A characteristic domain splitting method

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Abstract. This work treats a linear convection-diffusion problem. Diffusion and convection may be equally important or convection may dominate the problem. The method of characteristics is combined with an overlapping domain decomposition technique so that domain decomposition is naturally combined with the time stepping. In each time step, the algorithm first determines the characteristic solution. Then a diffusion problem is solved in parallel on each subdomain with the characteristic solution as boundary conditions. No iteration is needed between the subdomain problems. Adaptive time steps are used for the characteristic tracing. The time steps used for the diffusion problems can be large.

Introduction

In this work, we consider the following linear convection-diffusion problem

$$\begin{cases} u_t - \epsilon\nabla \cdot (a(x)\nabla u) + \vec{b}(x) \cdot \nabla u = f, & \text{in } \Omega \subset \mathrm{R}^n, \ \mathrm{n} = 1,2,3. \\ u(x,t) = 0 \quad \text{on } \ \partial\Omega \times [0,\mathrm{T}] \ , & \\ u(x,0) = u_0(x) \quad \text{in } \ \Omega \ \text{ at } \ \mathrm{t} = 0. & \end{cases}$$

Assume that the problem has been suitably scaled such that a and $\vec{b}$ are of the same order. The parameter ϵ can be very small, but it can also be large. We are going to use the method of characteristics, see [DR] [Piro], to treat the convection part, and then use an overlapping domain decomposition technique to treat a symmetric diffusion problem. An important feature of this method is that no iterations may be required between the subdomain problems. This is due to the way domain decomposition is

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combined with the method of characteristics.

Domain decomposition methods have been used for nonsymmetrical convection-diffusion problems, see [BLP]–[CW], [W], [X1]-[XC], However, the methods proposed in these papers are efficient only for diffusion dominated problems. For problems where diffusion and convection are equally important or convection is dominating, special care must be taken. In a recent work by Rannacher and Zhou [RZ], the streamline diffusion method is used with an overlapping domain decomposition for a linear convection dominated problem. This work has been motivated by [RZ]. Here we use the method of characteristics to treat the convection term. By doing this, we get symmetrical problems when working with the diffusion part. Even for diffusion dominated problems or problems where diffusion and convection are equally important, the proposed algorithm will give better results than the conventional finite element method. This is due to the characteristic treatment of the convection term.

The Algorithm

At a given time t, and for a given x, let $X = X(x, t; t_0)$ be the solution of:

$$\begin{cases} \frac{dX}{dt} = \vec{b}(X), \\ X(x, t_0; t_0) = x. \end{cases} \tag{1}$$

If $\vec{b}$ is smooth, then there always exists a $\tau_0 > 0$ such that (1) has a unique solution for $|t - t_0| < \tau_0$. Let us choose an integer $N > 0$ such that $\Delta t = \frac{T}{N} \leq \tau_0$ and divide $[0, T] = \cup_{n=1}^{N} [t^{n-1}, t^n]$, $t^n = n\Delta t$. Backward tracing will be used to approximate the solution of (1). Moreover, adaptivity of the step used in the backward tracing will be needed. Thus let Δt_x be the step used in the backward tracing from the point x, i.e.

$$\Delta t = t^{n+1} - t^n = m_x \Delta t_x.$$

Then define points at the approximate characteristic (streamline) backwards from x by:

$$\tilde{x}^{n+1} = x,$$
$$\tilde{x}^{n+\frac{m_x-k}{m_x}} = \tilde{x}^{n+\frac{m_x-k+1}{m_x}} - \vec{b}\left(\tilde{x}^{n+\frac{m_x-k+1}{m_x}}\right) \cdot \Delta t_x\ , \qquad \text{for } \mathrm{k} = 1, 2, \cdots, \mathrm{m_x}.$$

When $k = m_x$, one finds that

$$\tilde{x}^n = \tilde{x}^{n-\frac{m_x-k}{m_x}}\ .$$

The characteristic solution is now given by

$$\tilde{u}^n = u^n(\tilde{x}^n)\ ,$$

where u^n is the computed solution at time level t^n. When $\tilde{x}^n$ falls outside of Ω, one takes $\tilde{u}^n = 0$. If nonhomogenous boundary conditions are used, we need to determine the point where the characteristic curve hits the boundary, and take $\tilde{u}^n$ to be the boundary value at that point.

The above iterative procedure is simply using the explicit one step backward tracing $\tilde{x} = x - \vec{b}(x) \cdot \Delta t$ over many local steps. For the domain decomposition, we assume that Ω has been divided into finite elements $\Omega = \cup_{e \in \mathcal{T}_h} e$. Let $\Omega_i, i = 1, 2, \ldots, m$ be a nonoverlapping decomposition of $\bar{\Omega}$, such that each Ω_i is the union of some elements. To each Ω_i, we associate an enlarged subdomain

$$\Omega_i^\delta = \{e \in \mathcal{T}_h | \quad dist(e, \Omega_i) \le \delta\} \ .$$

Hence, Ω_i^δ forms an overlapping domain decomposition with overlapping size δ. With each Ω_i^δ, we use $S_h^0(\Omega_i^\delta)$ to denote the linear finite element space with zero traces on Ω_i^δ. Note that the decomposition of Ω can be different from time level to time level in order to follow possible shock fronts.

Algorithm 1 *(The characteristic domain splitting algorithm).*

1. *Choose* $u_h^0 \in S_h^0(\Omega)$ *to be an approximation for* u_0.
2. *If* u^n *is known, do characteristic tracing to find*

$$\tilde{u}^n = u^n(\tilde{x}^n) \ .$$

This can be done in parallel for each of the nodal points at time level t^{n+1}.

3. *On each subdomain* Ω_i^δ, *find* u_i^{n+1} *in parallel for* $i = 1, 2, \ldots, m$ *such that*

$$\begin{cases} \left(\frac{u_i^{n+1} - \tilde{u}^n}{\Delta t}, v_i\right) + \epsilon(a\nabla u_i^{n+1}, \nabla v_i) = (f, v_i), \quad \forall v_i \in S_h^0(\Omega_i^\delta), \\ u_i^{n+1} = \tilde{u}^n \quad \text{on} \quad \partial\Omega_i^\delta \ . \end{cases}$$

4. *From the patchwise solution* u_i^{n+1}, *a global single valued solution*

$$u^{n+1} = \mathcal{C}\left(\{u_i^{n+1}\}_{i=1}^m\right) \in S_h^0(\Omega)$$

is constructed such that

$$\|u^{n+1}\|_{L^2(\Omega)} \le \sum \|u_i^{n+1}\|_{L^2(\Omega_i)}. \tag{2}$$

5. *If* $t^{n+1} < T$, *go to the next time level.*

For the above algorithm, detailed convergence analysis is given in [TDE]. We assume that the local time step satisfies

$$\Delta t_x \le \frac{C(\vec{b})}{\|\nabla u^n\|_{\infty, V(x)}} \Delta t \ ,$$

where $C(\vec{b})$ is a constant depending on $\vec{b}$, u^n is the solution computed at the previous time level and $V(x)$ is a neighbourhood of x such that the characteristic curve starting from x and going backward is contained in $V(x) \times [t^n, t^{n+1}]$. Furthermore, if the overlapping size satisfies

$$\delta \ge c_0 \max(\sqrt{\Delta t \epsilon}, h) |\ln \Delta t| \ , \tag{3}$$

where c_0 is a constant depending on the maximum angles of the elements in the overlapping area, then the computed solution u^n satisfies

$$\|u(t^n) - u^n\|_{L^2(\Omega)} \le C(h^2 + \Delta t) \ .$$

Above, C does not depend on ϵ, which means that we get the same accuracy in a region where the gradient of u is sharp.

Numerical Experiments.

As is shown by (3), for a given h and a given time step size Δt, we need to have sufficient overlap to guarantee the computational results to be of accuracy $O(h^2 + \Delta t)$.

As a test example, we consider a shock moving in the characteristic direction. It is known that $\phi(x, y, t) = \frac{1}{4\pi\epsilon t} e^{-\frac{x^2+y^2}{4\epsilon t}}$ satisfies the heat equation $\phi_t - \epsilon\Delta\phi = 0$. When ϵ is small, ϕ is singular near the point $x = y = 0$ for $t > 0$. Defining $u(x, y, t) = \phi(x - t, y - t, t + 0.1)$, one may easily verify that u is a solution of

$$\begin{cases} u_t - \epsilon\Delta u + u_x + u_y = 0, \\ u(x, y, 0) = \phi(x, y, 0.1). \end{cases}$$

This solution represents a shock moving in the direction of $\vec{b} = (1, 1)$. See Figure 1 for the location of the shock at different times.

In the computations, the domain Ω is taken as $\Omega = [0, 1] \times [0, 1]$. It is first divided into coarse rectangular subdomains with size $H = H_x = H_y$, and then each subdomain is divided into fine mesh rectangular elements with size $h = h_x = h_y$. Both the fine and the coarse meshes are uniform. A bilinear finite element space is used in the computations. Each coarse subdomain is extended by L elements into its neighbours. This defines the overlap. In the following tables, $m \times m$ is the number of subdomains and $n \times n$ is the number of elements in each subdomain. Hence $H = 1/m$, $h = 1/nm$. We let $\| \cdot \|_2$ denote the discrete L^2-norm.

The effect of varying the overlapping size L, and the number of subdomains m, is investigated in Table 1. Thus, the values of ϵ, Δt and h are fixed. Choosing $m = 1$ means that we are solving the global problem without domain decomposition. The computed solution u^n based on domain decomposition ($m > 1$) is compared with the exact solution $u(t^n)$ and the global solution u_G^n ($m = 1$), at $t^n = 0.5065$. From the table, one may observe that for different m, just one or two elements of overlap is needed to give the same accuracy as the global solution. However, by increasing the overlapping size L, u^n is getting closer and closer to the global solution u_G^n.

In Table 2, different values of ϵ is tested. When ϵ is getting smaller, the shock becomes sharper as one should expect. We observe that for large ϵ, more overlap is needed to retain the accuracy of the global solution. By decreasing Δt, a relatively small overlap may also retain the accuracy. When ϵ is small ($\epsilon = 0.01$), just one element of overlap is sufficient. Figure 1 shows the computed solution for $\epsilon = 0.01$ at different times. The figure for the analytical solution looks exactly the same.

Conclusion

The method of characteristics is combined with an overlapping domain decomposition technique to solve a convection-diffusion problem. When ϵ is small or Δt is small, one or two elements, are enough overlapping to get as accurate solution as the global solution. No iteration is needed between the subdomain problems. However, by increasing the overlapping size, the domain decomposition solution gets closer to the global finite element solution. When ϵ is large, more overlap is needed or one should decrease the time step size in order to decrease the needed overlapping size. By combining the domain decomposition method with the method of characteristics, the algorithm is able to capture sharp travelling shocks.

m	n	L	$\|u(t^n)-u^n\|_\infty$	$\frac{1}{h}\|u(t^n)-u^n\|_2$	$\|u_G^n-u^n\|_\infty$	$\frac{1}{h}\|u_G^n-u^n\|_2$
1	40	-	$6.96\cdot10^{-2}$	1.1817	-	-
4	10	1	$3.96\cdot10^{-2}$	0.3629	0.1062	1.5284
4	10	3	$5.86\cdot10^{-2}$	1.0120	$1.15\cdot10^{-2}$	0.1670
4	10	5	$6.83\cdot10^{-2}$	1.1621	$1.30\cdot10^{-3}$	0.0197
5	8	1	$5.33\cdot10^{-2}$	0.8033	0.1175	1.9780
5	8	2	$3.30\cdot10^{-2}$	0.5448	$3.72\cdot10^{-2}$	0.6389
5	8	4	$6.54\cdot10^{-2}$	1.1070	$4.32\cdot10^{-3}$	$7.48\cdot10^{-2}$
8	5	1	$1.33\cdot10^{-1}$	1.9804	0.2026	3.1568
8	5	2	$8.96\cdot10^{-3}$	0.1605	$6.39\cdot10^{-2}$	1.030
8	5	3	$4.84\cdot10^{-2}$	0.8264	$2.17\cdot10^{-2}$	0.3558

Table 1 ϵ=0.1, Δt=1/160, t^n=81/160,
$u(t^n)$: known solution, u^n: computed solution, u_G^n: global finite element solution.

ϵ	Δt	nt	m	n	L	$\|u(t^n)-u^n\|_\infty$	$\frac{1}{h}\|u(t^n)-u^n\|_2$	$\|u_G^n-u^n\|_\infty$	$\frac{1}{h}\|u_G^n-u^n\|_2$
0.01	1/160	81	1	80	-	1.7365	21.8401	-	-
0.01	1/160	81	4	20	1	1.4983	18.9283	0.3439	3.3324
0.01	1/160	81	4	20	3	1.7226	21.6590	$2.05\cdot10^{-2}$	0.2012
0.01	1/160	81	4	20	5	1.7358	21.8293	$1.10\cdot10^{-3}$	$1.20\cdot10^{-2}$
0.01	1/160	81	8	10	1	1.1523	14.7338	0.7372	7.8819
0.01	1/160	81	8	10	3	1.7017	21.3977	$4.23\cdot10^{-2}$	0.4704
0.01	1/160	81	8	10	5	1.7347	21.8144	$2.20\cdot10^{-3}$	$2.74\cdot10^{-2}$
0.5	1/160	81	1	40	-	$7.89\cdot10^{-4}$	$1.86\cdot10^{-2}$	-	-
0.5	1/160	81	4	10	3	$1.63\cdot10^{-2}$	0.3306	$1.71\cdot10^{-2}$	0.3487
0.5	1/160	81	4	10	5	$5.46\cdot10^{-3}$	0.1087	$6.25\cdot10^{-3}$	0.1268
0.5	1/160	81	4	10	8	$7.19\cdot10^{-4}$	$1.28\cdot10^{-2}$	$1.50\cdot10^{-3}$	$3.04\cdot10^{-2}$
1.0	1/500	250	1	40	-	$1.74\cdot10^{-4}$	$4.07\cdot10^{-3}$	-	-
1.0	1/500	250	4	10	3	$2.26\cdot10^{-3}$	$4.54\cdot10^{-2}$	$2.43\cdot10^{-3}$	$4.94\cdot10^{-2}$
1.0	1/500	250	4	10	5	$5.64\cdot10^{-4}$	$1.09\cdot10^{-2}$	$7.38\cdot10^{-4}$	$1.49\cdot10^{-2}$
1.0	1/500	250	4	10	6	$2.40\cdot10^{-4}$	$4.37\cdot10^{-3}$	$4.14\cdot10^{-4}$	$8.35\cdot10^{-3}$
1.0	1/500	250	4	10	7	$5.87\cdot10^{-5}$	$9.00\cdot10^{-4}$	$2.33\cdot10^{-4}$	$4.69\cdot10^{-3}$

Table 2 nt: number of time steps, $u(t^n)$: known solution,
u^n: computed solution, u_G^n: global finite element solution.

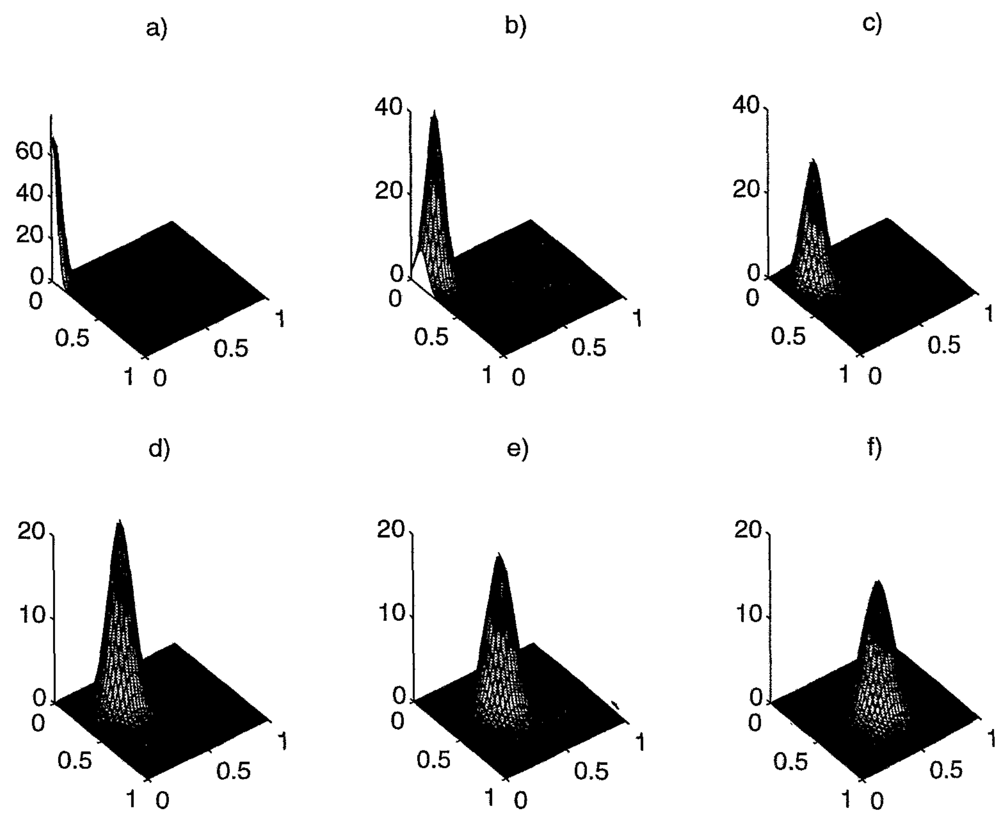

Figure 1 The computed solution with $H = 1/10, h = 1/200, L = 1$, $\epsilon = 0.01$, $\Delta t = 1/100$, at different times: a) t=0.01, b) t=0.1, c) t=0.2 d) t=0.3 e) t=0.4 f) t=0.5.

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