A Multigrid Method for Nonlinear Parabolic Problems

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1 Introduction

The finite element methods for solving nonlinear parabolic problems have been studied by many authors; see, e.g., Douglas and Dupont [5], Wheeler [4], Luskin [3]. These atthors have proposed various ways of solving the problems numerically and they have established optimal order convergence rates of methods, such as the linearized methods, the predictor-corrector methods, the extrapolation methods, the alternating direction methods and different iterative methods [2]. Multigrid methods for solving parabolic problems have been studied by some authors; see Hachbusch [14-15], Bank and Dupont [12], Brandt and Greenwald [16] as well as Yu [13]. But these methods are given mainly for linear parabolic equations. For nonlinear parabolic problems Hachbusch and Brandt in [14], [15], [16] have given multigrid methods by using integral differential equations and the frozen-τ technique.

In this paper, we present a multigrid procedure for two-dimensional nonlinear parabolic problems. The method is an extension of our earlier algorithm given in [13] for linear parabolic problems. The iterative methods for solving the system of nonlinear algebraic equations are avoided because the unknown function $U_k^{n+\theta}$ in the nonlinear coefficient $a(x, U_k^{n+\theta})$ and the right term $f(x, t, U_k^{n+\theta})$ in the system of nonlinear algebraic equations is replaced by $I_k U_{k-1}^{n+\theta}$ in the multigrid procedure, where I_k denotes a intergrid transfer operator, θ a weight function and $U_{k-1}^{n+\theta}$ the solutions of the equation on level $k-1$. We analyze the convergence of our algorithm and the computational cost for N time steps. The computational cost is asymptotically $O(NN_k)$ where N_k is the dimension of the discrete finite element space and N is

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Domain Decomposition Methods in Sciences and Engineering, edited by R. Glowinski *et al.*

the number of time steps. In addition, the methods can be applied to more general nonlinear parabolic problems.

2 Notations and preliminaries

We consider an initial value problem of the following nonlinear parabolic equation:

$$\begin{cases} \frac{\partial u}{\partial t} = \nabla(a(x,u)\nabla u) + f(x,t,u), & (x,t) \in \Omega \times [0,T], \\ u(x,t) = 0, & (x,t) \in \partial\Omega \times [0,T], \\ u(x,0) = u_0(x), & x \in \partial\Omega, \end{cases} \tag{2.1}$$

where $\Omega \subset R^2$ is a convex polygonal domain, ∇ is a gradient operator with respect to $x = (x_1, x_2)$. Assume that the nonlinear coefficient $a(x,p)$ satisfies the condition: There are constants $K_0,\ K_1 > 0$ such that

$$0 < K_0 \leq a(x,u) \leq K_1,\ \forall (x,p) \in \bar{\Omega} \times R^1. \tag{2.2}$$

$a(x,p)$ and $f(x,t,p)$ satisfy uniform Lipschitz conditions with respect to p, i.e., there is a constant $L > 0$ such that

$$\begin{aligned} &|a(x,p_1) - a(x,p_2)| \leq L|p_1 - p_2|,\ \forall (x,p) \in \bar{\Omega} \times R^1, \\ &|f(x,t,p_1) - f(x,t,p_2)| \leq L|p_1 - p_2|,\ \forall (x,t,p) \in \bar{\Omega} \times [0,T] \times R^1. \end{aligned} \tag{2.3}$$

The variational form of problem (2.1) is : Find a continuously differentiable mapping $u(t) = u(x,t) : [0,T] \to H_0^1(\Omega)$ such that

$$\begin{cases} (\frac{\partial u}{\partial t}, v) + a(u; u, v) = (f(u), v),\ \forall v \in H_0^1(\Omega), \\ (u(x,0), v) = (u_0(x), v), \end{cases} \tag{2.4}$$

where $a(u,v) = \int_\Omega a(x,u)\nabla u \nabla v dx$, $(f(u),v) = \int_\Omega f(x,t,u) v dx$. Assume that the solution of problem (2.4) exists and is unique, and that the solution is smooth enough for finite element analysis.

Let Γ_1 be an initial mesh partition of domain Ω (a triangulation or quadrilateral partition). Γ_k $(k \geq 1)$ is a partition obtained by connecting the midpoints of the edges of elements in Γ_{k-1}. Then $\Omega = \cup_{\tau \in \Gamma_k}$ and $h_k = \frac{1}{2} h_{k-1}$ where $h_k = \max_{\tau \in \Gamma_k} h_\tau$.

Let $\mathcal{M}_k (k \geq 1)$ be a finite element space of piecewise linear or quadratic functions associated with the decompositions Γ_k $(k \geq 1)$. Then $\mathcal{M}_{k-1} \subset \mathcal{M}_k \subset H_0^1(\Omega)$.

Let $\Delta t > 0$ be a time step size, $t_n = n\Delta t$ $(n = 1, 2, \cdots, N)$, $N = [\frac{T}{\Delta t}]$. Let

$$\begin{aligned} &t_{n+\theta} = \frac{1}{2}(1+\theta)t_{n+1} + \frac{1}{2}(1-\theta)t_n, U^n = U(x,t_n), \\ &U^{n+\theta} = \frac{1}{2}(1+\theta)U^{n+1} + \frac{1}{2}(1-\theta)U^n, \\ &f(U^{n+\theta}) = f(x, t_{n+\theta}, U^{n+\theta}), \end{aligned}$$

where $\theta \in [0,1]$.

The finite element method for solving the variational problem (2.4) is: Find $\{U^j\}_{j=1}^N : \bar{J} \to \mathcal{M}_k$ such that

$$\begin{cases} (\dfrac{U^{n+1}-U^n}{\Delta t}, v) + a(U^{n+\theta}; U^{n+\theta}, v) = (f(U^{n+\theta}), v), \\ (u(x,0), v) = (u_0(x), v), \ \forall v \in \mathcal{M}_k. \end{cases} \tag{2.5}$$

(2.5) is the Crank-Nicolson scheme for $\theta = 0$. (2.5) is the fully implicit scheme for $\theta = 1$. For $\forall \theta \in [0,1]$, obviously, (2.5) is a system of nonlinear algebraic equations for each time $t_j = j\Delta t$.

3 Time-Dependent Full Multigrid Method

Let I_k be an intergrid transfer operator, $I_k : \mathcal{M}_{k-1} \to \mathcal{M}_k$. I_k is defined as the piecewise linear function or as the average of values of the neighboring nodal points. Let I_k^t be the conjugate operator of I_k or the restriction operator, $I_k^t : \mathcal{M}_k \to \mathcal{M}_{k-1}$ which satisfies

$$(I_k^t u_k, v_{k-1}) = (u_k, I_k v_{k-1}), \quad \forall u_k \in \mathcal{M}_k, \ v_{k-1} \in \mathcal{M}_{k-1}. \tag{3.1}$$

By the nested property of the finite element spaces, there exists a matrix $B_k = [b_{ij}]_{N_{k-1}\times N_k}$ such that $I_k = B_k^T$, $I_k^t = B_k$[13].

If the solutions U_{k-1}^{n+1} and U_{k-1}^n on level $k-1$ as well as U_k^n on level k are known, then we obtain a system of linearized algebraic equations as follows:

$$\begin{cases} (\dfrac{U_k^{n+1}-U_k^n}{\Delta t}, v) + a(I_k U_{k-1}^{n+\theta}; U_k^{n+\theta}, v) = (f(I_k U_{k-1}^{n+\theta}), v), \\ (u(x,0), v) = (u_0(x), v), \ \forall v \in \mathcal{M}_k. \end{cases} \tag{3.2}$$

In the following, we will give the time-dependent k level algorithm for solving the system of linear algebraic equations (3.2). Assume that the solutions U_{k-1}^{n+1} and U_{k-1}^n on level $k-1$ and U_k^n on level k are known. Then an initial approximate value of the solution at $(n+1)$th step time on the k level is taken as:

$$U_{k,0}^{n+1} = U_k^n + I_k(U_{k-1}^{n+1} - U_{k-1}^n). \tag{3.3}$$

1) **Pre-smoothing**: Perform ν_1 time smoothing iterations on level k:

$$U_{k,\nu_1}^{n+1} = S_k^{\nu_1} U_{k,0}^{n+1} \tag{3.4}$$

where S_k is a smoothing operator, such as the Jacobi, Gauss-Seidel and the preconditioned conjugate gradient iteration.

2) **Coarse grid correction**: The coarse grid equation is that $\forall v \in \mathcal{M}_{k-1}$,

$$\begin{aligned} &(\frac{\hat{U}_{k-1}^{n+1} - U_{k-1}^n}{\Delta t}, v) + a(U_{k-1}^{n+\theta}; \hat{U}_{k-1}^{n+\theta}, v) = (f(U_{k-1}^{n+\theta}), v) + [(f(I_k U_{k-1}^{n+\theta}), I_k v) \\ &- (\frac{U_{k,\nu_1}^{n+1} - U_k^n}{\Delta t}, I_k v) - a(I_k U_{k-1}^{n+1}; \frac{1}{2}(1+\theta) U_{k,\nu_1}^{n+1} + \frac{1}{2}(1-\theta) U_k^n, I_k v)], \end{aligned} \tag{3.5}$$

where

$$\hat{U}_{k-1}^{n+\theta} = \frac{1}{2}(1+\theta)\hat{U}_{k-1}^{n+1} + \frac{1}{2}(1-\theta)U_{k-1}^{n}.$$

Let $\hat{U}_{k-1,p}^{n+1}$ be the solution of (3.5) obtained by using p time iterations and $\hat{U}_{k-1,0}^{n+1} = U_{k-1}^{n+1}$ as the initial value. Then, the corrected value U_{k,ν_1+1}^{n+1} of the iterative solution of (3.4) on level $k-1$ is defined as:

$$U_{k,\nu_1+1}^{n+1} = U_{k,\nu_1}^{n+1} + I_k(\hat{U}_{k-1,p}^{n+1} - U_{k-1}^{n+1}). \tag{3.6}$$

3)**Post-smoothing**: Perform ν_2 time smoothing iterations on level k:

$$U_{k,\nu_1+\nu_2+1}^{n+1} = S_k^{\nu_2} U_{k,\nu_1+1}^{n+1}. \tag{3.7}$$

Thus we obtain an approximate solution of the equation (3.2) at the $(n+1)'$s time step on level k as

$$U_k^{n+1} = U_{k,\nu_1+\nu_2+1}^{n+1}.$$

The full multigrid scheme is defined as a recursive process over the mesh level k. If we carry out the multigrid operation for each time step n, we get a time-dependent full multigrid method.

The k level algorithm depends on the solution U_{k-1}^{n+1}, U_{k-1}^{n} and U_k^n. Therefore the full multigrid iterative procedure depends on the solution U_k^0 for $k = 1, 2, \cdots$ and U_1^n for $n = 1, 2, \cdots, N$.

The approximate solutions $U_k^0 (k = 1, 2, \cdots)$ are determined by the following scheme.

1) For $k = 1$, $U_1^0 = \bar{U}_1^0$ is obtained by exactly solving equation (3.8).

2) For $k > 1$, U_k^0 is obtained by using $I_k U_{k-1}^0$ as the initial value of the multigrid iterations. The exact solution $\bar{U}_k^0 (k = 1, 2, \cdots)$ satisfies the equation:

$$(\bar{U}_k^0, v) + a(u_0; \bar{U}_k^0, v) = (f(u_0(x)), v), \ \forall v \in \mathcal{M}_k. \tag{3.8}$$

The solution U_1^n $(n = 1, 2, \cdots, N)$ for the different θ values will be considered in the following two situations.

1) When $\theta \neq 0$, U_1^{n+1} is obtained by solving the following linear equation:

$$(\frac{U_1^{n+1} - U_1^n}{\Delta t}, v) + a(U_1^n; U_1^{n+\theta}, v) = (f(U_1^n), v), \ \forall v \in \mathcal{M}_1, \tag{3.9}$$

for $n = 0, 1, 2, \cdots, N-1$.

2) When $\theta = 0$, U_1^1 is obtained by applying the predictor and corrector twice. Let U_1^* be a solution of the following predictor equation,

$$(\frac{U_1^* - U_1^0}{\Delta t}, v) + a(U_1^0; (U_1^* + U_1^0)/2, v) = (f(U_1^0), v), \ \forall v \in \mathcal{M}_1. \tag{3.10}$$

Here $U_1^{*\frac{1}{2}} = (U_1^* + U_1^0)/2$, and U_1^{**} the solution of the following corrector equation,

$$(\frac{U_1^{**} - U_1^0}{\Delta t}, v) + a(U_1^{*\frac{1}{2}}; (U_1^{**} + U_1^0)/2, v) = (f(U_1^{*\frac{1}{2}}), v), \ \forall v \in \mathcal{M}_1. \tag{3.11}$$

Set $U_1^{**\frac{1}{2}} = (U_1^{**} + U_1^0)/2$. Then U_1^1 is obtained by the equation:

$$(\frac{U_1^1 - U_1^0}{\Delta t}, v) + a(U_1^{**\frac{1}{2}}; U_1^{\frac{1}{2}}, v) = (f(U_1^{**\frac{1}{2}}), v),\ \forall v \in \mathcal{M}_1. \tag{3.12}$$

The solution $U_1^{n+1}(n = 1, 2, \cdots, N-1)$ is obtained by applying the modified Crank-Nicolson method.

$$(\frac{U_1^{n+1} - U_1^n}{\Delta t}, v) + a(EU_1^n; U_1^{n+\frac{1}{2}}, v) = (f(EU_1^n), v),\ \forall v \in \mathcal{M}_1, \tag{3.13}$$

where $EU_1^n = \frac{3}{2}U_1^n - \frac{1}{2}U_1^{n-1}$.

4 Convergence Analysis

Let u be the solution of (2.1) which satisfies

$$u \in L^\infty(H^3),\ \frac{\partial u}{\partial t} \in L^2(H^1) \cap L^\infty(H^2),\ \frac{\partial^2 u}{\partial t^2} \in L^\infty(H^1),\ \frac{\partial^3 u}{\partial t^3} \in L^2(L^2) \cap L^1(H^1). \tag{4.1}$$

Then under the conditions (2.2) and (2.3), the finite element solution of (2.5) has the following error estimate; see [3-5].

Lemma 1. *Let u be the solution of (2.4). Let $\tilde{U}_k^n(n \geq 1)$ and $\tilde{U}_k^0$ be the solutions of (2.5) and (3.8), respectively. Then for $\theta \in [0,1]$, there are constants $c^*, \tau_0 > 0$ independent of $h_k, \{\tilde{U}_k^n\}$ and Δt such that for $\Delta t \leq \tau_0$, we have*

$$\|u - \tilde{U}_k^n\|_{L^2} + h_k\|u - \tilde{U}_k^n\|_{H_0^1} \leq \begin{cases} c^*(h_k^2 + \Delta t^2),\ \theta = 0 \\ c^*(h_k^2 + \Delta t),\ \theta \neq 0 \end{cases}. \tag{4.2}$$

We will now prove that the finite element solution of the discrete equation (3.2) still satisfies (4.2).

Lemma 2. *Assume that we have obtained the finite element solutions $\bar{U}_{k-1}^{n+1}$, $\bar{U}_{k-1}^n$ on level $k-1$ and $\bar{U}_k^n$ on level k and let $\bar{U}_k^{n+1}$ be the finite element solution of (3.2), and let $\tilde{U}_k^{n+1}$ be the finite element solution of (2.5) on level k. Then, for $\theta \in [0,1]$, $\Delta t \sim O(h_k^2)$, there are constants $c^*, \tau_0 > 0$, independent of $h_k, \{\tilde{U}_k^n\}, \{\bar{U}_k^n\}$ and Δt, such that for $\Delta t \leq \tau_0$, we have*

$$\|\tilde{U}_k^n - \bar{U}_k^n\|_{L^2} + h_k\|\tilde{U}_k^n - \bar{U}_k^n\|_{H_0^1} \leq \begin{cases} c^*(h_k^2 + \Delta t^2),\ \theta = 0 \\ c^*(h_k^2 + \Delta t),\ \theta \neq 0 \end{cases}. \tag{4.3}$$

Applying Lemma 1, Lemma 2, and the triangle inequality, we obtain the following convergence result for the finite element solution of the equation (3.2).

Theorem 1. *Let u be the solution of (2.4) and satisfy conditions (2.2), (2.3) and (4.1). Let $\bar{U}_k^n(n \geq 2)$ be the solution of (3.2) and let $\bar{U}_1^n, \bar{U}_k^0$ be the solutions of (3.9)-(3.13) and (3.8), respectively. Then for $\theta \in [0,1]$ and $\Delta t \sim O(h_k^2)$, there are the constants $c^*, \tau_0 > 0$ independent of $h_k, \{\bar{U}_k^n\}$ and Δt such that for $\Delta t \leq \tau_0$, we have*

$$\|u - \bar{U}_k^n\|_{L^2} + h_k\|u - \bar{U}_k^n\|_{H_0^1} \leq \begin{cases} c^*(h_k^2 + \Delta t^2),\ \theta = 0 \\ c^*(h_k^2 + \Delta t),\ \theta \neq 0 \end{cases} \tag{4.4}$$

Lemma 3. *Assume that u satisfies conditions (2.2) (2.3) and (4.1). Let $\bar{U}_{k-1}^{n+1}$ be the solution of (3.2) on level $k-1$ and $\hat{U}_{k-1}^{n+1}$ be the solutions of (3.5). Then for $\theta \in [0,1]$, and $\Delta t \sim O(h_k^2)$, we have*

$$\begin{aligned}&\|\hat{U}_{k-1}^{n+1} - \bar{U}_{k-1}^{n+1}\|_{L^2} + h_k\|\hat{U}_{k-1}^{n+1} - \bar{U}_{k-1}^{n+1}\|_{H_0^1}\\ &\leq R_{k-1} + c[\|\bar{U}_k^{n+1} - U_{k,\nu_1}^{n+1}\|_{L^2} + \Delta t\|\nabla(\bar{U}_k^{n+1} - U_{k,\nu_1}^{n+1}\|_{L^2}]^{\frac{1}{2}},\end{aligned} \tag{4.5}$$

where

$$R_{k-1} = \begin{cases} c^*(h_{k-1}^2 + \Delta t^2), \ \theta = 0, \\ c^*(h_{k-1}^2 + \Delta t), \ \theta \neq 0. \end{cases}$$

The constants c^*, c depend on $K_0, K_1, L, \|\nabla u\|_{L^\infty(L^\infty)}$.

Let $\hat{U}_{k-1,p}^{n+1}$ be an approximate solution of equation (3.5) obtained by p smoothing iterations. Then there exists a constant $0 < \gamma < 1$ such that

$$\begin{aligned}&\|\hat{U}_{k-1}^{n+1} - \hat{U}_{k-1,p}^{n+1}\|_{L^2}^2 + \frac{1}{2}(1+\theta)\Delta t K_0\|\nabla(\hat{U}_{k-1}^{n+1} - \hat{U}_{k-1,p}^{n+1})\|_{L^2}^2\\ &\leq \gamma^p[\|\hat{U}_{k-1}^{n+1} - \bar{U}_{k-1}^{n+1}\|_{L^2}^2 + \frac{1}{2}(1+\theta)\Delta t K_1\|\nabla(\hat{U}_{k-1}^{n+1} - \bar{U}_{k-1}^{n+1})\|_{L^2}^2].\end{aligned} \tag{4.6}$$

Therefore, the error of the coarse corrective solution of (3.6) satisfies the inequality

$$\begin{aligned}&\|\bar{U}_k^{n+1} - U_{k,\nu_1+1}^{n+1}\|_{L^2}^2 + \frac{1}{2}(1+\theta)\Delta t K_0\|\nabla(\bar{U}_k^{n+1} - U_{k,\nu_1+1}^{n+1})\|_{L^2}^2\\ &\leq cI_1 + (1+\gamma^p)R_{k-1}^2\end{aligned} \tag{4.7}$$

where

$$R_{k-1} = \begin{cases} c^*(h_{k-1}^2 + \Delta t^2), \ \theta = 0, \\ c^*(h_{k-1}^2 + \Delta t), \ \theta \neq 0, \end{cases}$$

and

$$I_1 = \|\bar{U}_k^{n+1} - U_{k,\nu_1}^{n+1}\|_{L^2}^2 + \frac{1}{2}(1+\theta)\Delta t K_0\|\nabla(\bar{U}_k^{n+1} - U_{k,\nu_1}^{n+1}\|_{L^2}^2.$$

Inequality (4.7) shows that the error of the coarse corrective solution is bounded by the error of the solution of (3.2) and the error of the finite element solution of (3.5).

The smoothing iterative method of (3.2) satisfies the estimate:

$$\begin{aligned}&\|\bar{U}_k^{n+1} - U_{k,\nu_1}^{n+1}\|_{L^2}^2 + \frac{1}{2}(1+\theta)\Delta t K_0\|\nabla(\bar{U}_k^{n+1} - U_{k,\nu_1}^{n+1})\|_{L^2}^2\\ &\leq \rho(S_k^{\nu_1})[\|\bar{U}_k^{n+1} - U_{k,0}^{n+1}\|_{L^2}^2 + \frac{1}{2}(1+\theta)\Delta t K_1\|\nabla(\bar{U}_k^{n+1} - U_{k,0}^{n+1})\|_{L^2}^2].\end{aligned} \tag{4.8}$$

Thus by (4.7) and (4.8), the k level algorithm defined in (3.3)-(3.7) satisfies:

Theorem 2. *Let $\bar{U}_k^{n+1}$ be the exact solution of (3.2) and let $\bar{U}_{k,\nu_1+\nu_2+1}^{n+1}$ be the iterative solution of the k level algorithm for (3.2). If there exists a constant $0 < \gamma < 1$ such that (4.6) holds for the level $k-1$, then for $\nu_1 + \nu_2$ large enough, we have*

$$\begin{aligned}&\|\bar{U}_k^{n+1} - U_{k,\nu_1+\nu_2+1}^{n+1}\|_{L^2}^2 + \frac{1}{2}(1+\theta)K_0\Delta t\|\nabla(\bar{U}_k^{n+1} - U_{k,\nu_1+\nu_2+1}^{n+1})\|_{L^2}^2\\ &\leq \gamma R_{k-1}^2 + \gamma[\|\bar{U}_k^{n+1} - U_{k,0}^{n+1}\|_{L^2}^2 + \frac{1}{2}(1+\theta)K_0\Delta t\|\nabla(\bar{U}_k^{n+1} - U_{k,0}^{n+1})\|_{L^2}^2\end{aligned} \tag{4.9}$$

Theorem 3. *Let u be the solution of (2.4) and satisfy conditions (2.2), (2.3) and (4.1). Let $U^n_{k,\nu_1+\nu_2+1}$ be the k level iterative solution of (3.3)-(3.7) Then there are constants $c^*, \tau_0 > 0$ independent of h_k and Δt such that if $\Delta t \sim O(h_k^2)$ and $\Delta t \leq \tau_0$,*

$$\|u - U^{n+1}_{k,\nu_1+\nu_2+1}\|_{L^2} + h_k\|u - U^{n+1}_{k,\nu_1+\nu_2+1}\|_{H^1_0} \leq R_k. \tag{4.10}$$

Theorem 4. *Assume that conditions (2.2), (2.3) and (4.1) hold. Then the approximate solution defined by multigrid algorithm satisfies the inequality:*

$$\|u(t_{n+1}) - U_k^{n+1}\|_{L^2} + h_k\|u(t_{n+1}) - U_k^{n+1}\|_{H^1_0} \leq R_k \tag{4.11}$$

where the constant c^* is independent of $h_k, \Delta t$ and $\{U_k^n\}$.

REFERENCES

[1] Douglas J. Jr., Dupont T. and Ewing R. E. (1979) Incomplete iteration for time-stepping a Galerkin method for a quasilinear parabolic problem, *SIAM. J. Numer. Anal.* 16: 503-522.

[2] Douglas J. Jr. (1979) *Effective time-stepping methods for the numerical solution of nonlinear parabolic problems*, the Mathematics of Finite Elements and Applications III, 1978 (Whiteman J. R. ed.), Academic Press, New york, 289-304.

[3] Luskin M. (1979) A Galerkin method for nonlinear parabolic equations with nonlinear boundary conditions. *SIAM. J. Numer. Anal.* 16: 284-299.

[4] Wheeler M. F. (1973) A priori L^2 error estimates for Galerkin approximations to parabolic partial differential equations. *SIAM. J. Numer. Anal.* 10:723-759.

[5] Douglas J. Jr. and Doupont T. (1970) Galerkin methods for parabolic equations. *SIAM. J. Numer. Anal.* 7:575-626.

[6] Missirlis N. M. and Evans D. J. (1981) On the convergence of some generalized preconditioned iterative methods. *SIAM. J. Numer. Anal.* 18: 591-596.

[7] Axelsson O. (1974) On preconditioning and convergence acceleration in sparse matrix problems. *CERN European Organization for Nuclear Research,* Geneva.

[8] Douglas J. Jr. (1976) *Preconditioned conjugate gradient iteration applied to Galerkin methods for a mildly nonlinear Dirichlet problem*, Sparse Matrix Computations, Academic Press, Inc., New york, 333-348.

[9] Johson O. G., Micchelli C. A. and Paul G. (1983) Polynomial preconditioners for conjugate gradient calculations. *SIAM. J. Numer. Anal.* 20: 362-376.

[10] Hayes L. T. (1981) Galerkin alternating-direction methods for nonrectangular regions using patch approximations. *SIAM. J. Numer. Anal.* 18:627-643.

[11] Bramble J. H., Ewing R. E. and Li Gang (1989) Alternating direction multistep methods for parabolic problems – iterative stabilization. *SIAM. J. Numer. Anal.* 26:904-919.

[12] Bank R. E. and Dupont T. (1981) An optimal order process for solving finite element equations. *Math. Comp.* 36: 35-51.

[13] Yu Xijun, A parabolic multigrid method, (submitted).

[14] Hackbusch W. (1981) Fast numerical solution of time-periodic parabolic problems by a multigrid method, *SIAM. J. Sci. Stat. Comput.* 2: 198-206.

[15] — Numerical solution of linear and nonlinear parabolic control problems.

[16] Brandt A. and Greenwald J. (1991) parabolic multigrid revisited, *Intern. Series of Numer. Math.* 98: 143-154.

[17] Ciarlet P. G. (1978) *The finite element method for elliptic problems*, Netherlands.