A Multigrid Method for Nonlinear Parabolic Problems

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1 Introduction

The finite element methods for solving nonlinear parabolic problems have been studied by many authors; see, e.g., Douglas and Dupont [5], Wheeler [4], Luskin [3]. These authors have proposed various ways of solving the problems numerically and they have established optimal order convergence rates of methods, such as the linearized methods, the predictor-corrector methods, the extrapolation methods, the alternating direction methods and different iterative methods [2]. Multigrid methods for solving parabolic problems have been studied by some authors; see Hachbusch [14-15], Bank and Dupont [12], Brandt and Greenwald [16] as well as Yu [13]. But these methods are given mainly for linear parabolic equations. For nonlinear parabolic problems Hachbusch and Brandt in [14], [15], [16] have given multigrid methods by using integral differential equations and the frozen-$\tau$ technique.

In this paper, we present a multigrid procedure for two-dimensional nonlinear parabolic problems. The method is an extension of our earlier algorithm given in [13] for linear parabolic problems. The iterative methods for solving the system of nonlinear algebraic equations are avoided because the unknown function $U_{k}^{n+\theta}$ in the nonlinear coefficient $a(x, U_{k}^{n+\theta})$ and the right term $f(x, t, U_{k}^{n+\theta})$ in the system of nonlinear algebraic equations is replaced by $I_{k}U_{k-1}^{n+\theta}$ in the multigrid procedure, where $I_{k}$ denotes a intergrid transfer operator, $\theta$ a weight function and $U_{k-1}^{n+\theta}$ the solutions of the equation on level $k - 1$. We analyze the convergence of our algorithm and the computational cost for $N$ time steps. The computational cost is asymptotically $O(NN_{k})$ where $N_{k}$ is the dimension of the discrete finite element space and $N$ is
the number of time steps. In addition, the methods can be applied to more general nonlinear parabolic problems.

2 Notations and preliminaries

We consider an initial value problem of the following nonlinear parabolic equation:

\[
\begin{align*}
\frac{\partial u}{\partial t} &= \nabla (a(x, u) \nabla u) + f(x, t, u), \quad (x, t) \in \Omega \times [0, T], \\
u(x, 0) &= 0, \quad (x, t) \in \partial \Omega \times [0, T], \\
u(x, 0) &= u_0(x), \quad x \in \partial \Omega,
\end{align*}
\]  

(2.1)

where $\Omega \subset R^2$ is a convex polygonal domain, $\nabla$ is a gradient operator with respect to $x = (x_1, x_2)$. Assume that the nonlinear coefficient $a(x, p)$ satisfies the condition: There are constants $K_0$, $K_1 > 0$ such that

\[
0 < K_0 \leq a(x, u) \leq K_1, \quad \forall (x, p) \in \Omega \times R^1.
\]  

(2.2)

$a(x, p)$ and $f(x, t, p)$ satisfy uniform Lipschitz conditions with respect to $p$, i.e., there is a constant $L > 0$ such that

\[
\begin{align*}
|a(x, p_1) - a(x, p_2)| &\leq L|p_1 - p_2|, \quad \forall (x, p) \in \Omega \times R^1, \\
f(x, t, p_1) - f(x, t, p_2) &\leq L|p_1 - p_2|, \quad \forall (x, t, p) \in \Omega \times [0, T] \times R^1.
\end{align*}
\]  

(2.3)

The variational form of problem (2.1) is: Find a continuously differentiable mapping $u(t) = u(x, t) : [0, T] \rightarrow H_0^1(\Omega)$ such that

\[
\begin{align*}
\left( \frac{\partial u}{\partial t}, v \right) + a(u, u) \nabla u \nabla v dx + f(u), v \right) = (f(u), v), \quad \forall v \in H_0^1(\Omega), \\
(u(x, 0), v) &= (u_0(x), v),
\end{align*}
\]  

(2.4)

where $a(u, v) = \int_\Omega a(x, u) \nabla u \nabla v dx$, $(f(u), v) = \int_\Omega f(x, t, u) v dx$. Assume that the solution of problem (2.4) exists and is unique, and that the solution is smooth enough for finite element analysis.

Let $\Gamma_k$ be an initial mesh partition of domain $\Omega$ (a triangulation or quadrilateral partition). $\Gamma_k (k \geq 1)$ is a partition obtained by connecting the midpoints of the edges of elements in $\Gamma_{k-1}$. Then $\Omega = \cup_{r \in \Gamma_k}$ and $h_k = \frac{1}{2} h_{k-1}$ where $h_k = \text{max}_{r \in \Gamma_k} h_r$.

Let $\mathcal{M}_k (k \geq 1)$ be a finite element space of piecewise linear or quadratic functions associated with the decompositions $\Gamma_k (k \geq 1)$. Then $\mathcal{M}_{k-1} \subset \mathcal{M}_k \subset H_0^1(\Omega)$.

Let $\Delta t > 0$ be a time step size, $t_n = n\Delta t$ ($n = 1, 2, \cdots, N$), $N = \lfloor \frac{T}{\Delta t} \rfloor$. Let

\[
\begin{align*}
t_{n+\theta} &= \frac{1}{2}(1 + \theta) t_{n+1} + \frac{1}{2}(1 - \theta) t_n, \quad U^n = U(x, t_n), \\
U^{n+\theta} &= \frac{1}{2}(1 + \theta) U^{n+1} + \frac{1}{2}(1 - \theta) U^n, \\
f(U^{n+\theta}) &= f(x, t_{n+\theta}, U^{n+\theta}),
\end{align*}
\]

where $\theta \in [0, 1]$. 

The finite element method for solving the variational problem (2.4) is: Find \( \{U^j\}_{j=1}^N : \mathcal{J} \to \mathcal{M}_k \) such that

\[
\begin{align*}
\left\{ \begin{array}{l}
\left( \frac{U^{n+1} - U^n}{\Delta t}, v \right) + a(U^{n+\theta}, U^{n+\theta}, v) = (f(U^{n+\theta}), v), \\
(u(x, 0), v) = (u_0(x), v), \quad \forall v \in \mathcal{M}_k.
\end{array} \right.
\end{align*}
\tag{2.5}
\]

(2.5) is the Crank-Nicolson scheme for \( \theta = 0 \). (2.5) is the fully implicit scheme for \( \theta = 1 \). For \( \forall \theta \in [0, 1] \), obviously, (2.5) is a system of nonlinear algebraic equations for each time \( t_j = j\Delta t \).

3 Time-Dependent Full Multigrid Method

Let \( I_k \) be an intergrid transfer operator, \( I_k : \mathcal{M}_{k-1} \to \mathcal{M}_k \). \( I_k \) is defined as the piecewise linear function or as the average of values of the neighboring nodal points. Let \( I_k^T \) be the conjugate operator of \( I_k \) or the restriction operator, \( I_k^T : \mathcal{M}_k \to \mathcal{M}_{k-1} \) which satisfies

\[
(I_k^T u_k, v_{k-1}) = (u_k, I_k v_{k-1}), \quad \forall u_k \in \mathcal{M}_k, \ v_{k-1} \in \mathcal{M}_{k-1}.
\tag{3.1}
\]

By the nested property of the finite element spaces, there exists a matrix \( B_k = [b_{ij}]_{N_k \times N_k} \) such that \( I_k = B_k^T \), \( I_k^T = B_k^{[1]} \).

If the solutions \( U_{k-1}^{n+1} \) and \( U_{k-1}^n \) on level \( k-1 \) as well as \( U_k^n \) on level \( k \) are known, then we obtain a system of linearized algebraic equations as follows:

\[
\begin{align*}
\left\{ \begin{array}{l}
\left( \frac{U_k^{n+1} - U_k^n}{\Delta t}, v \right) + a(I_k U_k^{n+\theta}, U_k^{n+\theta}, v) = (f(I_k U_k^{n+\theta}), v), \\
(u_x(x, 0), v) = (u_0(x), v), \quad \forall v \in \mathcal{M}_k.
\end{array} \right.
\end{align*}
\tag{3.2}
\]

In the following, we will give the time-dependent \( k \) level algorithm for solving the system of linear algebraic equations (3.2). Assume that the solutions \( U_{k-1}^{n+1} \) and \( U_{k-1}^n \) on level \( k-1 \) and \( U_k^n \) on level \( k \) are known. Then an initial approximate value of the solution at \( (n+1) \)th step time on the \( k \) level is taken as:

\[
U_{k,0}^{n+1} = U_k^n + I_k(U_{k-1}^{n+1} - U_{k-1}^n).
\tag{3.3}
\]

1) **Pre-smoothing:** Perform \( \nu \) time smoothing iterations on level \( k \):

\[
U_{k,\nu+1}^{n+1} = S_k^{\nu} U_{k,\nu}^{n+1}
\tag{3.4}
\]

where \( S_k \) is a smoothing operator, such as the Jacobi, Gauss-Seidel and the preconditioned conjugate gradient iteration.

2) **Coarse grid correction:** The coarse grid equation is that \( \forall v \in \mathcal{M}_{k-1} \),

\[
\begin{align*}
\left( \frac{\hat{U}_{k-1}^{n+1} - U_{k-1}^n}{\Delta t}, v \right) + a(U_{k-1}^{n+\theta}, \hat{U}_{k-1}^{n+\theta}, v) & = (f(U_{k-1}^{n+\theta}), v) + [(f(I_k U_{k-1}^{n+\theta}), I_k v)] \\
& - \left( \frac{U_{k-1}^{n+1} - U_k^n}{\Delta t}, I_k v \right) - a(I_k U_{k-1}^{n+\theta}, \frac{1}{2}(1 + \theta)U_{k,\nu+1}^{n+1} + \frac{1}{2}(1 - \theta)U_k^n, I_k v).
\end{align*}
\tag{3.5}
\]
where
\[
\hat{U}^{n+1}_{k-1} = \frac{1}{2}(1 + \theta)\hat{U}^{n+1}_{k-1} + \frac{1}{2}(1 - \theta)U^n_{k-1}.
\]

Let \( \hat{U}^{n+1}_{k-1,p} \) be the solution of (3.5) obtained by using \( p \) time iterations and \( \hat{U}^{n+1}_{k-1,0} = U^n_{k-1} \) as the initial value. Then, the corrected value \( U^{n+1}_{k,\nu+1} \) of the iterative solution of (3.4) on level \( k - 1 \) is defined as:
\[
U^{n+1}_{k,\nu+1} = U^{n+1}_{k,\nu_1} + I_k(\hat{U}^{n+1}_{k-1,\nu_2} - U^n_{k-1}). \tag{3.6}
\]

3) Post-smoothing: Perform \( \nu_2 \) time smoothing iterations on level \( k \):
\[
U^{n+1}_{k,\nu_1+\nu_2+1} = S^\nu_k U^{n+1}_{k,\nu_1+1}. \tag{3.7}
\]

Thus we obtain an approximate solution of the equation (3.2) at the \((n + 1)\)’s time step on level \( k \) as
\[
U^{n+1}_{k} = U^{n+1}_{k,\nu_1+\nu_2+1}.
\]

The full multigrid scheme is defined as a recursive process over the mesh level \( k \). If we carry out the multigrid operation for each time step \( n \), we get a time-dependent full multigrid method.

The \( k \) level algorithm depends on the solution \( U^{n+1}_{k-1}, U^n_{k-1} \) and \( U^n_k \). Therefore the full multigrid iterative procedure depends on the solution \( U^n_k \) for \( k = 1, 2, \cdots \) and \( U^n_1 \) for \( n = 1, 2, \cdots, N \).

The approximate solutions \( U^n_k (k = 1, 2, \cdots, N) \) are determined by the following scheme:

1) For \( k = 1 \), \( U^n_1 = U^n_1 \) is obtained by exactly solving equation (3.8).

2) For \( k > 1 \), \( U^n_1 \) is obtained by using \( I_k U^n_{k-1} \) as the initial value of the multigrid iterations. The exact solution \( U^n_k (k = 1, 2, \cdots) \) satisfies the equation:
\[
(\hat{U}^0_k, v) + a(u_0; U^0_k, v) = (f(u_0(x)), v), \quad \forall v \in \mathcal{M}_k. \tag{3.8}
\]

The solution \( U^n_1 (n = 1, 2, \cdots, N) \) for the different \( \theta \) values will be considered in the following two situations.

1) When \( \theta \neq 0 \), \( U^{n+1}_1 \) is obtained by solving the following linear equation:
\[
\left( \frac{U^{n+1}_1 - U^n_1}{\Delta t} \right) + a(U^n_1; U^{n+\theta}_1, v) = (f(U^n_1), v), \quad \forall v \in \mathcal{M}_1. \tag{3.9}
\]

for \( n = 0, 1, 2, \cdots, N - 1 \).

2) When \( \theta = 0 \), \( U^n_1 \) is obtained by applying the predictor and corrector twice. Let \( U^*_1 \) be a solution of the following predictor equation,
\[
\left( \frac{U^*_1 - U^n_1}{\Delta t} \right) + a(U^n_1; (U^*_1 + U^n_1)/2, v) = (f(U^n_1), v), \quad \forall v \in \mathcal{M}_1. \tag{3.10}
\]

Here \( U^{1.5}_1 = (U^*_1 + U^n_1)/2 \), and \( U^{1**}_1 \) the solution of the following corrector equation,
\[
\left( \frac{U^{1**}_1 - U^n_1}{\Delta t} \right) + a(U^{1.5}_1; (U^{1**}_1 + U^n_1)/2, v) = (f(U^{1.5}_1), v), \quad \forall v \in \mathcal{M}_1. \tag{3.11}
\]
Set \( U_1^{** \frac{1}{2}} = (U_1^{**} + U_1^0)/2 \). Then \( U_1^1 \) is obtained by the equation:

\[
\frac{U_1^{n+1} - U_1^n}{\Delta t} + a(U_1^{** \frac{1}{2}}; U_1^1, v) = (f(U_1^{** \frac{1}{2}}), v), \forall v \in M_1.
\] (3.12)

The solution \( U_1^{n+1} (n = 1, 2, \cdots, N - 1) \) is obtained by applying the modified Crank-Nicolson method.

\[
\frac{U_1^{n+1} - U_1^n}{\Delta t} + a(EU_1^n; U_1^{n+\frac{1}{2}}, v) = (f(EU_1^n), v), \forall v \in M_1,
\] (3.13)

where \( EU_1^n = \frac{3}{2} U_1^n - \frac{1}{2} U_1^{n-1} \).

4 Convergence Analysis

Let \( u \) be the solution of (2.1) which satisfies

\[
u \in L^\infty(H^3), \quad \frac{\partial u}{\partial t} \in L^2(H^1) \cap L^\infty(H^2), \quad \frac{\partial^2 u}{\partial t^2} \in L^\infty(H^1), \quad \frac{\partial^3 u}{\partial t^3} \in L^2(L^2) \cap L^1(H^1).
\] (4.1)

Then under the conditions (2.2) and (2.3), the finite element solution of (2.5) has the following error estimate; see [3-5].

**Lemma 1.** Let \( u \) be the solution of (2.4). Let \( \bar{U}_k^n (n \geq 1) \) and \( \bar{U}_k^0 \) be the solutions of (2.5) and (3.8), respectively. Then for \( \theta \in [0, 1] \), there are constants \( c^*, \tau_0 > 0 \) independent of \( h_k, \{\bar{U}_k^n\} \) and \( \Delta t \) such that for \( \Delta t \leq \tau_0 \), we have

\[
\|u - \bar{U}_k^n\|_{L^2} + h_k\|u - \bar{U}_k^n\|_{H^3} \leq \begin{cases} c^*(h_k^2 + \Delta t^2), \theta = 0 \\ c^*(h_k^2 + \Delta t), \theta \neq 0 \end{cases}.
\] (4.2)

We will now prove that the finite element solution of the discrete equation (3.2) still satisfies (4.2).

**Lemma 2.** Assume that we have obtained the finite element solutions \( \bar{U}_{k-1}^{n+1}, \bar{U}_k^n \) on level \( k - 1 \) and \( \bar{U}_k^n \) on level \( k \) and let \( \bar{U}_k^{n+1} \) be the finite element solution of (2.2), and let \( \bar{U}_k^{n+1} \) be the finite element solution of (2.5) on level \( k \). Then, for \( \theta \in [0, 1] \), \( \Delta t \sim O(h_k^2) \), there are constants \( c^*, \tau_0 > 0 \), independent of \( h_k, \{\bar{U}_k^n\}, \{\bar{U}_k^0\} \) and \( \Delta t \), such that for \( \Delta t \leq \tau_0 \), we have

\[
\|\bar{U}_k^n - \bar{U}_k^n\|_{L^2} + h_k\|\bar{U}_k^n - \bar{U}_k^n\|_{H^3} \leq \begin{cases} c^*(h_k^2 + \Delta t^2), \theta = 0 \\ c^*(h_k^2 + \Delta t), \theta \neq 0 \end{cases}.
\] (4.3)

Applying Lemma 1, Lemma 2, and the triangle inequality, we obtain the following convergence result for the finite element solution of the equation (3.2).

**Theorem 1.** Let \( u \) be the solution of (2.4) and satisfy conditions (2.2), (2.3) and (4.1). Let \( \bar{U}_k^n (n \geq 2) \) be the solution of (3.2) and let \( \bar{U}_k^n, \bar{U}_k^0 \) be the solutions of (3.9)-(3.13) and (3.8), respectively. Then for \( \theta \in [0, 1] \) and \( \Delta t \sim O(h_k^2) \), there are the constants \( c^*, \tau_0 > 0 \) independent of \( h_k, \{\bar{U}_k^n\} \) and \( \Delta t \) such that for \( \Delta t \leq \tau_0 \), we have

\[
\|u - \bar{U}_k^n\|_{L^2} + h_k\|u - \bar{U}_k^n\|_{H^3} \leq \begin{cases} c^*(h_k^2 + \Delta t^2), \theta = 0 \\ c^*(h_k^2 + \Delta t), \theta \neq 0 \end{cases}.
\] (4.4)
Lemma 3. Assume that \( u \) satisfies conditions (2.2) (2.3) and (4.1). Let \( \bar{U}^{n+1} \) be the solution of (3.2) on level \( k-1 \) and \( \hat{U}^{n+1} \) be the solutions of (3.5). Then for \( \theta \in [0, 1] \), and \( \Delta t \sim \mathcal{O}(h_k^2) \), we have

\[
\| \hat{U}^{n+1}_{k-1} - \bar{U}^{n+1}_{k-1} \|_{L^2} + h_k \| \hat{U}^{n+1}_{k-1} - \bar{U}^{n+1}_{k-1} \|_{H^1_0} \\
\leq R_{k-1} + c(\| \hat{U}^{n+1}_{k,v_1} - U^{n+1}_{k,v_1} \|_{L^2} + \Delta t \| \nabla (\hat{U}^{n+1}_{k,v_1} - U^{n+1}_{k,v_1}) \|_{L^2})^{\frac{1}{2}},
\]  

(4.5)

where

\[
R_{k-1} = \begin{cases} 
  c^*(h_{k-1}^2 + \Delta t^2), & \theta = 0, \\
  c^*(h_{k-1}^2 + \Delta t), & \theta \neq 0. 
\end{cases}
\]

The constants \( c^*, c \) depend on \( K_0, K_1, L, \| \nabla u \|_{L^\infty(L^\infty)} \).

Let \( \hat{U}^{n+1}_{k,v_1} \) be an approximate solution of equation (3.5) obtained by \( p \) smoothing iterations. Then there exists a constant \( 0 < \gamma < 1 \) such that

\[
\| \hat{U}^{n+1}_{k,v_1} - \bar{U}^{n+1}_{k,v_1} \|_{L^2} + \frac{1}{2} (1 + \theta) \Delta t K_0 \| \nabla (\hat{U}^{n+1}_{k,v_1} - \bar{U}^{n+1}_{k,v_1}) \|_{L^2} \\
\leq \gamma R_{k-1} + cI_1 + (1 + \gamma^p) R_{k-1}^2 \]

(4.6)

Therefore, the error of the coarse corrective solution of (3.6) satisfies the inequality

\[
\| \hat{U}^{n+1}_{k} - U^{n+1}_{k,v_1} \|_{L^2} + \frac{1}{2} (1 + \theta) \Delta t K_0 \| \nabla (\hat{U}^{n+1}_{k} - U^{n+1}_{k,v_1}) \|_{L^2} \\
\leq cI_1 + (1 + \gamma^p) R_{k-1} \]

(4.7)

where

\[
R_{k-1} = \begin{cases} 
  c^*(h_{k-1}^2 + \Delta t^2), & \theta = 0, \\
  c^*(h_{k-1}^2 + \Delta t), & \theta \neq 0. 
\end{cases}
\]

and

\[
I_1 = \| \hat{U}^{n+1}_{k} - U^{n+1}_{k,v_1} \|_{L^2} + \frac{1}{2} (1 + \theta) \Delta t K_0 \| \nabla (\hat{U}^{n+1}_{k} - U^{n+1}_{k,v_1}) \|_{L^2}.
\]

Inequality (4.7) shows that the error of the coarse corrective solution is bounded by the error of the solution of (3.2) and the error of the finite element solution of (3.5).

The smoothing iterative method of (3.2) satisfies the estimate:

\[
\| \hat{U}^{n+1}_{k} - U^{n+1}_{k,v_1} \|_{L^2} + \frac{1}{2} (1 + \theta) \Delta t K_0 \| \nabla (\hat{U}^{n+1}_{k} - U^{n+1}_{k,v_1}) \|_{L^2} \\
\leq \rho(S_k^{n+1}) \| \hat{U}^{n+1}_{k} - U^{n+1}_{k,v_1} \|_{L^2} + \frac{1}{2} (1 + \theta) \Delta t K_0 \| \nabla (\hat{U}^{n+1}_{k} - U^{n+1}_{k,v_1}) \|_{L^2}.
\]

(4.8)

Thus by (4.7) and (4.8), the \( k \) level algorithm defined in (3.3)-(3.7) satisfies:

Theorem 2. Let \( \hat{U}^{n+1}_{k} \) be the exact solution of (3.2) and let \( \hat{U}^{n+1}_{k,v_1} \) be the iterative solution of the \( k \) level algorithm for (3.2). If there exists a constant \( 0 < \gamma < 1 \) such that \( (4.6) \) holds for the level \( k-1 \), then for \( v_1, v_2 \) large enough, we have

\[
\| \hat{U}^{n+1}_{k} - U^{n+1}_{k,v_1,v_2} \|_{L^2} + \frac{1}{2} (1 + \theta) K_0 \| \nabla (\hat{U}^{n+1}_{k} - U^{n+1}_{k,v_1,v_2}) \|_{L^2} \\
\leq \gamma R_{k-1}^2 + \gamma \| \hat{U}^{n+1}_{k} - U^{n+1}_{k,v_1} \|_{L^2}^2 + \frac{1}{2} (1 + \theta) K_0 \| \nabla (\hat{U}^{n+1}_{k} - U^{n+1}_{k,v_1}) \|_{L^2}.
\]

(4.9)
Theorem 3. Let $u$ be the solution of (2.4) and satisfy conditions (2.2), (2.3) and (4.1). Let $U_{k,\nu_1,\nu_2+1}^{n+1}$ be the $k$ level iterative solution of (3.3)-(3.7) Then there are constants $c^*, \tau_0 > 0$ independent of $h_k$ and $\Delta t$ such that if $\Delta t \sim O(h_k^3)$ and $\Delta t \leq \tau_0$,

$$\|u - U_{k,\nu_1,\nu_2+1}^{n+1}\|_L^2 + h_k \|u - U_{k,\nu_1,\nu_2+1}^{n+1}\|_{H_0^1} \leq R_k$$

(4.10)

Theorem 4. Assume that conditions (2.2), (2.3) and (4.1) hold. Then the approximate solution defined by multigrid algorithm satisfies the inequality:

$$\|u(t_{n+1}) - U_k^{n+1}\|_L^2 + h_k \|u(t_{n+1}) - U_k^{n+1}\|_{H_0^1} \leq R_k$$

(4.11)

where the constant $c^*$ is independent of $h_k$, $\Delta t$ and $\{U_k^n\}$.

REFERENCES


