Preconditioners for Spectral and Mortar Finite Element Methods

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1 Introduction

Domain decomposition methods have been developed quite systematically for conforming lower order finite element approximations of elliptic problems. Less attention has been paid to the large, often very ill-conditioned, linear algebraic systems of equations that arise in discretizations based on spectral elements, p–version finite elements, and mortar finite element methods.

In this paper, a brief review is given of some of our relatively recent work on domain decomposition methods for spectral elements, carried out jointly with Luca Pavarino, and more recent work on mortar elements which is joint with Yves Achdou, Mario Casarin, and Yvon Maday.

The iterative substructuring (Schur complement) algorithms form a main family of domain decomposition methods for elliptic problems. Like other domain decomposition methods, they are preconditioned conjugate gradient methods. The region Ω of the given elliptic problem is subdivided into non-overlapping subregions Ω_i and a discretization is introduced. In the pure form of the iterative substructuring algorithms, the interior variables of all the subregions are first eliminated using a direct method. A preconditioner, for the remaining, Schur complement, system of linear algebraic equations, is then constructed from solvers of certain local problems

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and, in addition, a solver of a coarse, global problem similar to that of the coarsest level of a multigrid algorithm. The coarse problem is often the key to good performance, and it can be quite exotic; cf. Widlund [Wid94]. Very large systems of linear algebraic equations, arising when elliptic problems are discretized by finite elements, finite differences, or spectral methods, have been solved successfully by such iterative methods. A number of systematic experimental studies have been carried out with domain decomposition algorithms for both lower and higher order finite elements. Early experimental work for quite large lower order finite element problems is reported in Smith [Smi93]. Very large spectral problems have been solved, using similar preconditioners, by Paul Fischer and Einar Rønquist; see, e.g., [FR94]. For pioneering work on $p-$version finite elements, see Mandel [Man90, Man94]. We note that problems based on higher order discretizations are often exceptionally ill-conditioned. Therefore the challenge to construct good preconditioners is greater than for lower order methods.

A theory for $h-$version finite element methods in three dimensions is summarized in Dryja, Smith, and Widlund [DSW94]; see also Dryja and Widlund [DW95] for the underlying general *Schwarz theory* and an analysis of Neumann-Neumann methods. The author has also completed several papers on spectral elements in joint work with Luca Pavarino; see, in particular, [PW94b, PW94a], and section 4 of this paper. For further work on domain decomposition methods for spectral elements, including Schwarz methods with overlap and iterative substructuring algorithms for Stokes and Navier-Stokes equations, see the recent doctoral dissertation of Mario Casarin [Cas96].

Mortar elements form an interesting class of nonconforming finite elements; see section 5 of this paper for a brief description. Several studies on domain decomposition methods for mortar finite elements have already been completed by Achdou, Kuznetsov, [AK95], and together with Pironneau, [AKP95], Dryja [Dry96], and LeTallec and Sassi [LTS93]. An alternative algorithm, a *hierarchical basis* domain decomposition method has been developed by the author for problems in the plane in joint work with Mario Casarin; see [CW95] and section 6 of this paper. There is also an ongoing joint study, with Yves Achdou and Yvon Maday, [AMW96b, AMW96a], also for plane elliptic problems. Here, in addition, we will describe an algorithm, developed with Yvon Maday, [MW96] for problems in three dimensions.

We recall that the number of iterations required to decrease the energy norm of the error in a conjugate gradient iteration, by a fixed factor, is proportional to $\sqrt{\kappa(B^{-1}A)}$. Here κ is the spectral condition number, A the coefficient matrix of the original system, and B that of the preconditioner. Our central goals are the design of methods with small condition numbers and the development of tools for estimating upper and lower bounds for the spectrum of $B^{-1}A$. All good results in our theory show that the condition number of the iteration operator is independent of the number of subregions and grows only polylogarithmically with the number of degrees of freedom of an individual local problem; the bounds are proportional to $(1+\log(H/h))^q$ or $(1+\log p)^q$, with q a small integer. Here, for the $h-$version, H and h are the diameters of a typical subregion and its elements, respectively. For the spectral case, p is the degree of the polynomials used in each subregion of a spectral approximation.

2 The elliptic problem and spectral finite elements

We begin by formulating a linear, elliptic model problem on a bounded domain Ω in R^3, with a Lipschitz continuous boundary $\partial\Omega$, as a classical calculus of variation problem: Find $u \in V$ such that

$$a(u,v) = \int_\Omega k(x)\nabla u \cdot \nabla v \, dx \;\; = \;\; f(v), \; \forall \; v \in \; V \, . \tag{1}$$

The coefficient $k(x) > 0$ can be discontinuous. For simplicity, let $k(x) = k_i$, a constant, in each subregion Ω_i. We impose a homogeneous Dirichlet condition on a nonempty set $\partial\Omega_D \subset \partial\Omega$; this condition is incorporated into the definition of the space V. A Neumann boundary condition is given on $\partial\Omega_N = \partial\Omega \backslash \partial\Omega_D$; inhomogeneous Neumann boundary data are incorporated into the right hand side of (1).

Here, we only consider one variant of the spectral finite element methods. We assume that Ω is the union of subregions Ω_i which are cubes or images of a reference cube under reasonable smooth mappings. These subregions serve as elements of the spectral element discretization. We use conforming Q_p elements and replace the integrals of the bilinear form of (1), element by element, by Gauss-Lobatto-Legendre (GLL) quadrature. The resulting finite element space is denoted by $V^p \subset V$ and the new bilinear form, $a_Q(\cdot,\cdot)$, is given by,

$$a_Q(u,v) = (k\nabla u, \nabla v)_Q = \sum_i k_i (\nabla u, \nabla v)_{Q,\Omega_i}, \quad \forall u, v \in V^p(\Omega).$$

Here, on the reference cube $\overline{\Omega}_{ref} = [-1,1]^3$, we use the inner product

$$(u,v)_{Q,\Omega_{ref}} = \sum_{i=0}^{p}\sum_{j=0}^{p}\sum_{k=0}^{p} u(\xi_i,\xi_j,\xi_k)v(\xi_i,\xi_j,\xi_k)\rho_i\rho_j\rho_k,$$

where the ξ_i and the ρ_i are the GLL nodes and weights, respectively.

It is known that ellipticity is preserved; cf. [BM92]. A convenient nodal basis on the reference cube is provided by

$$l_i(x)l_j(y)l_k(z), \qquad 0 \le i,j,k \le p.$$

The $l_i(x)$ are the basic Lagrange interpolating polynomials of degree p defined by $l_i(\xi_j) = \delta_{ij}, 0 \le i,j \le p$; see further [BM92], where the special structure of the stiffness matrix is also discussed. We note that this set of basis functions naturally divides into interior, face, edge, and vertex functions.

The finite element problem is thus obtained by replacing, in (1), the bilinear form $a(\cdot,\cdot)$ by $a_Q(\cdot,\cdot)$ and by restricting u and v to the space V^p. Numerical quadrature can also be used to replace the right hand side of (1).

The finite element variational problem is turned into a linear system of algebraic equations, $Kx = b$, where K is the stiffness matrix and b the load vector. As a direct consequence of the ellipticity and symmetry of the bilinear form, we have $K^T = K > 0$.

3 Block-Jacobi and related domain decomposition methods

There are a number of ways of introducing domain decomposition methods; here we select the one we believe to be the simplest. We work in a framework of block-Jacobi/conjugate gradient methods. The stiffness matrix K is then preconditioned by a matrix K_J, which is the direct sum of diagonal blocks of K. Each block corresponds to a set of degrees of freedom with corresponding basis elements, which span a subspace V_i. The space V^p is a direct sum of the subspaces $V_i, i = 0, \cdots, N$. A good choice of the subspaces, after a suitable change of basis in V^p, is the key to success.

Associated with each subspace V_i is an orthogonal projection P_i onto V_i defined by

$$a_Q(P_i u, v) = a_Q(u, v), \quad \forall v \in V_i, \quad u \in V^p,$$

or an operator T_i, defined in terms of an alternative bilinear form $\tilde{a}_i(\cdot, \cdot)$,

$$\tilde{a}_i(T_i u, v) = a_Q(u, v), \quad \forall v \in V_i, \quad u \in V^p.$$

In simple cases, where a subspace V_i corresponds to a group of variables associated with adjacent mesh points, P_i corresponds to the inverse of a diagonal block of K embedded among zero blocks, times K. To obtain T_i, we can replace the inverse of the designated matrix block by a preconditioner of a local problem associated with these variables.

The global subspace can be chosen as the finite element problem of degree one, i.e. as V^1, or, more generally, as another suitable subspace with just one, or a few, degrees of freedom for each element Ω_i of the triangulation of Ω. We have considered the consequences of choosing V^1 as well as more exotic choices. Extra care has to be taken if V^1 is selected; we refer to Pavarino and Widlund [PW95] for a discussion of how it is possible to combine the use of V^1 and fast convergence.

The spectrum relevant to the iterative methods is that of the operator

$$T = \sum_{i=0}^{N} T_i.$$

The eigenvalues of $K^{-1}K_J$, and of T^{-1}, can be shown to be the stationary values of the Rayleigh quotient

$$\frac{\sum_{i=0}^{N} \tilde{a}_i(u_i, u_i)}{a(u, u)}, \quad u = \sum_{i=0}^{N} u_i, \quad u_i \in V_i.$$

Typically, the most challenging part, in work of this kind, is to provide an upper bound for this expression. An upper bound on $a(u_i, u_i)/\tilde{a}_i(u_i, u_i), \forall u_i \in V_i$, for each i, is also required to obtain a lower bound of the eigenvalues of $K_J^{-1}K$. Good bounds can only be obtained by carefully selecting the subspaces V_i, and the bilinear forms $\tilde{a}_i(\cdot, \cdot)$, which together fully specify the iterative method.

Our class of preconditioner can be extended to the case when the finite element space is not a direct sum of subspaces that correspond to blocks of a block-Jacobi splitting. The relevant operators T_i are still well defined for any set of subspaces V_i.

The basic formula, which is easy to prove, is then

$$a(T^{-1}u, u) = inf_{u=\sum u_i} \sum_{i=0}^{N} \tilde{a}_i(u_i, u_i);$$

see Zhang [Zha91]. We note that if the subspaces V_i do not form a direct sum, then there is some freedom of choice in the representation of u at least for some elements of V. This can be turned into a considerable advantage. We can, e.g., start with a direct sum decomposition of the finite element space and then enrich some or all of the subspaces, increasing their dimensions. This is often related to an increase of the overlap of a decomposition of the given region Ω into subdomains. We can also strengthen a preconditioner by adding additional subspaces, e.g., a coarse, global space.

There is an underlying general theory. We note that any block-Jacobi method can be considered as a special *additive Schwarz* method where a direct sum decomposition is used. There are also Gauss-Seidel-like, *multiplicative*, and several *hybrid* Schwarz methods. The derivation of bounds for these families of algorithms, once bounds are proven for the additive case, is by now completely routine; see, e.g., [DSW94, DW95].

4 A choice of subspaces for spectral elements in three dimensions

Our choice of subspaces is directly related to important geometric objects: *interiors, faces, edges,* and *vertices* of the elements. We can merge edges and vertices, creating *wire baskets* and we will describe a *wire basket based* global space V_0; see below. We recall that the subspaces V_i and the energy forms $\tilde{a}_i(\cdot, \cdot)$ define an iterative method completely.

The subspaces and bilinear forms are defined as follows:

We use an interior space for each element: $Q_p \cap H_0^1(\Omega_i)$. Exact solvers are used for these subspaces.

There is also a space for each face Γ_{ij}, where $\overline{\Gamma}_{ij} = \overline{\Omega}_i \cap \overline{\Omega}_j$. The functions of this space vanish on and outside $\partial\Omega_{ij} = \partial(\Omega_i \cup \Gamma_{ij} \cup \Omega_j)$. It is crucial to have a good recipe for the extension of the values on the designated face to the interior of the two relevant elements. The minimal energy (discrete harmonic) extension is used. This corresponds to a change of basis and it greatly improves the performance of the resulting block-Jacobi preconditioner. We note that the use of exact solvers for these subspaces can be relatively expensive but that good alternatives have been developed; see, e.g., Casarin [Cas96].

Our coarse space, of piecewise discrete harmonic functions, is associated with the wire baskets of the elements. Any function in this space is defined fully by its values on the wire basket. In our algorithm, we simply extend the values given on the boundary of a face by assigning the average of the boundary values to all the interior GLL points of the face and complete the extension by using discrete harmonic functions in the interior of the subregions. A simple bilinear form, defined by

$$\tilde{a}_0(u, u) \;=\; C(1 + \log p) \sum_i k_i \inf_{c_i} \|u - c_i\|^2_{L_2(W_i)},$$

is used leading to a special linear system of algebraic equations with one degree of freedom per element, c_i, and a larger subsystem with a diagonal matrix.

Our method is closely modeled on an algorithm developed for lower order elements by Barry Smith; see [Smi91, Smi93]. The following result is established in [PW94a].

Theorem 1

$$\kappa(K_J^{-1}K) \leq C(1+\log p)^2.$$

Here C is a constant, which is independent of p, the values of the k_i, as well as the number of subregions.

Our paper also contains a strict numerical upper bound for the condition number of the method as a function of p. There is also a second, quite different proof due to Mario Casarin, see [Cas95a, Cas95b, Cas96]. Similar bounds hold for a number of other algorithms. We also note that if V^1 is used as the coarse space, and it is combined with carefully chosen local spaces, then the same type of result holds provided that the coefficient function $k(x)$ is quasi-monotone; see Pavarino and Widlund [PW95]. For related work on multigrid methods and a definition of quasi-monotonicity, see Dryja, Sarkis, and Widlund [DSW96].

5 Mortar finite elements

We give a brief description of the mortar elements first introduced by Christine Bernardi, Yvon Maday, and Anthony Patera in [BMP94]. We will focus exclusively on the second generation of these methods; see Ben Belgacem and Maday [BB93, BBM93b]. We confine our study to the case of Poisson's equation, i.e. $k_i = 1$, $\forall i$.

The mortar elements have a number of interesting features. They are nonconforming finite element methods where the partitioning of the region Ω into subregions Ω_i is not necessarily geometrically conforming. Thus, in three dimensions, vertices and edges of one subregion can intersect the interior of edges and/or faces of its neighbors: in two dimensions vertices can intersect edges of neighboring subregions. In each subregion, we are free to choose a standard finite element or spectral element method without much regard to its neighbors. Even if the subregions geometrically conform, the finite element meshes need not. In the spectral case, we can use polynomial spaces of different order in different subregions and we can also mix finite elements and spectral elements.

The mortar element functions are, generally, discontinuous across the interface

$$\Gamma = \overline{\cup_{i=1}^{I} \partial\Omega_i \setminus \partial\Omega}.$$

between the subregions. This interface defines the partition of the region into subregions, and it can sometimes serve as a coarse finite element mesh. The original bilinear form $a(\cdot,\cdot)$ must now be replaced by a bilinear form $a^\Gamma(\cdot,\cdot)$, which corresponds to a broken norm. It is defined as the sum of contributions from the individual subregions:

$$a^\Gamma(u_h, v_h) = \sum_i a_{\Omega_i}(u_h, v_h). \tag{2}$$

Our analysis only involves arguments about individual subregions and their next neighbors. The subregions are assumed to be shape regular and all neighboring

subregions to have diameters of comparable size. However, there is no need to assume that all subregions have comparable diameters.

In the rest of this section, we will focus on piecewise linear finite elements and on two dimensions. To simplify our discussion, we also assume that all of the subregions are triangular (or quadrilateral) and that the triangulation of each subregion is quasi-uniform and that it is obtained by successive refinements of the subregions; cf., e.g., [Yse86]. This, in particular, will be the framework for the preconditioner discussed in the next section. We denote the diameter of the subregion Ω_i by H_i, and the smallest diameter of any of its elements by h_i. Our results depend only on the minimal angle of the overall triangulation, and ℓ, the maximum of the number of refinement levels $\ell(i)$ of the subregions Ω_i.

Thus, each Ω_i is subdivided by a nested family of standard conforming finite element triangulations: $\mathcal{T}_0^i = \{\Omega_i\}, \mathcal{T}_1^i, \mathcal{T}_2^i, \ldots, \mathcal{T}_{\ell(i)}^i$. The quasi-uniform triangulation $\mathcal{T}_{k+1}^i$ is obtained from the next coarser triangulation, $\mathcal{T}_k^i$, by subdividing each of its triangles into four shape-regular, but not necessarily equal, triangles. We assume that the triangles of level $k+1$ have diameters of an order approximately one half of the diameter of those of level k.

A set of *mortars* $\{\gamma_m\}_{m=1}^M$ is obtained by selecting open edges of the subregions such that

$$\Gamma = \cup_{m=1}^M \overline{\gamma}_m, \quad \gamma_m \cap \gamma_n = \emptyset \text{ if } m \neq n.$$

Each edge is viewed as belonging to just one subregion. The other edges are the *nonmortars* and are denoted by δ_n. The restrictions of the triangulations of the different subregions to the mortars and nonmortars will typically not match; they are denoted by γ_m^h and δ_n^h, respectively. Discontinuous mortar finite element functions have two different traces on the interface Γ given by one-sided limits of finite element functions defined on the individual subregions. The continuity across the interface of a conforming finite element method is replaced by weak continuity across the individual nonmortars:

For each n, we define a space of test functions $\mathcal{W}^h(\delta_n)$ given by the restriction, to the nonmortar δ_n, of the finite element space defined on the subregion of which δ_n is an edge. The elements of $\mathcal{W}^h(\delta_n)$ are subject to the further constraints that they are constant in the first and last mesh intervals of δ_n^h.

The *mortar projection* π_n maps all of $L_2(\delta_n)$ onto the finite element space defined on the nonmortar mesh δ_n^h. Given $w \in L_2(\delta_n)$ and values $u^{(n)}(v_{n_1})$ and $u^{(n)}(v_{n_2})$ at the two endpoints v_{n_1} and v_{n_2} of δ_n, we determine the values of $\pi_n(w, u^{(n)}(v_{n_1}), u^{(n)}(v_{n_2}))$ on δ_n^h by

$$\int_{\delta_n} (w - \pi_n(w, u^{(n)}(v_{n_1}), u^{(n)}(v_{n_2}))\psi ds = 0, \quad \forall \psi \in \mathcal{W}^h(\delta_m). \tag{3}$$

We note that only the values at the nodes interior to δ_n are determined by this condition; the values $u^{(n)}(v_{n_1})$ and $u^{(n)}(v_{n_2})$ are genuine degrees of freedom of the finite element model.

We note that in the spectral case, a class of polynomials are used as test functions. They are of a degree two less than the space of traces of the spectral element functions given on the subregion of which δ_n is an edge.

After these preparations, the mortar finite element space V^h can now be fully defined: The restriction of V^h to Ω_i, $V^h(\Omega_i)$, is a regular conforming finite element

space as described above, and the jump across each nonmortar δ_n satisfies the set of constraints given by (3).

The discrete problem is then: Find $u \in V^h$ such that

$$a^\Gamma(u,v) = f^\Gamma(v), \quad \forall v \in V^h, \tag{4}$$

where $a^\Gamma(u,v)$ is defined in formula (2) and, similarly, $f^\Gamma(v)$ is the sum of contributions from the different subregions.

The rate of convergence of the solution of (4) to the solution of (1) is comparable to that of a conforming discretization; cf. [BBM93a], [BMP94], and references therein for theoretical and experimental results.

6 A preconditioner for geometrically conforming problems

We now consider the case of a geometrically conforming decomposition of a plane region. We note that even though we will use a hierarchical basis in the design of our preconditioner, we can primarily work with a nodal basis. We will use a nodal basis of the mortar finite element space which is associated with all nodes interior to the subregions and on $\partial\Omega_N$, all nodes interior to the mortars, and all nodes at all the vertices of the subregions except those which fall on $\partial\Omega_D$.

Our algorithm, fully described and analyzed in [CW95], is based on earlier work by Smith and Widlund [SW90]. We note that the analysis of the new algorithm is considerably more intricate than that of the earlier paper in which only conforming finite elements were considered.

At the expense of an exact solution of a homogeneous Dirichlet problem for each subregion, we reduce problem (4) to that of finding the piecewise discrete harmonic part of the solution. We recall that a finite element function u is discrete harmonic in the subregion Ω_i if

$$a^\Gamma(u,v) = 0 \quad \forall v \in V^h \cap H_0^1(\Omega_i),$$

and that a discrete harmonic function provides the unique minimal energy extension of finite element boundary data given on the boundary $\partial\Omega_i$. This static condensation step reduces the size of the discrete system. We note that it is not necessary to compute a matrix representation of the related Schur complement. We only need matrix-vector products with the Schur complement and these can be computed at the expense of solving a Dirichlet problem for each subregion. After finding sufficiently accurate values on Γ, the solution of (4) is then computed everywhere by solving a finite element problem for each subregion Ω_i with Dirichlet data given on $\partial\Omega_i \setminus \partial\Omega_N$.

The values interior to each subregion correspond to a local subspace subspace of V^h. The other subspaces, which define our iterative substructuring method, are associated with the vertices of the subregions, their edges, and there is also a simple global subspace.

The set of vertices of the subregions associated with degrees of freedom of V^h, i.e. those with values not given by the Dirichlet data on $\partial\Omega_D$, is denoted by $\mathcal{V}$. Each crosspoint of Γ corresponds to several nodes of $\mathcal{V}$ and to one degree of freedom for each of the subregions that meet at that point; these nodes are in the same geometrical position, but are assigned to different subregions. In order to describe and analyze our

algorithm, we define a special vertex basis function ϕ_{v_ℓ} for each of these degrees of freedom and derive estimates of their norms; for these estimates, see [CW95].

For each vertex v_ℓ of $\mathcal{V}$, let $\phi_{v_\ell} \in V^h(\Omega)$ be given the value 1 at v_ℓ, while all other degrees of freedom on Γ are set to zero. This completely defines ϕ_{v_ℓ} since the interior nodal values on the nonmortars are given by the mortar projections, and those in the interior of the Ω_i by discrete harmonic extensions.

A one-dimensional vertex space is associated with each $v_\ell \in \mathcal{V}$:

$$V_{v_\ell} = \text{span of } \phi_{v_\ell}.$$

We use the exact bilinear form $a^\Gamma(\cdot,\cdot)$ for these spaces.

Before we can introduce the edge subspaces, and their bilinear forms, of our iterative method, we need to review some aspects of Yserentant's hierarchical basis method; cf. [Yse86]. We denote by $\mathcal{N}_k^i$, $k = 0, 1, \ldots, \ell(i)$, the set of vertices of the triangles of $\mathcal{T}_k^i$, by V_k^i the space of continuous functions on $\overline{\Omega}_i$ that are linear in the triangles of $\mathcal{T}_k^i$, and by V^i the most refined space $V_{\ell(i)}^i$. All elements of all V_k^i vanish on $\partial\Omega_i \cap \partial\Omega_D$. An interpolation operator $I_k^i : V^i \to V_k^i$, is defined by

$$I_k^i u(x) = u(x) \quad \forall x \in \mathcal{N}_k^i.$$

Following Yserentant [Yse86], we define a discrete norm, for any set $\Lambda \subset \overline{\Omega}_i$, by

$$|||u|||_\Lambda^2 = \sum_{k=1}^{\ell(i)} \sum_{x \in \mathcal{N}_k^i \setminus \mathcal{N}_{k-1}^i \cap \overline{\Lambda}} |(I_k^i u - I_{k-1}^i u)(x)|^2. \tag{5}$$

Let W_k^i be the image of $I_k^i - I_{k-1}^i$; this is the subspace of functions of V_k^i that vanish on $\mathcal{N}_{k-1}^i$. A hierarchical basis of V^i can now be defined recursively. The hierarchical basis of V_0^i is the standard finite element nodal basis restricted to the single triangle Ω_i. It is clear that $V_k^i = V_{k-1}^i + W_k^i$, $k \geq 1$. In each step, we augment the hierarchical basis of V_{k-1}^i by the level k nodal basis functions which span $W_k^i \subset V_k^i$. For a function u represented in this basis, the discrete norm $|||u|||_\Lambda^2$ is simply the Euclidean norm of its coefficients and thus very easy to compute. Moreover, the transformation between the standard nodal basis and the hierarchical basis is very fast and easy to implement; see [SW90] and [Yse86]. This is especially true in the present context since the change of basis can be carried out edge by edge, and in parallel.

A subspace $V_{i(m)} = V_{\gamma_m}$ is now associated with each mortar γ_m. The bilinear form for this subspace is given by $\tilde{a}_{i(m)}(u,u) = |||u|||_{\gamma_m}^2$. The elements of this local space vanish on $\Gamma \setminus \gamma_m$ and they are continued into the interior of the subregions as discrete harmonic functions.

Finally, a coarse space, which is conforming, is given by

$$V_0 = \{u \in V^h |\ u \text{ is linear on each } \Omega_i\} \cap V.$$

The bilinear form associated with V_0 is $a^\Gamma(\cdot,\cdot)$ which coincides with $a(\cdot,\cdot)$ on this subspace since the restriction of any element of V_0 to an edge is a linear function and therefore satisfies the mortar jump condition.

The Schwarz framework provides a preconditioned equation $Tu = b$, in terms of these spaces and bilinear forms, which has the same solution as (4). The main result of [CW95] is the following theorem.

Theorem 2 *The condition number of T satisfies*

$$\kappa(T) \leq C(1+\ell)^2.$$

In our relatively extensive numerical experiments, we have found that the rate of convergence tends to be slightly better than for the algorithm described in [SW90]; see [CW95].

7 Algorithms for geometrically nonconforming problems

We now give a short description of recent joint work with Yves Achdou and Yvon Maday; see [AMW96b, AMW96a]. This work addresses more general classes of problems; the partition of the two-dimensional region into subregions can be geometrically non-conforming and spectral elements, as well as lower order finite elements are considered. These algorithms are also iterative substructuring methods with interior local subspaces and complementary families of subspaces with piecewise discrete harmonic elements.

In these algorithms, the piecewise discrete harmonic part of the mortar element space is partitioned into a direct sum of subspaces. The coarse space is of higher dimension than that of the method described in the previous section, but it is also well defined in the geometrically non-conforming case. One degree of freedom of the coarse space is associated with each vertex. The corresponding basis function takes on the value one at the designated vertex and vanishes at all the others. On each mortar it is a linear function; the values at the nonmortars are, as always, determined by those on the mortars and at the vertices. We note that in a geometrically conforming, lower order finite element case, this coarse space is strictly contained in the sum of the coarse space and the vertex spaces considered in the previous section. A main technical challenge is to provide a sufficiently good bound on the energy norm of these basis functions, and a general element of the coarse space. For the spectral case, several new technical tools were required; see [AMW96a] for full details.

In addition, we use one subspace for each mortar γ_m. Just as in the previous section, an element in the space $V_{i(m)} = V_{\gamma_m}$ is defined by its values in the interior of γ_m and it vanishes on $\Gamma \setminus \gamma_m$. The basic results of [AMW96b, AMW96a] concerns algorithms which use exact solvers for these spaces, but a somewhat weaker result is also obtained for a class of inexact solvers in the lower order finite element case.

The resulting polylogarithmic bounds, see [AMW96b, AMW96a], are quite similar to those of the other sections of this paper.

8 An algorithm for mortar elements in three dimensions

We now turn to a brief discussion of an iterative substructuring method for problems in three dimensions, which has been designed and analyzed jointly with Yvon Maday. It is a *face based* method, a family of methods originating in the 1994 PhD thesis of Marcus Sarkis; see [Sar94]. Our method is described by families of subspaces, which fully define the domain decomposition method; in this algorithm the original bilinear

form, $a^{\Gamma}(\cdot,\cdot)$, and exact solvers are used for all the subspaces. Our results, so far, are only for lower order finite elements, but we hope to extend them to the spectral case.

The coarse space has discrete harmonic basis functions, one for each mortar. A basis function is equal to 1 at all of the interior mesh points of one mortar, γ_m, and vanishes at all other nodes on the interface Γ except those on the boundary of γ_m. On each edge, and at each vertex, of γ_m, they take on positive constant values chosen so that the basis functions form a partition of unity. The values on the nonmortars are determined by insisting that the coarse space is a subspace of the original mortar finite element space.

There is one interior space for each subregion with functions that vanish on, and outside, the boundary of the subregion; this subspace is quite similar to an interior subspace for a conforming finite element space.

For each mortar, there is a local space defined by arbitrary values at the interior nodes of the mortar in question, with all other degrees of freedom on the interface Γ set to zero. This is a natural generalization of the local spaces discussed previously.

For each degree of freedom on the wire basket, there is a one-dimensional subspace defined by one standard mortar basis function.

We note that these subspaces do not form a direct sum decomposition of the mortar finite element space; the local spaces themselves span the entire space.

The following result will be established in a forthcoming joint paper with Maday.

Theorem 3 *For the method described in this section,*

$$\kappa(T) \leq C(1 + \log(H/h))^2.$$

Here C is a constant, which is independent of H, h, as well as the number of subregions.

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