A Parallel Domain Decomposition Procedure for Convection Diffusion Problems

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1 Introduction

For simplicity we consider the following model convection-diffusion equation

$$\begin{array}{ll} -\nabla \cdot (\varepsilon \nabla u - \vec{\beta} u) + \alpha u = f & \text{in } \Omega, \\ u = g & \text{on } \Gamma = \partial\Omega, \end{array} \tag{1.1}$$

where Ω is an open bounded domain on the plane, and α, f, g are given functions on Ω and Γ, $\vec{\beta}$ is a vector-valued function, and ε is a 2×2 matrix which is symmetric and positive definite.

In this paper, we assume that the problem (1.1) is convection-dominated so that the equation is of hyperbolic type and the solution possesses boundary and interior layers. Such problems are known to be difficult to discretize and to implement in practical computation.

Our objective here is to outline a stable finite element scheme and its parallel implementation by using domain decomposition techniques for (1.1). The finite element method is based on the standard discontinuous Galerkin procedure [10] combined with mixed finite element technique. The discretization scheme and some error estimates were discussed in [9]. We will report some new error estimates that have been derived in [12]. Our parallel domain decomposition algorithm [13] and its convergence analysis follow the work of Després, Joly, Robert [6] and Douglas, Paes Leme, Roberts, Wang [7] for the standard mixed finite element method.

The reader is referred to [12] and [13] for details of this research.

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2 A Mixed Discontinuous Galerkin Scheme

Without loss of generality, assume that the matrix-valued function ε is a (small) constant. Then a mixed formulation for (1.1) can be obtained by introducing a variable:

$$\vec{q} = -\varepsilon^{\frac{1}{2}} \nabla u .$$

The problem (1.1) is equivalent to seeking $(\vec{q}, u)$ with $u = g$ on Γ such that

$$\begin{aligned} &\vec{q} + \varepsilon^{\frac{1}{2}} \nabla u = 0 && \text{in } \Omega, \\ &\nabla \cdot (\varepsilon^{\frac{1}{2}} \vec{q}) + \nabla \cdot (\vec{\beta} u) + \alpha u = f && \text{in } \Omega. \end{aligned} \tag{2.1}$$

For each real parameter $h > 0$, let $\mathcal{T}_h$ be a finite element partition of $\bar{\Omega}$ consisting of triangles or quadrilaterals e with diameters bounded by h. To derive a variational form for (2.1), we introduce the following functional spaces:

$$\begin{aligned} \mathbf{V} &= \{\vec{v} \in \left[L^2(\Omega)\right]^2, \nabla \cdot \vec{v} \in L^2(\Omega)\}, \\ W &= \{w \in L^2(\Omega), w|_e \in H^1(e), \forall e \in \mathcal{T}_h\}. \end{aligned}$$

Also introduce the following linear and bilinear forms:

$$\begin{aligned} A(\vec{q}, \vec{v}) &= (\vec{q}, \vec{v}), \vec{q}, \vec{v} \in \mathbf{V}, \\ B(\vec{q}, w) &= \varepsilon^{\frac{1}{2}} (\nabla \cdot \vec{q}, w), \vec{q} \in \mathbf{V}, w \in W, \\ D(u, w) &= \sum_{e \in \mathcal{T}_h} (-\int_e u \vec{\beta} \cdot \nabla w + \int_{\partial e_-} u_+ [w] \vec{n} \cdot \vec{\beta} ds) + (\alpha u, w) u, w \in W, \\ g(\vec{v}) &= -\varepsilon^{\frac{1}{2}} \int_\Gamma g\, \vec{v} \cdot \vec{n}\, ds \vec{v} \in \mathbf{V}, g \in H^{\frac{1}{2}}(\Gamma), \end{aligned}$$

where $(\cdot, \cdot)$ is the standard inner product in $L^2(\Omega)$ or $\left[L^2(\Omega)\right]^2$ as appropriate, $\vec{n}$ is the outward normal direction on ∂e,

$$\partial e_- = \{l \in \partial e, \quad \vec{n} \cdot \vec{\beta}|_l < 0\},$$

and

$$[w] = w_+ - w_-, \quad w_+(\vec{x}) = \lim_{t \to 0^+} w(\vec{x} + t\vec{\beta}), \quad w_-(\vec{x}) = \lim_{t \to 0^-} w(\vec{x} + t\vec{\beta}).$$

With the above notation, a weak form for (2.1) which seeks $(\vec{q}, u) \in \mathbf{V} \times W$ such that

$$\begin{aligned} A(\vec{q}, \vec{v}) - B(\vec{v}, u) &= g(\vec{v}), && \forall \vec{v} \in \mathbf{V}, \\ B(\vec{q}, w) + D(u, w) &= (f, w), && \forall w \in W. \end{aligned} \tag{2.2}$$

It is not hard to verify that if the solution of (1.1) is smooth enough, then the problem (2.1) is equivalent to (2.2).

The Ritz-Galerkin procedure can be applied to yield a mixed discontinuous Galerkin method for (1.1). To this end, let $\mathbf{V}_h \times W_h \subset \mathbf{V} \times W$ be appropriately defined finite element spaces associated with $\mathcal{T}_h$. Then the Ritz-Galerkin approximation is the solution of the following linear system:

$$\begin{aligned} A(\vec{q}_h, \vec{v}) - B(\vec{v}, u_h) &= g(\vec{v}), && \forall \vec{v} \in \mathbf{V}_h, \\ B(\vec{q}_h, w) + D(u_h, w) &= (f, w), && \forall w \in W_h. \end{aligned} \tag{2.3}$$

Now we comment briefly on the construction of $\mathbf{V}_h \times W_h$. Since there is no continuity requirement for functions in W_h, it is natural to include in W_h piecewise polynomials only. Consequently, the standard mixed finite element spaces are good candidates for $\mathbf{V}_h$ in order to provide a stable and accurate approximation by using (2.3). If the diffusion effect is negligible, one could replace $\mathbf{V}_h$ by continuous piecewise polynomials. For more information on the construction of $\mathbf{V}_h \times W_h$, we refer to [5], [11], [4], [8]. Other possibilities can be found in [12].

If the mixed finite element spaces are employed in (2.3), then the following global error estimates can be derived [12]:

Theorem 2.1 *Let $(\vec{q}, u)$ be the unique solution of (2.2), and $(\vec{q}_h, u_h)$ be the finite element approximation by using the Raviart-Thomas element of order $i \geq 0$. Assume that $\vec{\beta} \in \mathbf{V}$, $\alpha \in L^\infty(\Omega)$, and $\alpha + \frac{1}{2}\nabla \cdot \vec{\beta} \geq \alpha_0 > 0$. Then,*

$$\|\vec{q} - \vec{q}_h\| + \|u - u_h\| \leq Ch^{i+1}(\|\vec{q}\|_{i+1} + h^{-\frac{1}{2}}\|u\|_{i+1}), \tag{2.4}$$

where $\|\cdot\|$ denotes the L^2-norm and $\|\cdot\|_{i+1}$ stands for the norm in $H^{i+1}(\Omega)$.

Interior error estimates are also available for the mixed discontinuous Galerkin method. Intuitively speaking, if the solution u does not change rapidly in Ω except in a small region of boundary layers, then one would have the following error estimate:

$$\|u - u_h\|_{L^2(D)} \leq C(\varepsilon^{1/2}h^{-1/2} + h^{1/2}),$$

where $D \subset \Omega$ is any subregion excluding the boundary layer of u and C is a constant independent of ε.

3 A Parallel Iterative Procedure

Our iterative algorithm can be considered as a modification of the parallel procedure studied in [7]. In this section, we outline the algorithm as well as some convergent results. Details of the analysis can be found in [13].

Let $\{\Omega_j, j = 1, \ldots, M\}$ be a partition of Ω:

$$\bar{\Omega} = \bigcup_{j=1}^{M} \bar{\Omega}_j : \qquad \Omega_j \cap \Omega_k = \emptyset,\ j \neq k.$$

Assume that $\partial\Omega_j, j = 1, \ldots, M$, is Lipschitz and that Ω_j is star-shaped. Set

$$\Gamma_j = \Gamma \cap \partial\Omega_j, \qquad \Gamma_{jk} = \Gamma_{kj} = \partial\Omega_j \cap \partial\Omega_k.$$

Let us consider decomposition of (2.1) or (2.2) over $\{\Omega_j\}$. In addition to requiring $\{\mathbf{q}_j, u_j\}, j = 1, \ldots, M$, to satisfy

$$\begin{array}{ll} \mathbf{q}_j + \epsilon^{\frac{1}{2}}\nabla u_j = 0, & \text{in } \Omega_j, \\ \epsilon^{\frac{1}{2}}\nabla \cdot \mathbf{q}_j + \nabla \cdot (\vec{\beta} u_j) + \alpha u_j = f, & \text{in } \Omega_j, \\ u_j = g, & \text{on } \Gamma_j, \end{array} \tag{3.1}$$

it is necessary to impose the consistency conditions

$$\begin{aligned} u_j &= u_k, & x \in \Gamma_{jk}, \\ \mathbf{q}_j \cdot \mathbf{n}_j + \mathbf{q}_k \cdot \mathbf{n}_k &= 0, & x \in \Gamma_{jk}, \end{aligned} \tag{3.2}$$

where $\mathbf{n}_j$ is the unit outward normal to Ω_j. It is convenient to replace (3.2) by the Robin boundary condition

$$\begin{aligned} -\eta \mathbf{q}_j \cdot \mathbf{n}_j + u_j &= \eta \mathbf{q}_k \cdot \mathbf{n}_k + u_k, & x \in \Gamma_{jk} \subset \partial\Omega_j, \\ -\eta \mathbf{q}_k \cdot \mathbf{n}_k + u_k &= \eta \mathbf{q}_j \cdot \mathbf{n}_j + u_j, & x \in \Gamma_{jk} \subset \partial\Omega_k, \end{aligned} \tag{3.3}$$

where η is a positive (normally chosen to be a constant) function on $\bigcup \Gamma_{jk}$. Following an idea in [6] and [7], we can define a parallel iterative procedure for (3.1) by introducing Lagrange multipliers on the edges $\{\Gamma_{jk}\}$.

Let $\mathbf{V}_h \times W_h$ be a mixed finite element space over $\{\Omega_j\}$; any of the usual choices is acceptable, see [5], [11], [4], [3], [8]. Each of these spaces defined through local spaces $\mathbf{V}_j \times W_j = \mathbf{V}(\Omega_j) \times W(\Omega_j)$, and setting

$$\begin{aligned} \mathbf{V}_h &= \{\mathbf{v} \in H(div, \Omega) : \mathbf{v}|_{\Omega_j} \in \mathbf{V}_j\}, \\ W_h &= \{w : w|_{\Omega_j} \in W_j\}, \\ \Lambda_h &= \{\lambda : \lambda|_{\Gamma_{jk}} \in P_m(\Gamma_{jk}) = \Lambda_{jk}, \Gamma_{jk} \neq \emptyset\}, \end{aligned}$$

where m is the order of polynomial on Γ_{jk} such that $\mathbf{q}_j \cdot \mathbf{n}_j|_{\Gamma_{jk}} \in \Lambda_{jk}$. Then the hybridized mixed discontinuous Galerkin method is given by seeking

$$\{\mathbf{q}_j \in \mathbf{V}_j, u_j \in W_j, \lambda_{jk} \in \Lambda_{jk} : j = 1, \dots, M; k = 1, \dots, M\}$$

such that for all $(\mathbf{v}, w, \mu) \in \mathbf{V}_j \times W_j \times \Lambda_{jk}$,

$$\begin{aligned} &(\mathbf{q}_j, \mathbf{v})_{\Omega_j} - \epsilon^{\frac{1}{2}} (\nabla \cdot \mathbf{v}, u_j)_{\Omega_j} + \epsilon^{\frac{1}{2}} \sum_k \langle \lambda_{jk}, \mathbf{v} \cdot \mathbf{n}_j \rangle_{\Gamma_{jk}} = -\epsilon^{\frac{1}{2}} \langle g, \mathbf{v} \cdot \mathbf{n}_j \rangle_{\Gamma_j}, \\ &\epsilon^{\frac{1}{2}} (\nabla \cdot \mathbf{q}_j, w)_{\Omega_j} + d(u_j, w)_{\Omega_j} \\ &= (f, w)_{\Omega_j} - \langle g, \mathbf{n}_j \cdot \vec{\beta} w_+ \rangle_{\Gamma_{j-}} - \sum_k \langle u_{k-}, \mathbf{n}_j \cdot \vec{\beta} w_+ \rangle_{\Gamma_{jk-}} \\ &\langle \mu, \mathbf{q}_j \cdot \mathbf{n}_j + \mathbf{q}_k \cdot \mathbf{n}_k \rangle_{\Gamma_{jk}} = 0. \end{aligned} \tag{3.4}$$

Substitute u_j and u_k in (3.3) by λ_{jk} and λ_{kj}, so that

$$\langle \lambda_{jk}, \mathbf{v} \cdot \mathbf{n}_j \rangle_{\Gamma_{jk}} = \langle \eta(\mathbf{q}_j \cdot \mathbf{n}_j + \mathbf{q}_k \cdot \mathbf{n}_k) + \lambda_{kj}, \mathbf{v} \cdot \mathbf{n}_j \rangle_{\Gamma_{jk}}. \tag{3.5}$$

Then, the iterative process can be defined as follows: let, for all j and k,

$$\mathbf{q}_j^0 \in \mathbf{V}_j, \;\; u_j^0 \in W_j, \;\; \lambda_{jk}^0 \in \Lambda_{jk}$$

arbitrarily. ($\lambda_{jk}^0 = \lambda_{kj}^0$ seems natural) and then compute $\{\mathbf{q}_j^n, u_j^n, \lambda_{jk}^n\} \in \mathbf{V}_j \times W_j \times \Lambda_{jk}$

recursively as the solution of the equations

$$\begin{aligned}
&(\mathbf{q}_j^n, \mathbf{v})_{\Omega_j} - \epsilon^{\frac{1}{2}}(\nabla\cdot\mathbf{v}, u_j^n)_{\Omega_j} + \epsilon^{\frac{1}{2}}\sum_k \langle \eta\mathbf{q}_j^n\cdot\mathbf{n}_j, \mathbf{v}\cdot\mathbf{n}_j\rangle_{\Gamma_{jk}} \\
&= -\epsilon^{\frac{1}{2}}\sum_k \langle \eta\mathbf{q}_k^{n-1}\cdot\mathbf{n}_k + \lambda_{kj}^{n-1}, \mathbf{v}\cdot\mathbf{n}_j\rangle_{\Gamma_{jk}} - \epsilon^{\frac{1}{2}}\langle g, \mathbf{v}\cdot\mathbf{n}_j\rangle_{\Gamma_j}, \forall \mathbf{v}\in\mathbf{V}_j, \\
&\epsilon^{\frac{1}{2}}(\nabla\cdot\mathbf{q}_j^n, w)_{\Omega_j} + d(u_j^n, w)_{\Omega_j} \\
&= (f, w)_{\Omega_j} - \langle g, \mathbf{n}_j\cdot\vec{\beta}w_+\rangle_{\Gamma_{j-}} - \sum_k \langle u_{k-}^{n-1}, \mathbf{n}_j\cdot\vec{\beta}w_+\rangle_{\Gamma_{jk-}}, \forall w\in W_j, \\
&\lambda_{jk}^n = \eta(\mathbf{q}_j^n\cdot\mathbf{n}_j + \mathbf{q}_k^{n-1}\cdot\mathbf{n}_k) + \lambda_{kj}^{n-1}.
\end{aligned} \tag{3.6}$$

Note that in (3.6), the first and the second equations are independent of λ_{jk}^n. Thus, λ_{jk}^n can be evaluated by the third equation after the determination of $\mathbf{q}_j^n$ and u_j^n.

The following convergence result has been obtained for the above algorithm.

Theorem 3.1. *If $\alpha + \frac{1}{2}\nabla\cdot\vec{\beta} \geq 0$ holds, then the iterate solution $\{\mathbf{q}_j^n, u_j^n, \lambda_{jk}^n\} \in \mathbf{V}_j \times W_j \times \Lambda_j$ of (3.6) converges to the solution $\{\mathbf{q}_j, u_j, \lambda_{jk}\}$ of the global hybridized mixed discontinuous Galerkin procedure (3.4) in the following senses:*

$$\begin{aligned}
&\mathbf{q}_j^n \to \mathbf{q}_j = \mathbf{q}^*|_{\Omega_j} && \text{in } L^2(\Omega_j), \\
&u_j^n \to u_j = u^*|_{\Omega_j} && \text{in } L^2(\Omega_j), \\
&\lambda_{jk}^n \text{ and } \lambda_{kj}^n \to \lambda_{jk} && \text{in } L^2(\Gamma_{jk}),
\end{aligned}$$

where $\{\mathbf{q}^*, u^*\} \in \mathbf{V}_h \times W_h$ is the solution of the global mixed discontinuous Galerkin method (2.3).

To estimate the rate of convergence, let $T_{f,g}$ be the affine mapping from $\mathbf{V}_h\times W_h\times\Lambda_h$ to itself such that, for any $(\mathbf{s}, p, \theta) \in \mathbf{V}_h\times W_h\times\Lambda_h$, $(\mathbf{r}, e, \mu) \equiv T_{f,g}(\mathbf{s}, p, \theta)$ is the solution of the following linear system:

$$\begin{aligned}
&(\mathbf{r}_j, \mathbf{v})_{\Omega_j} - \epsilon^{\frac{1}{2}}(\nabla\cdot\mathbf{v}, e_j)_{\Omega_j} + \epsilon^{\frac{1}{2}}\sum_k \langle \eta\mathbf{r}_j\cdot\mathbf{n}_j, \mathbf{v}\cdot\mathbf{n}_j\rangle_{\Gamma_{jk}} \\
= \; &-\epsilon^{\frac{1}{2}}\sum_k \langle \eta\mathbf{s}_k\cdot\mathbf{n}_k + \theta_{kj}, \mathbf{v}\cdot\mathbf{n}_j\rangle_{\Gamma_{jk}} - \epsilon^{\frac{1}{2}}\langle g, \mathbf{v}\cdot\mathbf{n}_j\rangle_{\Gamma_j} \qquad \forall\mathbf{v}\in\mathbf{V}_j, \\
&\epsilon^{\frac{1}{2}}(\nabla\cdot\mathbf{r}_j, w)_{\Omega_j} + d(e_j, w)_{\Omega_j} \\
= \; &(f, w)_{\Omega_j} - \langle g, \mathbf{n}_j\cdot\vec{\beta}w_+\rangle_{\Gamma_{j-}} - \sum_k \langle p_{k-}, \mathbf{n}_j\cdot\vec{\beta}w_+\rangle_{\Gamma_{jk-}} \qquad \forall w\in W_j, \\
&\mu_{jk} = \eta(\mathbf{r}_j\cdot\mathbf{n}_j + \mathbf{s}_k\cdot\mathbf{n}_k) + \theta_{kj}.
\end{aligned} \tag{3.7}$$

The convergence rate of the iterative scheme can be characterized by the spectral radius of the linear operator $T_{0,0}$ [7].

Theorem 3.2. *Let $\rho(T_0)$ be the spectral radius of $T_0 = T_{0,0}$. Assume that*

$$\alpha + \frac{1}{2}\nabla\cdot\vec{\beta} \geq 0,$$

then

$$\rho(T_0) < 1\,.$$

Thus, the iterative procedure (3.6) is convergent.

Theorem 3.3 *Assume that there exists an $\alpha_0 > 0$ such that*

$$\alpha + \frac{1}{2}\nabla \cdot \vec{\beta} \geq \alpha_0.$$

Assume that the partition $\{\Omega_j\}$ is quasiregular, and the parameter η in the iterative procedure (3.6) satisfies $\eta = \sqrt{h(C_1\epsilon^{-1} + \rho\alpha_0^{-1})}$. Then,

$$\rho(T_0) \leq 1 - \frac{Ch}{\sqrt{C_1 h + \epsilon h \rho \alpha_0^{-1} + \alpha_0^{-1}|\gamma|^{-2}}} = \gamma_0\,,$$

where $\rho = \max_j |\partial\Omega_j|/|\Omega_j|$, γ is the eigenvalue of T_0. The iteration (3.6) converges with an error in the n^{th} iteration bounded asymptotically by $O(\gamma_0^n)$.

The following are some particular cases of the domain decomposition.

Case One: Assume the triangulation Ω_j of Ω into elements to be quasiregular and that the subdomains in the domain decomposition to coincide with $\{\Omega_j\}$. In addition, assume $\alpha_0 = O(1)$. Then, by choosing the parameter $\eta = O(\epsilon^{-\frac{1}{2}})$, it follows that (3.6) converges with rate bounded by $\gamma_0 = 1 - Ch$.

Case Two: Let us consider the convection-diffusion problem with "good" convective direction in the sense that any streamline passes through only a finite number of subdomains. Also assume that $\alpha_0 = O(1)$ and $|\Omega_j| = O(1)$. Then, choosing $\eta = O(\epsilon^{-\frac{1}{2}}h^{\frac{1}{2}})$, leads to the estimate $\gamma_0 = 1 - C\sqrt{h}$.

Case Three: Suppose $\epsilon \leq h^{1+\omega}$ with $\omega \geq 0$. If the convective direction $\vec{\beta}$ is "good", and $\alpha_0 = O(1)$, $|\Omega_j| = O(1)$. By choosing $\eta = O(\epsilon^{-\frac{1}{2}}h)$, one arrives at the following estimate

$$\gamma_0 = 1 - \frac{Ch}{\sqrt{h^2+\epsilon}} = \begin{cases} 1 - Ch^{\frac{1-\omega}{2}}, & 0 \leq \omega < 1; \\ 1 - c\ (c < 1), & \omega > 1, \end{cases}$$

which shows a uniform convergence for the domain decomposition algorithm when $\epsilon \leq h^2$.

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