Domain Decomposition Methods for Unbounded Domains

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1 Introduction

In many fields of scientific and engineering computing it is necessary to solve boundary value problems of partial differential equations over unbounded domains. There the standard techniques such as the finite element method will meet some difficulties, even if they are very effective for bounded domains. The standard results for convergence and error estimates of finite element methods are not valid on unbounded domains. No doubt it would be feasible to restrict the domain within some bounded subdomain simply, but the limit of finite element solutions is not necessarily the solution of original problem. In order to get enough accuracy a very large bounded subdomain should be taken, and a very high cost of computation must be paid. Then in the recent twenty years many new methods for solving problems over unbounded domains, such as the infinite element method [1, 11, 13], the boundary element method [2, 5, 7, 17], the coupling method of finite elements and boundary elements [8, 12, 14], the finite element method with approximate conditions on an artificial boundary [3, 15], etc., have been developed. However, each of them has its own advantages and disadvantages. We believe that the coupling of the finite element method and the natural boundary element method has some advantages over other coupling methods [14, 16]. All these methods lead to complicated linear algebraic equations.

The domain decomposition methods are important computational techniques developed rapidly in recent years [6, 10]. A domain is divided into subdomains, a large, difficult and complicated problem is reduced to some small, easy and simple subproblems. But up to now most of the theoretical analyses and algorithms are only valid for bounded domains. In fact, it is more important to develop domain decomposition method for unbounded domains. But when an unbounded domain is divided into subdomains, there is still at least one unbounded subdomain. In this case, the finite element method by itself is not enough; the boundary reduction method,

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which is an effective measure for handling some problems on unbounded domains, will be needed [9, 20].

There are many different ways to do the boundary reduction. The natural boundary reduction suggested and developed by K. Feng and D. Yu is one of them, and it has many distinctive advantages [4, 17]. Based on the natural boundary reduction, an overlapping and a non-overlapping domain decomposition methods for unbounded domains are discussed in this paper. Using one or two circles as artificial boundaries, we can divided an unbounded domain with a closed boundary into non-overlapping or overlapping subdomains. For the interior, small, bounded subdomain, the standard finite element method can be used without any difficulty. In the exterior circular subdomain, the results on the natural boundary reduction [17] can be applied directly. Then a Schwarz alternating method and a Dirichlet-Neumann method for unbounded domains are developed. Especially, for Poisson equation, some careful estimates of contraction factors and convergence rates are given. The theoretical and numerical results show that their convergences are very fast [19, 20].

In this paper, a harmonic boundary value problem is discussed in detail. Since many results on the natural boundary reduction of the biharmonic, plane elasticity and Stokes equations are also given in [17], developing domain decomposition methods for those problems over unbounded domains is also possible.

Steklov-Poincare operators play an important role in domain decomposition methods. The relationship between Steklov-Poincare operators and natural integral operators, and the inverse formulas of Steklov-Poincare operators are discussed in [21].

2 Schwarz Alternating Method Based on Natural Boundary Reduction

Let Ω be an unbounded domain with a closed curve Γ_0 as its boundary. The circles Γ_1 and Γ_2 are in Ω and their radii $R_1 > R_2 > 0$. Let Ω_1 be the bounded domain between Γ_0 and Γ_1, and Ω_2 the unbounded domain outside Γ_2. Then the original problem is decomposed into two subproblems over Ω_1 and Ω_2 with $\Omega_1 \cap \Omega_2 \neq \emptyset$. Taking some initial boundary value on Γ_1, e.g. zero, combining it with the given boundary condition on Γ_0, we solve the problem over Ω_1, get the value of solution on Γ_2, and then solve the problem over Ω_2, get the value of solution on Γ_1, and then solve the problem over Ω_1 again, and so on. This is a Schwarz alternating algorithm: the standard finite element method is applied to Ω_1, and the Poisson integral formula, which is obtained by the natural boundary reduction, is used for Ω_2.

Consider an exterior boundary value problem of the harmonic equation

$$-\Delta w = 0, \text{ in } \Omega, \quad w = g, \text{ on } \Gamma_0. \tag{1}$$

where Ω is the domain exterior to the closed curve Γ_0. The problem is equivalent to

$$\begin{cases} -\Delta u &= f, \text{ in } \Omega, \\ u &= 0, \text{ on } \Gamma_0 \end{cases} \tag{2}$$

for some proper f. With the condition that u is bounded in infinity, (2) has unique solution. Define the following Schwarz alternating algorithm:

$$\begin{cases} -\Delta u_1^{(k)} &= f, & \text{in} & \Omega_1, \\ u_1^{(k)} &= 0, & \text{on} & \Gamma_0, \quad k=1,2,\cdots \\ u_1^{(k)} &= u_2^{(k-1)}, & \text{on} & \Gamma_1, \end{cases} \tag{3}$$

and

$$\begin{cases} -\Delta u_2^{(k)} &= f, & \text{in} & \Omega_2, \\ u_2^{(k)} &= u_1^{(k)}, & \text{on} & \Gamma_2, \end{cases} \quad k=1,2,\cdots, \tag{4}$$

where $u_2^{(0)} \in H^{\frac{1}{2}}(\Gamma_1)$ is arbitrarily given, e. g. $u_2^{(0)} = 0$. The solution of problem (2) is in the space

$$V = \{v \in W_0^1(\Omega) | v = 0 \quad \text{on} \quad \Gamma_0\}.$$

Let

$$V_1 = \{v \in H^1(\Omega_1) | v = 0 \quad \text{on} \quad \Gamma_0 \cup \Gamma_1\}, \ V_2 = \{v \in W_0^1(\Omega_2) | v = 0 \quad \text{on} \quad \Gamma_2\},$$

then

$$u_1^{(k)} - u_2^{(k-1)} \in V_1, \qquad u_2^{(k)} - u_1^{(k)} \in V_2.$$

From the bilinear form

$$D(u,v) = \int_\Omega \nabla u \cdot \nabla v dx, \tag{5}$$

the inner product $(u,v)_1$ and the norm $\|\cdot\|_1$ in V can be defined. Then (3) and (4) are equivalent to the variational problems

$$\begin{cases} \text{Find } u_1^{(k)} \in V_1 + u_2^{(k-1)} & \text{such that} \\ D(u_1^{(k)} - u, v_1) = 0, & \forall v_1 \in V_1 \end{cases} \tag{6}$$

and

$$\begin{cases} \text{Find } u_2^{(k)} \in V_2 + u_1^{(k)} & \text{such that} \\ D(u_2^{(k)} - u, v_2) = 0, & \forall v_2 \in V_2, \end{cases} \tag{7}$$

respectively. Let $P_{V_i^\perp} : V \to V_i^\perp$, $i = 1,2$, denote the projectors in $(\cdot,\cdot)_1$,

$$e_i^{(k)} = u - u_i^{(k)}, \qquad i = 1,2$$

be the errors, we have

$$\begin{cases} e_1^{(k)} &= P_{V_1^\perp} e_2^{(k-1)}, \\ e_2^{(k)} &= P_{V_2^\perp} e_1^{(k)}, \end{cases} \qquad k = 1,2,\cdots. \tag{8}$$

Then,

$$\begin{cases} e_1^{(k+1)} &= P_{V_1^\perp} P_{V_2^\perp} e_1^{(k)}, & k = 1,2,\cdots, \\ e_2^{(k+1)} &= P_{V_2^\perp} P_{V_1^\perp} e_2^{(k)}, & k = 0,1,\cdots. \end{cases} \tag{9}$$

Theorem 1

$$\lim_{k\to\infty} \|e_i^{(k)}\|_1 = 0, \qquad i = 1,2, \tag{10}$$

and there is a constant $\alpha \in [0,1)$ *such that*

$$\|P_{V_1^\perp}P_{V_2^\perp}\| \le \alpha, \qquad \|P_{V_2^\perp}P_{V_1^\perp}\| \le \alpha, \tag{11}$$

Furthermore,

$$\|e_1^{(k)}\|_1 \le \alpha^{k-1}\|e_1^{(1)}\|_1, \qquad \|e_2^{(k)}\|_1 \le \alpha^k\|e_2^{(0)}\|_1. \tag{12}$$

Theorem 1 shows that the above Schwarz alternating method converges geometrically.

It is difficult to estimate the contraction factor for a general unbounded domain Ω. Here let Ω be an exterior domain of a circle Γ_0 with radius R_0, and $R_0 < R_2 < R_1$.

Let $\gamma' : W_0^1(\Omega_2) \to H^{\frac{1}{2}}(\Gamma_1)$ and $\gamma'' : H^1(\Omega_1) \to H^{\frac{1}{2}}(\Gamma_2)$ be the Dirichlet trace operators, $P_1 : H^{\frac{1}{2}}(\Gamma_1) \to H_0^1(\Omega_1) = \{v \in H^1(\Omega_1)|v = 0 \text{ on } \Gamma_0\}$ and $P_2 : H^{\frac{1}{2}}(\Gamma_2) \to W_0^1(\Omega_2)$ be the Poisson integral operators. Then,

$$\gamma''P_1\gamma'P_2 : H^{\frac{1}{2}}(\Gamma_2) \to H^{\frac{1}{2}}(\Gamma_2),$$

$$\gamma'P_2\gamma''P_1 : H^{\frac{1}{2}}(\Gamma_1) \to H^{\frac{1}{2}}(\Gamma_1).$$

Theorem 2 *The operators* $\gamma''P_1\gamma'P_2$ *and* $\gamma'P_2\gamma''P_1$ *are contraction mappings:*

$$\|\gamma''P_1\gamma'P_2f\|_{\frac{1}{2},\Gamma_2} \le \delta\|f\|_{\frac{1}{2},\Gamma_2}, \qquad \forall f \in H^{\frac{1}{2}}(\Gamma_2), \tag{13}$$

$$\|\gamma'P_2\gamma''P_1g\|_{\frac{1}{2},\Gamma_1} \le \delta\|g\|_{\frac{1}{2},\Gamma_1}, \qquad \forall g \in H^{\frac{1}{2}}(\Gamma_1), \tag{14}$$

where $0 < \delta < 1$, *and we can take*

$$\delta = \max\left(\frac{\ln(R_2/R_0)}{\ln(R_1/R_0)}, \left(\frac{R_2}{R_1}\right)^2\right). \tag{15}$$

From theorem 2, we can see that, the contraction factor δ only depends on R_0, R_1, and R_2, and the larger the ratio R_1/R_2, the smaller the factor δ. From the proof of the theorem, we can also see that $\delta \to 0$ when $R_2/R_0 \to 1$, and that

$$\frac{|\alpha_n^*|}{|\alpha_n|} = \frac{R_2^{2|n|} - R_0^{2|n|}}{R_1^{2|n|} - R_0^{2|n|}} \le \left(\frac{R_2}{R_1}\right)^{2|n|}, \qquad n = \pm 1, \pm 2, \cdots, \tag{16}$$

where α_n and α_n^* are coefficients of Fourier expansion of f and $\gamma''P_1\gamma'P_2f$ (or g and $\gamma'P_2\gamma''P_1g$), respectively. This means that the contraction factor is exponentially attenuate with the frequency.

Applying theorem 2 to the above alternating algorithm, we have

$$\begin{cases} \|u_2^{(k)} - u\|_{1,\Omega_2} & \le & C\delta^k \to 0, \\ \|u_1^{(k)} - u\|_{1,\Omega_1} & \le & C\delta^k \to 0 \end{cases}, \tag{17}$$

when $k \to \infty$, where C depends on u, Ω_1, Ω_2 and the initial value on the artificial boundary.

For proofs of these theorems are given in [20].

3 A Dirichlet-Neumann Method Based On Natural Boundary Reduction

Consider the Dirichlet problem for the Poisson's equation

$$\begin{cases} -\Delta u &= f, \quad \text{in} \quad \Omega, \\ u &= g, \quad \text{on} \quad \Gamma_0, \end{cases} \tag{18}$$

where Ω is an unbounded domain with a closed curve Γ_0 as its boundary. Adding the proper boundary condition at infinity, (18) has unique solution.

Let circle Γ_1, with radius R_1, encircle Γ_0 such that dist $(\Gamma_1, \Gamma_0) > 0$. Then Ω is divided into an interior subdomain Ω_1 and an exterior subdomain Ω_2. Γ_1 is the artificial boundary. We define the following Dirichlet-Neumann alternating domain decomposition method.

Step 1. Choose original $\lambda^0 \in H^{\frac{1}{2}}(\Gamma_1)$, $n = 0$.

Step 2. Solve a Dirichlet problem in Ω_2:

$$\begin{cases} -\Delta u_2^n &= f, \quad \text{in} \quad \Omega_2, \\ u_2^n &= \lambda^n, \quad \text{on} \quad \Gamma_1. \end{cases} \tag{19}$$

Step 3. Solve a mixed boundary value problem in Ω_1:

$$\begin{cases} -\Delta u_1^n &= f, \quad \text{in} \quad \Omega_1, \\ \frac{\partial u_1^n}{\partial n_1} &= -\frac{\partial u_2^n}{\partial n_2}, \quad \text{on} \quad \Gamma_1, \\ u_1^n &= g, \quad \text{on} \quad \Gamma_0. \end{cases} \tag{20}$$

Step 4. Set $\lambda^{n+1} = \theta_n u_1^n + (1-\theta_n)\lambda^n$, on Γ_1.

Step 5. Set $n = n+1$, goto step 2.

By the theory of the natural reduction [17], the solution of (19) is given by following the Poisson integral formula:

$$u_2^n(r,\varphi) = \frac{r^2 - R_1^2}{2\pi} \int_0^{2\pi} \frac{\lambda^n(\varphi')}{R_1^2 + r^2 - 2R_1 r\cos(\varphi - \varphi')} d\varphi' + \iint_{\Omega_2} G(p,p') f(p') dp', \tag{21}$$

and there is a natural integral equation on Γ_1:

$$\frac{\partial u_2^n}{\partial n_2}(\varphi) = -\frac{1}{4\pi R_1} \int_0^{2\pi} \frac{\lambda^n(\varphi')}{\sin^2 \frac{\varphi - \varphi'}{2}} d\varphi' + \iint_{\Omega_2} [\frac{\partial}{\partial n} G(p,p')] f(p') dp', \tag{22}$$

where $G(p,p')$ is the Green function for Ω_2:

$$G(p,p') = \frac{1}{4\pi} \ln \frac{R_1^4 + r^2 r'^2 - 2rr' R_1^2 \cos(\varphi - \varphi')}{R_1^2[r^2 + r'^2 - 2rr'\cos(\varphi - \varphi')]}.$$

Then in fact we need not solve (19) by the finite or boundary element method in step 2. Applying (22), $\frac{\partial u_2^n}{\partial n_2}$ can be found directly from λ^n, and obviously, the computation is fully parallel. As for the numerical computation of the hypersingular integral given by (22), see [17, 18]. Since Ω_1 is a small bounded subdomain, it is no difficulty to solve (20) by the standard finite element method in step 3.

Let $e_i^n = u - u_i^n$, $i = 1, 2$, $\mu = u|_{\Gamma_1} - \lambda^n$, then the errors e_1^n and e_2^n satisfy

$$\begin{cases} -\Delta e_2^n = 0, & \text{in} \quad \Omega_2, \\ e_2^n = \mu^n, & \text{on} \quad \Gamma_1 \end{cases} \tag{23}$$

and

$$\begin{cases} -\Delta e_1^n = 0, & \text{in} \quad \Omega_1, \\ \frac{\partial e_1^n}{\partial n_1} = -\frac{\partial e_2^n}{\partial n_2}, & \text{on} \quad \Gamma_1, \\ e_1^n = 0, & \text{on} \quad \Gamma_0. \end{cases} \tag{24}$$

Moreover, it is important to choose θ_n properly. If θ_n is badly chosen, the algorithm will diverge.

By the standard domain decomposition theorey [10] this Dirichlet-Neumann method is equivalent to a precondition Richardson iteration method. Then the convergence analysis can be reduced to the estimation of the characteristic values of $S_1^{-1}S$, i.e., estimating upper and lower bounds for

$$\frac{(S\mu, \mu)}{(S_1\mu, \mu)} = 1 + \frac{(S_2\mu, \mu)}{(S_1\mu, \mu)} = 1 + \frac{a_2(H_2\mu, H_2\mu)}{a_1(H_1\mu, H_1\mu)}. \tag{25}$$

Here $S = S_1 + S_2$ is the Steklov-Poincare operator on Γ_1, H_1 and H_2 are harmonic extension operators of functions on Γ_1 to Ω_1 and Ω_2, respectively,

$$a_i(u, v) = \iint_{\Omega_i} \nabla u \nabla v dx dy, \qquad i = 1, 2. \tag{26}$$

Theorem 3 *If circle Γ_0' with radius $R_0' < R_1$ is the smallest circle which encloses surrounds Γ_0 and has the same center as Γ_1, then*

$$a_2(H_2\lambda, H_2\lambda) \le a_1(H_1\lambda, H_1\lambda) \le \frac{R_1^2 + (R_0')^2}{R_1^2 - (R_0')^2} a_2(H_2\lambda, H_2\lambda), \tag{27}$$

$$\frac{1}{\sqrt{2}} \|\lambda\|^2_{\frac{1}{2},\Gamma_1} \le a_1(H_1\lambda, H_1\lambda) \le \frac{R_1^2 + (R_0')^2}{R_1^2 - (R_0')^2} \|\lambda\|^2_{\frac{1}{2},\Gamma_1}, \tag{28}$$

$$\frac{1}{\sqrt{2}} \|\lambda\|^2_{\frac{1}{2},\Gamma_1} \le a_2(H_2\lambda, H_2\lambda) \le \|\lambda\|^2_{\frac{1}{2},\Gamma_1}, \tag{29}$$

where

$$\lambda \in H_0^{\frac{1}{2}}(\Gamma_1) = \{\mu \in H^{\frac{1}{2}}(\Gamma_1), \int_{\Gamma_1} \mu ds = 0\}.$$

Theorem 4 *Under the assumption of theorem 3 our Dirichlet-Neumann method and the corresponding preconditioned Richardson iteration method converge when $0 < \theta < 1$. In particular, when $\theta = \frac{R_1^2 + (R_0')^2}{2R_1^2 + (R_0')^2}$, the contraction factor satisfes $\delta \le \frac{(R_0')^2}{2R_1^2 + (R_0')^2}$, the convergence rate $-\ln \delta \ge \ln[2(\frac{R_1}{R_0'})^2 + 1]$, the condition number*

$$cond\ (S_1^{-1}S) \le 1 + (\frac{R_0'}{R_1})^2 < 2.$$

4 Conclusion

1. The boundary reduction is a forceful means for handling problems over unbounded domains. Based on the boundary reduction and the boundary element methods, some domain decomposition methods, where subdomains are overlapping or non-overlapping, are developed. These methods are applicable to unbounded domains and have the advantages of both the finite and the boundary element methods, while standard domain decomposition method can only be used for bounded domain.
2. Comparing with other kinds of boundary reduction, the natural boundary reduction has many advantages. Since a circle can be taken as an artificial boundary, the Poisson integral formulas and the natural boundary integral equations can be applied directly in the exterior subdomain Ω_2, where we do not need solve any equations, and a full parallel computation can be implemented. It is totally different from the finite element computation.
3. Only the subproblem over interior subdomain Ω_1 should be solved by the finite element method. Many available standard programs can be used. Since Ω_1 can be taken quite as small as possible, only few elements are necessary for solving the subproblem over Ω_1. It is much simpler than the coupling method of FEM and BEM.
4. Both the theoretical analysis and the numerical experiment show that these methods are feasible and converges very quickly, provided that the ratio R_1/R_2 (for overlapping subdomains) or R_1/R_0' (for non-overlapping subdomains) is not too close to 1. The larger the ratio, the faster the convergence.
5. These methods are suitable not only for solving Poisson equation, but also for solving biharmonic, plane elasticity and Stokes equations. Since many results of the natural boundary reduction for those equations are already given in [17], the extension of these methods is not difficult. Moreover, they can also be extended to more general problems when some more general boundary reduction is used.

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