0^h(\Omega_1) \oplus \cdots \oplus S_0^h(\Omega_N)$ satisfies

$$a_h^k(w_P, \phi) = a_h^k(w, \phi), \quad \forall \phi \in S_0^h(\Omega_k), \text{ for each } k,$$

where

$$a_h^k(u,v) = \sum_{T\in J_h, T\subset\Omega_k} \sum_{|\alpha|=2} \int_T D^\alpha u D^\alpha v dx.$$

Notice that w_P is determined on Ω_k by the nodal parameters of w on Ω_k and that

$$a_h^k(w_H, \phi) = 0 \text{ for all } \phi \in S_0^h(\Omega_k).$$

Thus on each Ω_k, w is decomposed into a function w_P whose nodal parameters vanish on $\partial\Omega_k$ and a function $w_H \in S^h(\Omega_k)$ which satisfies the above homogeneous equations and has the same nodal parameters as w at $\bigcup_k \partial\Omega_k$. We shall refer to such a function w_H as "discrete a_h^k–biharmonic".

We note that the above decomposition is orthogonal with respect to the inner-product $a_h(\cdot,\cdot)$ and hence, $a_h(w,w) = a_h(w_P,w_P) + a_h(w_H,w_H)$.

To define the bilinear form $B(\cdot,\cdot)$, we introduce a linear interpolation operator E_h, and intergrid transfer operators I_h and I_H. The conforming relative of Morley element is the Argyris quintic element. Let $AR^h(\Omega)$ and $AR^H(\Omega)$ be the Argyris quintic element space associated with J_h and J_H, respectively; see [2].

For an arbitrary vertex p of J_h, we assign to it one of its adjacent edge midpoints e_p. If $p \in \bigcup\Gamma_{ij}$, we assign to it e_p which belongs to $\bigcup\Gamma_{ij}$. If $p \in \partial\Omega$, we assign to it e_p which belongs to $\partial\Omega$. For $v \in S_0^h(\Omega)$, we define $E_h v \in AR^h(\Omega)$ such that

$$\begin{aligned} &E_h v(p) = v(p), \quad \forall \text{ verties } p \\ &\partial_n E_h v(m) = \partial_n v(m), \quad \forall \text{ midpoint } m \\ &D^\alpha E_h v(p) = 0, \quad |\alpha| = 2; \end{aligned} \tag{2.1}$$

and

$$\begin{aligned} \partial_x E_h v(p) &= \partial_n v(e_p)\cos\beta + \frac{v(p)-v(a)}{l_{ap}}\sin\beta, \\ \partial_y E_h v(p) &= \partial_n v(e_p)\sin\beta + \frac{v(a)-v(p)}{l_{ap}}\cos\beta, \end{aligned} \tag{2.2}$$

where $n = (\cos\beta, \sin\beta), s = (-\sin\beta, \cos\beta)$ are the unit normal and tangential vector respectively, and l_{ap} is the length of the segment ap (cf. Figure 2.1). We note that (2.1) is defined as in Brenner [2] but that (2.2) is different.

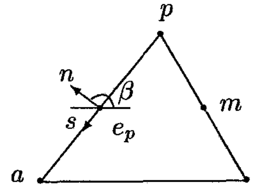

Figure 2.1

From the definition of the conforming interpolation operator E_h, we can see that nodal parameters of $E_h v_h$ on $\cup\Gamma_{ij}$ depend only on those of v_h on $\cup\Gamma_{ij}$. This property is important in our discussion.

The intergrid transfer operator $I_H : AR^h(\Omega) \longrightarrow AR^H(\Omega)$ is defined by (cf.[9])

$$\begin{cases} D^\alpha I_H v(p) = D^\alpha v(p), \text{ for } |\alpha| \le 1 \\ D^\alpha I_H v(p) = 0, \text{ for } |\alpha| = 2 \\ \partial_n I_H v(m) = \partial_n v(m), \text{ for all internal midpoints } m \in J_H, \end{cases}$$

for $v \in AR^h(\Omega)$. The intergrid transfer operator $I_h : AR^H(\Omega) \longrightarrow AR^h(\Omega)$ is defined similarly.

Now we construct a preconditioner. From the decomposition of Argyris quintic element space

$$AR^h(\Omega) = I_h I_H AR^h(\Omega) \oplus AR^h(\Omega)',$$

we decompose $w_H \in S^h(\Omega_k)$ into

$$w_H = w_E + w_V,$$

where $w_V \in S^h(\Omega_k)$ is a discrete a_h^k–biharmonic function such that the nodal parameters of $E_h w_V$ on $\partial\Omega_k$ are those of $I_h I_H E_h w_H$ along each Γ_{ij}. Thus w_E is a discrete a_h^k–biharmonic function in Ω_k for each k such that the nodal parameters of $E_h w_E$ vanish at all nodes of coarse meshes. Let

$$\bar{S}_0^h(\Omega) = \{v_h \in S_0^h(\Omega);\ \text{nodal parameters of } E_h v_h|_{\Gamma_{ij}} = \ \text{those of } I_h I_H E_h v_h|_{\Gamma_{ij}}\}$$

for all Γ_{ij}. Then, we have a space decomposition

$$S_0^h(\Omega) = \bar{S}_0^h(\Omega) \oplus (S_0^h(\Omega))'.$$

Using this decomposition, we now define the bilinear form $B(\cdot,\cdot)$ as follows

$$\begin{aligned}
B(w,\phi) &= a_h(w_P,\phi_P) \\
&+ \sum_{\Gamma_{ij}} \left\{ \langle \partial_s \bar{w}_E, \partial_s \bar{\phi}_E \rangle_{H_{00}^{1/2}(\Gamma_{ij})} + \langle \partial_n \bar{w}_E, \partial_n \bar{\phi}_E \rangle_{H_{00}^{1/2}(\Gamma_{ij})} \right\} \\
&+ \sum_{\Gamma_{ij}} \{ (w_V(v_i) - w_V(v_j) - D\bar{w}_V(v_i)(v_i - v_j)) \\
&\quad .(\phi_V(v_i) - \phi_V(v_j) - D\bar{\phi}_V(v_i)(v_i - v_j))H^{-2} \\
&+ (D\bar{w}_V(v_i) - D\bar{w}_V(v_j))(D\bar{\phi}_V(v_i) - D\bar{\phi}_V(v_j)) \} \\
&+ \sum_{T \in J_H} \sum_m (\partial_n(w_V - (w_V)_I)(m)(\partial_n(\phi_V - (\phi_V)_I)(m)),
\end{aligned} \tag{2.3}$$

where and from now on $\bar{v} = E_h v$, and $\langle\cdot,\cdot\rangle_{H_{00}^{1/2}(\Gamma_{ij})}$ means $H_{00}^{1/2}(\Gamma_{ij})$-inner product which is defined by

$$\begin{aligned}
\langle v, w \rangle_{H_{00}^{1/2}(\Gamma_{ij})} &= \int_{\Gamma_{ij}} \int_{\Gamma_{ij}} \frac{(v(x)-v(y))(w(x)-w(y))}{|x-y|^2} ds(x) ds(y) \\
&+ \int_{\Gamma_{ij}} v(x) w(x) \left(\frac{1}{|x - v_i|} + \frac{1}{|x - v_j|} \right) ds(x), v, w \in H_{00}^{1/2}(\Gamma_{ij}).
\end{aligned}$$

We shall demonstrate how the linear system $Bw = g$ can be solved efficiently.

Given g, the problem of solving $Bw = g$ reduces to finding the functions w_P and w_H. The function w_P restricted to Ω_k satisfies

$$a_h^k(w_P, \phi) = (g, \phi) \text{ for all } \phi \in S_0^h(\Omega_k). \tag{2.4}$$

Thus it can be obtained by solving in parallel the corresponding biharmonic Dirichlet problem (2.4) on each subdomain. With w_P known, we are left with the equation

$$\begin{aligned}
&\sum_{\Gamma_{ij}}\left\{\langle\partial_s\bar{w}_E,\partial_s\bar{\phi}_E\rangle_{H_{00}^{1/2}(\Gamma_{ij})}+\langle\partial_n\bar{w}_E,\partial_n\bar{\phi}_E\rangle_{H_{00}^{1/2}(\Gamma_{ij})}\right\}\\
&+\sum_{\Gamma_{ij}}\{(w_V(v_i)-w_V(v_j)-D\bar{w}_V(v_i)(v_i-v_j))\\
&\quad.(\phi_V(v_i)-\phi_V(v_j)-D\bar{\phi}_V(v_i)(v_i-v_j))H^{-2}\\
&+(D\bar{w}_V(v_i)-D\bar{w}_V(v_j))(D\bar{\phi}_V(v_i)-D\bar{\phi}_V(v_j))\}\\
&+\sum_{T\in J_H}\sum_m(\partial_n(w_V-(w_V)_I)(m))(\partial_n(\phi_V-(\phi_V)_I)(m))\\
&=(g,\phi)-a_h(w_P,\phi).
\end{aligned}\tag{2.5}$$

(The last equality holds since $a_h(w_P,\phi_H)=0$). Notice that the value of $(g,\phi)-a_h(w_P,\phi)$ for each ϕ depends only on the nodal parameters of $\bar{\phi}$ on all Γ_{ij}. From the definition of the interpolation operator E_h, we see that the value of $(g,\phi)-a_h(w_P,\phi)$ for each ϕ depends only on the nodal parameters of ϕ on all Γ_{ij}. Thus (2.5) gives rise to a set of equations which can be treated as follows: for each Γ_{ij}, choose ϕ in a subspace of $S_0^h(\Omega)$ such that the nodal parameters of $\bar{\phi}$ vanish in the all interior mesh points of every Ω_k and on all other Γ_{ij}. Thus, on this subspace, (2.5) decouples into independent problems of finding $\bar{w}_E\in AR_0^h(\Gamma_{ij})$, $I_H\bar{w}_E=0$ given by

$$\begin{aligned}
&\langle\partial_s\bar{w}_E,\partial_s\bar{\phi}\rangle_{H_{00}^{1/2}(\Gamma_{ij})}+\langle\partial_n\bar{w}_E,\partial_n\bar{\phi}\rangle_{H_{00}^{1/2}(\Gamma_{ij})}\\
&=(g,\phi)-a_h(w_P,\phi),\ \forall\phi\in S_0^h(\Omega),\ I_H\bar{\phi}=0,\ \bar{\phi}\in AR_0^h(\Gamma_{ij})
\end{aligned}\tag{2.6}$$

for each Γ_{ij}. Note that these are local problems with unknowns corresponding to the nodes on Γ_{ij} and may be solved in parallel.

Next we solve for $\bar{w}_V$ on the edges. We consider the subspace $\{\phi;$ nodal parameters of $\bar{\phi}|_{\Gamma_{ij}}=$ those of $I_hI_H\bar{g}|_{\Gamma_{ij}}, g\in S_0^h(\Omega)\}$. Then, (2.5) reduces to

$$\begin{aligned}
&\sum_{\Gamma_{ij}}\{(w_V(v_i)-w_V(v_j)-D\bar{w}_V(v_i)(v_i-v_j))\\
&\quad.(\phi_V(v_i)-\phi_V(v_j)-D\bar{\phi}_V(v_i)(v_i-v_j))H^{-2}\\
&+(D\bar{w}_V(v_i)-D\bar{w}_V(v_j))(D\bar{\phi}_V(v_i)-D\bar{\phi}_V(v_j))\}\\
&+\sum_{T\in J_H}\sum_m(\partial_n(w_V-(w_V)_I)(m))(\partial_n(\phi_V-(\phi_V)_I)(m))\\
&=(g,\phi)-a_h(w_P,\phi).
\end{aligned}\tag{2.7}$$

The nodal parameters of $\bar{w}_V$ at nodes of $T\in J_H$ determine those of w_V on all edges Γ_{ij}, and hence $w_H=w_E+w_V$ is known on all edges Γ_{ij}.

The last step consists of determining w_H in each Ω_k so that

$$a_h^k(w_H,\phi)=0 \text{ for } \phi\in S_0^h(\Omega_k).\tag{2.8}$$

This problem is similar to (2.4), which can also be solved in parallel on each subdomain. Hence the solution of $Bw=g$ is determined by $w=w_P+w_H$.

We summarize the process by outlining the steps for obtaining the solution of

$$B(w,\phi) = (g,\phi) \text{ for all } \phi \in S_0^h(\Omega),$$

and hence for computing the action of B^{-1}.

Algorithm.

1. Find w_P by solving biharmonic Dirichlet problems on the subdomains. The solution of each individual Dirichlet problem on subdomains may be done in parallel.
2. Find $\bar{w}_E$ on Γ_{ij} by solving a one-dimensional equation on each Γ_{ij}; this may be done in parallel.
3. Find $\bar{w}_V$ on $\bigcup \Gamma_{ij}$ by solving a coarse mesh equation and then extending it to all edges Γ_{ij} by operator I_h.
4. Find w_H by extending the nodal values of $w_E + w_V$ on $\cup\Gamma_{ij}$ to all subdomains. As in step 1, the solution may be done in parallel.

3 Estimates of the Condition Number

We have the following theorem.

Theorem. *There are positive constants λ_0, λ_1 and C such that*

$$\lambda_0 B(w,w) \le a_h(w,w) \le \lambda_1 B(w,w), \quad \forall w \in S_0^h(\Omega),$$

where $\lambda_1/\lambda_0 \le C(1+\log^2 H/h)$. If all of the nodes of Ω_k lie on $\partial\Omega$, then $\lambda_1/\lambda_0 \le C$.

The proof can be found in [6,8]. It means that the condition number grows at most like $(1+\log^2 H/h)$ as h tends to zero. Therefore the preconditioned iteration converges rapidly.

Remark. We can easily get similar results for many other nonconforming plate elements [4].

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