Two–level Method with Coarse Space Size Independent Convergence

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Dedicated to Honza.

1 INTRODUCTION

The basic disadvantage of the standard two-level method is the strong dependence of its convergence rate on the size of the coarse-level problem. In order to obtain the optimal convergence result, one is limited to using a coarse space which is only a few times smaller than the size of the fine-level one. Consequently, the asymptotic cost of the resulting method is the same as in the case of using a coarse-level solver for the original problem. Today's two-level domain decomposition methods typically offer an improvement by yielding a rate of convergence which depends on the ratio of fine and coarse level only polylogarithmically ([BPS86], [BPS89], [DSW94], [Man93], [FR91], [MT]). However, these methods require the use of local subdomain solvers for which straightforward application of iterative methods is problematic, while the usual application of direct solvers is expensive.

We suggest a method diminishing significantly these difficulties. Following the

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unpublished technical report [VK95], we develop a simple abstract framework based on the concept of smoothed aggregation introduced in [VMB] with aggregates derived from the system of nonoverlapping subdomains. We show that the smoothing of the coarse-space by an appropriate polynomial of degree about N_{es}^{-d} (symbol d denotes the dimension of the problem to be solved and N_{es} is the characteristic number of degrees of freedom per subdomain) can assure the coarse-space size independent convergence. The associated cost is significantly smaller than that of the local solvers in the case of standard domain decomposition. Moreover, it decreases as d increases.

Because of the page limit, we only apply the abstract framework to for the case of scalar equation with jumps in coefficients. More general problems and numerical experiments will be treated in [TVB].

2 ABSTRACT FRAMEWORK

We are interested in a numerical solution of a system of linear algebraic equations

$$A\mathbf{x} = \mathbf{b} \tag{1}$$

with a symmetric positive definite $n \times n$ matrix A. Let $P : \mathbb{R}^m \to \mathbb{R}^n$, $m << n$ be a linear injective tentative prolongator and $S \in [\mathbb{R}^n]$ a symmetric smoother commuting with A. Let us set

$$A_S = S^2 A, \quad S' = I - \frac{\omega}{\rho} A_S, \quad \omega \in (0, 2), \quad \rho = \rho(A_S). \tag{2}$$

Furthermore, let $\mathbf{x} \leftarrow \mathcal{S}_S(\mathbf{x}, \mathbf{b})$ and $\mathbf{x} \leftarrow \mathcal{S}_{S'}(\mathbf{x}, \mathbf{b})$ be relaxation methods consistent with (1) such that their linear parts are matrices S and S'. Our algorithm is a standard variational two-level method with a smoothed prolongator SP, a pre-smoother $\mathcal{S}_S$ and a post-smoother $\mathcal{S}_{S'}$.

Algorithm 1 *Given the initial approximation* $\mathbf{x}$,

repeat

1. $\mathbf{x} \leftarrow \mathcal{S}_S(\mathbf{x}, \mathbf{b})$,
2. *solve* $(P^T A_S P)\mathbf{v} = P^T S(A\mathbf{x} - \mathbf{b})$,
3. $\mathbf{x} \leftarrow \mathbf{x} - SP\mathbf{v}$,
4. $\mathbf{x} \leftarrow \mathcal{S}_{S'}(\mathbf{x}, \mathbf{b})$

until convergence;

5. *Post process* $\mathbf{x} \leftarrow \mathcal{S}_S(\mathbf{x}, \mathbf{b})$.

In the following, we will prove that, for a suitable S and P, steps 1–4 of the algorithm ensure convergence independent of the dimension ratio n/m in the A_S-norm. The postprocessing step 5 of the algorithm enables us to prove the same result in the energy norm of the original problem (1). The main disadvantage of the convergence estimate in A_S-norm is its indirect coarse-space dependence as for a smaller coarse-space we need a more powerfull smoother S to get the optimal convergence result.

Assumption 2 *Let the smoother S be a symmetric matrix that commutes with A, and $\rho(S) \leq 1$. We assume that the tentative prolongator P satisfies the weak approximation property in the following form: For every $\mathbf{u} \in \mathbb{R}^n$, there exists $\mathbf{v} \in \mathbb{R}^m$ such that*

$$\|\mathbf{u} - P\mathbf{v}\| \leq C_1 C_D(m,n)\rho^{-1/2}(A)\|\mathbf{u}\|_A. \tag{3}$$

For the prolongator smoother S, we require

$$\rho(S^2 A) \leq C_2^2 C_D^{-2}(m,n)\rho(A), \tag{4}$$

where $C_D(m,n), C_1, C_2 > 0$, and C_1, C_2 do not depend on m and n.

Remark 3 *For second order problems, we typically have $C_D(m,n) = H/h$ (the ratio of local meshsizes on the coarse and the fine level). In Section 3 we construct S as a suitable polynomial in A for which $\rho(S^2A) \approx N^{-2}\rho(A)$, where N denotes the degree of S. In order to satisfy Assumption 2, we need $N \approx H/h$. This choice yields a coarse level matrix $(SP)^T A(SP)$ with a number of nonzero entries per row uniformly bounded with respect to H/h. Detailed arguments will be given in Section 3.*

The following theorem shows that, under Assumption 2, the convergence rate of Algorithm 1 is independent of dimensions m and n of the coarse and fine spaces.

Theorem 4 *Let $\mathbf{e}_i$ denote the error after i iterations given by steps 1–4 of Algorithm 1 , and $\mathbf{e}_i^S = S\mathbf{e}_0$ the error smoothed by step 5. Then, it holds that*

$$\|\mathbf{e}_{i+1}\|_{A_S}^2 \leq (1 - C_3)\|\mathbf{e}_i\|_{A_S}^2, \quad \text{and} \quad \|\mathbf{e}_i^S\|_A^2 \leq (1-C_3)^i\|\mathbf{e}_0\|_A^2, \tag{5}$$

where $C_3 = \frac{(C_1C_2)^{-2}\omega(2-\omega)}{1+(C_1C_2)^{-2}\omega(2-\omega)} > 0$. Here ω is the damping parameter from (2).

Proof. Since $\rho(S) \leq 1$ and $\mathbf{e}_i^S = S\mathbf{e}_i$, we have $\|\mathbf{e}_i^S\|_A = \|\mathbf{e}_i\|_{A_S}$ and $\|\mathbf{e}_0\|_{A_S} \leq \|\mathbf{e}_0\|_A$. Therefore, the second inequality in (5) follows from the first one.

It is obvious that the linear part of the steps 1–4 is given by

$$S'[I - SP(P^TA_SP)^{-1}P^TSA]S = S'S[I - P(P^TA_SP)^{-1}P^TA_S].$$

Thus, the method can be viewed as a standard two-level method for solving a problem with matrix A_S (in place of A) and prolongator P (in place of SP).

Since $I - P(P^TA_SP)^{-1}P^TA_S$ is the A_S–orthogonal projection onto A_S–orthogonal complement of Range(P) (i.e., projection onto Ker(P^TA_S)), $S'S = SS'$, $\rho(S) \leq 1$, and $\rho(S') \leq 1$, we have

$$\|S'S[I - P(P^TA_SP)^{-1}P^TA_S]\|_{A_S}^2 \leq \sup_{\mathbf{x} \in \mathrm{Ker}\,(P^TA_S)} \min\left\{\frac{\|S\mathbf{x}\|_{A_S}^2}{\|\mathbf{x}\|_{A_S}^2}, \frac{\|S'\mathbf{x}\|_{A_S}^2}{\|\mathbf{x}\|_{A_S}^2}\right\}. \tag{6}$$

In the rest of the proof, we will show that at least one of the expressions in the minimum above is bounded by $1 - C_3$ for any $\mathbf{x} \in \mathrm{Ker}\,(P^TA_S)$.

We first express $\|S'\mathbf{x}\|_{A_S}/\|\mathbf{x}\|_{A_S}$ in terms of $\|S\mathbf{x}\|_{A_S}/\|\mathbf{x}\|_{A_S}$. It is easy to see that

$$\frac{\|A_S\mathbf{x}\|^2}{\|\mathbf{x}\|_{A_S}^2} \geq C_E\rho(A_S) \quad \text{implies} \quad \frac{\|S'\mathbf{x}\|_{A_S}^2}{\|\mathbf{x}\|_{A_S}^2} \leq 1 - C_E\omega(2-\omega). \tag{7}$$

Let us recall that $A_S = AS^2$, and A and S commute. Hence, we can write $\|S^2\mathbf{x}\|_A^2 = \|S\mathbf{x}\|_{A_S}^2$, and

$$\frac{\|A_S\mathbf{x}\|^2}{\|\mathbf{x}\|_{A_S}^2} = \frac{\|A_S\mathbf{x}\|^2}{\|S^2\mathbf{x}\|_A^2}\frac{\|S^2\mathbf{x}\|_A^2}{\|\mathbf{x}\|_{A_S}^2} = \frac{\|AS^2\mathbf{x}\|^2}{\|S^2\mathbf{x}\|_A^2}\frac{\|S\mathbf{x}\|_{A_S}^2}{\|\mathbf{x}\|_{A_S}^2}. \tag{8}$$

Now, consider $\mathbf{x} \in \operatorname{Ker}(P^T A_S) = \operatorname{Ker}(P^T AS^2)$. Then, setting $\mathbf{u} = S^2\mathbf{x}$, we have $\mathbf{u} \in \operatorname{Ker}(P^T A) = \operatorname{Range}(P)^{\perp_A}$. From the weak approximation condition (3), we estimate the ratio $\frac{\|AS^2\mathbf{x}\|^2}{\|S^2\mathbf{x}\|_A^2}$ using the standard orthogonality argument: For $\mathbf{v} \in \mathbb{R}^m$ from (3), we obtain

$$\|\mathbf{u}\|_A^2 = (A\mathbf{u}, \mathbf{u}) = (A\mathbf{u}, \mathbf{u} - P\mathbf{v}) \le \|A\mathbf{u}\|\,\|\mathbf{u} - P\mathbf{v}\| \le C_1 C_D(m,n)\rho^{1/2}(A)\|A\mathbf{u}\|\,\|\mathbf{u}\|_A.$$

Therefore,

$$\frac{\|A\mathbf{u}\|}{\|\mathbf{u}\|_A} \ge C_1^{-1}C_D^{-1}(m,n)\rho^{1/2}(A).$$

Substituting this estimate into (8) and using Assumption 2, we get

$$\frac{\|A_S\mathbf{x}\|^2}{\|\mathbf{x}\|_{A_S}^2} \ge C_1^{-2}C_D^{-2}(m,n)\rho(A)\frac{\|S\mathbf{x}\|_{A_S}^2}{\|\mathbf{x}\|_{A_S}^2} \ge (C_1C_2)^{-2}\rho(A_S)\frac{\|S\mathbf{x}\|_{A_S}^2}{\|\mathbf{x}\|_{A_S}^2}. \tag{9}$$

Thus, by (7)

$$\frac{\|S'\mathbf{x}\|_{A_S}^2}{\|\mathbf{x}\|_{A_S}^2} \le [1 - \frac{\|S\mathbf{x}\|_{A_S}^2}{\|\mathbf{x}\|_{A_S}^2}(C_1C_2)^{-2}\omega(2-\omega)].$$

Since $\|S\mathbf{x}\|_{A_S}^2/\|\mathbf{x}\|_{A_S}^2 \le 1$, we may finally write using (6)

$$\|S'S[I - P(P^TA_SP)^{-1}P^TA_S]\|_{A_S}^2 \le \sup_{\alpha\in[0,1]} \min\left\{\alpha, 1 - \alpha(C_1C_2)^{-2}\omega(2-\omega)\right\}.$$

The expression on the right hand side is bounded by $\frac{1}{1+(C_1C_2)^{-2}\omega(2-\omega)}$ which completes the proof. □

3 EXAMPLE OF A TENTATIVE COARSE SPACE AND PROLONGATOR SMOOTHER

Let $\Omega \subset \mathbb{R}^2$ be a Lipschitz domain and $\mathcal{T}$ be a shape-regular (locally quasiuniform) finite element mesh on Ω. Let $V_{\mathcal{T}}$ be $P1$ or $Q1$ finite element space associated with the mesh $\mathcal{T}$ with zero Dirichlet boundary conditions imposed at some nodes of $\mathcal{T} \cap \partial\Omega$. Note that, for the purpose of numerical solution of the discretized problem, we do not need any assumptions on the form or measure of the part of the boundary with Dirichlet conditions imposed. For simplicity, we assume that the finite element basis functions φ_i are scaled so that $\|\varphi_i\|_{L^\infty} = 1$. We consider the following elliptic model problem: Find $u \in V_{\mathcal{T}}$ such that

$$A_\Omega(u,v) = (f,v)_{L^2(\Omega)} \quad \forall v \in V_{\mathcal{T}}, \quad A_\Omega(u,v) = \sum_{i=1}^{2}\int_\Omega a(x)\frac{\partial u(x)}{\partial x_i}\frac{\partial v(x)}{\partial x_i}dx. \tag{10}$$

We allow large variation of $a(x)$ between the subdomains; more detailed assumptions on $a(x)$ will be specified below.

Finite element discretization of (10) on $V_{\mathcal{T}}$ leads to a system of linear algebraic equations with a symmetric positive definite matrix $A_{\mathcal{T}}$. In order to accomodate discontinuities of the coefficient $a(x)$, we will solve the system with a diagonally scaled matrix $A = D^{-1/2} A_{\mathcal{T}} D^{-1/2}$ instead, where $D = \mathrm{diag}(A_{\mathcal{T}})$.

For the sake of construction of the tentative prolongator P we need a system $\{\Omega_i\}_{i=1}^m$ of closed disjoint subdomains of Ω such that each subdomain Ω_i is a simply connected closure of an aggregate of elements. We assume that each node of the underlying finite element mesh belongs to exactly one of the subdomains and that there is a layer one element wide between two neighboring subdomains. Further, we assume that the family of subdomains $\{\Omega_i\}$ satisfies the following properties:

Assumption 5

(i) We assume that there is about the same number of elements in each subdomain Ω_i. Let us denote the characteristic number of elements per subdomain by N_{es} and set $\hat{h} = N_{es}^{-1/2}$. We require that subdomain Ω_i can be mapped onto a reference subdomain $\hat{\Omega} = [0,1] \times [0,1]$ by a one-to-one locally Lipschitz mapping G_i:

$$||\partial G_i(x)|| \le C\frac{\hat{h}}{h(x)}, \quad ||\partial G_i^{-1}(\hat{x})|| \le \frac{h(x)}{c\hat{h}}, \quad \forall \hat{x} = G_i(x), x \in \Omega_i, \tag{11}$$

where $h(x)$ is the local meshsize in the neighborhood of x and $c, C > 0$ are constants uniform with respect to i. Symbol $||\cdot||$ denotes a matrix operator norm.

(ii) The coefficient $a(x)$ is allowed only a modest variation within each subdomain in the sense that $a(x) \approx a_i > 0, \quad \forall x \in \Omega_i$.

(iii) If Ω_i and Ω_j are adjacent subdomains (there exists $T \in \mathcal{T}$ so that $\partial T \cap \partial\Omega_i \neq \emptyset$ and $\partial T \cap \partial \Omega_j \neq \emptyset$) and $a_i >> a_j$, the jump in $a(x)$ occurs along $\partial\Omega_i$. In other words, the discontinuity is located on the boundary of the subdomain with the larger value of $a(x)$.

Conditions (11) imply that an element $T \subset \Omega_i$ of size $h(x)$ is mapped by G_i onto $G_i(T)$ of size about $\hat{h}$. Thus, G_i maps locally quasi-uniform mesh on Ω_i onto a quasi-uniform mesh of meshize $\hat{h}$, and subdomains Ω_i are reasonable aggregates consisting of about $N_{es} = \hat{h}^{-2}$ elements. If the mesh $\mathcal{T}$ is quasi-uniform ($h(x) \approx h$), then $h(x)/\hat{h}$ can be viewed as the characteristic size of Ω_i. Note that, if a subdomain decomposing algorithm uses only the adjacency of elements or nodes and generates shape regular subdomains in the case of quasiuniform mesh, then for locally quasi-uniform meshes it can be expected to generate subdomains satisfying (11).

The purpose of assumption *(iii)* is to ensure that the basis function φ_j associated with a node $v_j \in \Omega_i$, satisfies

$$a(\varphi_j, \varphi_j) \approx a_i. \tag{12}$$

If *(iii)* were not satisfied, (12) could be violated for the basis functions corresponding to the nodes on $\partial\Omega_j$ adjacent to a subdomain Ω_i with $a_i >> a_j$.

The tentative prolongator based on scaled aggregations is defined as follows.

Algorithm 6

(i) Set $P_{ij} = \begin{cases} 1, & \text{if the node } v_i \text{ belongs to subdomain } \Omega_i, \\ 0, & \text{otherwise.} \end{cases}$

(ii) Set $P \leftarrow D^{1/2}P$, *where* $D = \mathrm{diag}(A_{\mathcal{T}})$.

For each subdomain, we introduce an index set F_i of all unconstrained (with no Dirichlet boundary conditions imposed) degrees of freedom associated with Ω_i. Let $\Pi : \mathbb{R}^n \to V_\tau$ denotes the finite element interpolator given by $\Pi\mathbf{x} = \sum_{j=1}^n x_j\varphi_j$, the local interpolator $\Pi_i\mathbf{x} = \sum_{j\in F_i} x_j\varphi_j$, and discrete $l^2(F_i)$–norm $\|\mathbf{x}\|^2_{l^2(F_i)} = \sum_{j\in F_i} x_j^2, \mathbf{x} \in \mathbb{R}^n$.

Let Ω_i' be a subset of Ω_i consisting of all elements $T \subset \Omega_i$ such that all degrees of freedom on T are unconstrained. On each subdomain we define a linear mapping $Q_i : \mathbb{R}^n \to \mathbb{R}^n$ (acting on the degrees of freedom of F_i only) to be the $l^2(F_i)$-orthogonal projection onto the one-dimesional space of vectors spanned by $\mathbf{c} \in \mathbb{R}^n$ such that $c_j = 1$ for $j \in F_i$, zero elsewhere.

Lemma 1 (Discrete scaled Poincaré-Friedrichs inequality) *For every* $\mathbf{u} \in \mathbb{R}^n$, *it holds that*

$$\|\mathbf{u} - Q_i\mathbf{u}\|_{l^2(F_i)} \le CN_{es}^{1/2}|\Pi_i\mathbf{u}|_{H^1(\Omega_i)}, \tag{13}$$

the constant $C > 0$ *depends on the constants from* (11) *and on the aspect ratios of elements in* Ω_i *only.*

Proof. Consider a transformed function $\hat{u} = u \circ G_i^{-1}$ i.e. $\hat{u}(\hat{x}) = u(G_i^{-1}(\hat{x})), \hat{x} \in \hat{\Omega}$. Let us define a weighted L^2–norm by $\|u\|_{L^2_h} = \|u(x)/h(x)\|_{L^2}$. Owing to (11), H^1–seminorm scales uniformly, i.e. $|u|_{H^1(\Omega_i)} \approx |\hat{u}|_{H^1(\hat{\Omega})}$. It can be easily seen that

$$\|u\|_{L^2_h(\Omega_i)} \approx \hat{h}^{-1}\|\hat{u}\|_{L^2(\hat{\Omega})} \quad \text{and} \quad \|\Pi_i\mathbf{x}\|_{L^2_h(\Omega')} \approx \|\mathbf{x}\|_{l^2(F_i)}.$$

Let $\mathbf{c} \in \mathbb{R}^n$ be the vector given by $c_j = 1$ for $j \in F_i$, zeroes elsewhere (as in the definition of Q_i). Then, by the equivalence of $l^2(F_i)$ and $L^2_h(\Omega_i')$ for finite element functions , $\Omega_i' \subset \Omega_i$, and the scaling above, we have for $\alpha \in \mathbb{R}$

$$\begin{aligned} \|\mathbf{u} - \alpha\mathbf{c}\|_{l^2(F_i)} &\approx \|\Pi_i\mathbf{u} - \alpha\|_{L^2_h(\Omega_i')} \\ \le \|\Pi_i\mathbf{u} - \alpha\|_{L^2_h(\Omega_i)} &\approx \hat{h}^{-1}\|(\Pi_i\mathbf{u} - \alpha) \circ G_i^{-1}\|_{L^2(\hat{\Omega})}, \end{aligned} \tag{14}$$

Using the definition of Q_i, inequality (14), Poincaré-Friedrichs inequality on $\hat{\Omega}$ and the uniform scaling of H^1 seminorm, we obtain

$$\begin{aligned} \|\mathbf{u} - Q_i\mathbf{u}\|_{l^2(F_i)} &= \inf_{\alpha\in\mathbb{R}^1}\|\mathbf{u} - \alpha\mathbf{c}\|_{l^2(F_i)} \le C\hat{h}^{-1}\inf_{\alpha\in\mathbb{R}^1}\|(\Pi_i\mathbf{u}) \circ G_i^{-1} - \alpha\|_{L^2(\hat{\Omega})} \\ &\le C\hat{h}^{-1}|(\Pi_i\mathbf{u}) \circ G_i^{-1}|_{H^1(\hat{\Omega})} \le C\hat{h}^{-1}|\Pi_i\mathbf{u}|_{H^1(\Omega_i)}, \end{aligned}$$

concluding the proof. □

Now we are ready to prove the weak approximation property (3).

Lemma 2 (Weak approximation property) *Under Assumption 5, the inequality* (3) *is satisfied with* $C_D(m,n) = N_{es}^{1/2}$ *and* C_1 *that depends only on constants from* (11) *and aspect ratios of elements.*

Proof. We set $Q = D^{1/2}(\sum_{i=1}^{m} Q_i)D^{-1/2}$. Let $\mathbf{u} \in \mathbb{R}^n$, $\mathbf{x} = D^{1/2}\mathbf{u}$. Then, using (12),

$$\|\mathbf{u}\|_A^2 = \|\mathbf{x}\|_{A_T}^2 = A_\Omega(\Pi\mathbf{x}, \Pi\mathbf{x}) \geq \sum_{i=1}^{m} A_{\Omega_i}(\Pi_i\mathbf{x}, \Pi_i\mathbf{x}) \geq C\sum_{i=1}^{m} a_i|\Pi_i\mathbf{x}|_{H^1(\Omega_i)},$$

$$\|\mathbf{u} - Q\mathbf{u}\|^2 = \|D^{1/2}(I - \sum_{i=1}^{m} Q_i)\mathbf{x}\|^2 \leq C\sum_{i=1}^{m} a_i\|\mathbf{x} - Q_i\mathbf{x}\|_{l^2(F_i)}^2.$$

From here, setting $P\mathbf{v} = Q\mathbf{u}$ and using Lemma 1 and $\rho(A) \leq C$, the statement follows. □

In the rest of this section we will discuss the choice of the prolongator smoother S. Let $\bar{\rho}$ be the estimate of $\rho(A)$ satisfying

$$\rho(A) \leq \hat{\rho} \leq C_\rho \rho(A). \tag{15}$$

For any integer $i \geq 0$, we define $\hat{\rho}_i = \frac{\hat{\rho}}{9^i}$, $A_0 = A$ and

$$S_i = \prod_{j=0}^{i-1} W_j, \quad W_j = I - \frac{4}{3}\hat{\rho}_j^{-1}A_j, \quad A_j = W_{j-1}^2 A_{j-1}. \tag{16}$$

It is easy to see that $\deg(S_i) \leq \frac{3}{2}3^i$. We choose the prolongator smoother $S = S_K$ for K such that

$$\deg(S_{K+1}) \geq qN_{es}^{1/2} \geq \deg(S_K), \tag{17}$$

where $q \in (0, 1]$ is a given parameter.

Theorem 7 *Let the tentative prolongator P be given by Algorithm 6 with the system of subdomains $\{\Omega_i\}_{i=1}^m$ satisfying Assumption 5. Let the prolongator smoother S be defined by (15)–(17). Then, the statement of Theorem 4 is valid with the constant C_3 independent of the meshsize, coefficients a_i, constant N_{es}, and boundary conditions. Moreover, the coarse-level matrix and the smoothed prolongator SP have a uniformly bounded number of nonzero entries per row.*

Proof. Due to Lemma 2, the approximation property (3) is satisfied with $C(m, n) = N_{es}^{1/2}$. Let us show that (4) holds with the same $C(m, n)$. From definition (16), we have $S^2A = A_{K+1}$. By induction, we can prove $\rho(A_i) \leq \hat{\rho}_i$: For $i = 0$, the inequality holds by (15); assume it holds for $j \leq i$. Then, by (16)

$$\rho(A_{i+1}) = \max_{t\in\sigma(A_i)} (1 - \frac{4}{3}\hat{\rho}_i^{-1}t)^2 t \leq \max_{t\in[0,\hat{\rho}_i]} (1 - \frac{4}{3}\hat{\rho}_i^{-1}t)^2 t \leq \hat{\rho}_{i+1}.$$

Hence, $\rho(A_K) \leq 9^{-K}\hat{\rho}$. Considering that, by (17), $K \approx \log_3 N_{es}^{1/2}$, we get (4). The optimal convergence result now follows from Theorem 4.

Let us show that the number of nonzero entries per row of the coarse-level matrix $A_c = (SP)^T A(SP)$ is bounded uniformly with respect to N_{es}. It is easy to see that $[A_c]_{ij}$ can be nonzero only if $\mathrm{supp}(\Pi SP\mathbf{e}^i) \cap \mathrm{supp}(\Pi SP\mathbf{e}^j) \neq \emptyset$, where $\mathbf{e}^i$ is the i–th canonical basis vector of $\mathbb{R}^m$. Clearly, $\mathrm{supp}(\Pi P\mathbf{e}^i)$ is the domain Ω_i with one belt of surrounding elements added. Bounded overlaps of such supports are obvious. The smoother S adds at most qN_{es} strips of elements. Consequently, each support has a nonempty intersection with only a bounded number of other supports. □

Theorem 8 *Let the assumptions of Theorem 7 be fulfilled and the Choleski factorization be used to solve the coarse-level problem. Then, the optimal number of elements per subdomain is* $N_{es} \approx n^{2/5}$ *and the system* (1) *can be solved to the level of truncation error in* $O(n^{1.2})$ *operations.*

Proof. We only consider the components of the algorithm which cost more than $O(n)$ operations. During the setup, such procedures involve evaluation of SP ($O(N_{es}^{1/2}n)$ operations) and Choleski factorization of the coarse-level matrix, which costs $O(m^2) = O(n^2/N_{es})$ operations. As Theorem 7 assures the optimal convergence result, we only have to perform $O(1)$ iterations. Nonscalable procedures during each iteration are the smoothing ($O(N_{es}^{1/2}n)$ operations) and the back substitution ($O(m^{1.5}) = O((n/N_{es}^{1/2})^{1.5})$. The statement follows by trivial manipulations. □

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