# An Additive Schwarz Algorithm for a Variational Inequality

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## 1 Introduction

Let $\Omega$ be a bounded polygonal domain, $V$ be a subspace of the Sobolev space $H^k(\Omega), a(\cdot,\cdot)$ be a continuous, coercive and symmetric bilinear form on $V \times V$, $f \in V^*$. For simplicity, we assume that the elements of $V$ satisfy homogeneous boundary condition on $\partial\Omega$. Consider the variational inequality: find $u \in K$ such that

$$a(u, v-u) \geq f(v-u), \quad \forall v \in K \tag{1}$$

where

$$K = \{v \in V : v \geq \phi \text{ in } \Omega, \phi \in H^1(\Omega), \phi \leq 0 \text{ on } \partial\Omega\} \tag{2}$$

or

$$K = \{v \in V : \phi \leq v \leq \psi \text{ in } \Omega, \phi, \psi \in H^1(\Omega), \phi \leq 0 \leq \psi \text{ on } \partial\Omega\}. \tag{3}$$

Assume that $V^h \subset H_0^1(\Omega)$ is the finite element approximation of $V$ and that the set of nodal parameters includes the value of the function in $V_h$ at the nodes. Suppose $\phi, \psi \in C^0(\bar{\Omega})$. The finite element approximation of problem (1), (2) or problem (1), (3) is: find $u_h \in K^h$ such that

$$a_h(u_h, v-u_h) \geq f_h(v-u_h), \quad \forall v \in K^h, \tag{4}$$

where

$$K^h = \{v \in V^h : v \geq \phi \text{ at all the nodes }\}, \tag{5}$$

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or

$$K^h = \{v \in V^h : \phi \le v \le \psi \text{ at all the nodes }\}, \tag{6}$$

$a_h(.,.)$ is a continuous, coercive and symmetric form on $V^h \times V^h, f_h \in (V^h)^*$. In [2] a multiplicative Schwarz algorithm for solving problem (1), (2) and problem (1), (3) with $k = 1$ was studied. In [5] a multiplicative Schwarz algorithm for solving problem (1), (3) with $k = 2$ was proposed. Combining the methods of [2, 5] and the so-called average method of [3, 4], mainly for solving problem (1), (3) with $k = 1$, an additive Schwarz algorithm was given in [5]. A differential additive Schwarz algorithm has been discussed for solving (4) in [3, 4] provided that $K^h$ is a cone. Based on the work just mentioned, we propose an additive Schwarz algorithm for solving problem (4), (5) and problem (4), (6) in more general cases. We give the algorithm and the convergence theorem and get so-called finite step convergence for coincident components.

A similar algorithm has been studied in [8] and [9]. But they require a special choice of the initial value. In [8], monotone convergence was proven and in [9] the $h$ independent convergence rate has been obtained.

## 2 An Additive Schwarz Algorithm

We use the two-level triangulation of $\Omega$, given by Dryja and Widlund (ref. [1]). In this way, we get overlapping open subregions $\Omega_i, i = 1, \cdots, m$. Let $V_i = V^h \cap H_0^1(\Omega_i)$. The algorithm is defined as follows.

Algorithm I

Step 1. Given $\omega_i > 0, i = 1, \cdots, m$ with $\Sigma_{i=1}^m \omega_i = 1$. Take $u^0 \in K^h$ and $n := 0$;

Step 2. For $i = 1, \cdots, m$ solve the following subproblems: find $u^{n_i} \in K_i^n$ such that

$$a_h(u^{n,i}, v - u^{n,i}) \ge f_h(v - u^{n,i}), \quad \forall v \in K_i^n,$$

where $K_i^n = (u^n + v_i) \cap K^h$,

Step 3 $u^{n+1} = \sum_{i=1}^m \omega_i u^{n,i}$ ;

Step 4 $n := n + 1$, go to step 2.

In order to prove the convergence of Algorithm I, we consider the following variational inequality: find $u^* \in K^h$, such that

$$a_h(u^*, v - mu^*) \ge f_h(v - mu^*), \quad \forall v \in K_1^* + \cdots + K_m^*, \tag{7}$$

where $K_i^* = (u * + V_i) \cap K^h, i = 1, \cdots, m$.

**Lemma 1.** *Problem (7) is equivalent to Problem (4), (5) or Problem (4), (6).*

**Proof.** It is easy to prove that the solution of Problem (4), (5) (or (4), (6)) satisfies (7). Now we prove that any solution of (7) solves (4), (5) (or (4), (6) ). Assume that $u^*$ solves (7). It is sufficient to prove, for any $v \in K^h$, that

$$a_h(u^*, v - u^*) \ge f_h(v - u^*). \tag{8}$$

Let $\Omega_\varepsilon$ be the open subset of $\Omega$ with $\bar{\Omega}_\varepsilon \subset \Omega$, containing all the interior nodes. Then, there exists $\{\theta_i\}$ such that $0 \le \theta_i \le 1, \theta_i \in C_0^\infty(\Omega_i)$ and $\sum_{i=1}^m \theta_i = 1$ on $\bar{\Omega}_\varepsilon$. Therefore, we have

$$v = \sum_{i=1}^m I^h(\theta_i v) \text{ on } \bar{\Omega}$$

where $I^h$ is the $V^h$–interpolation operator. Obviously $v_i = I^h(\theta_i v) \in V_i$ and

$$v - u^* = \sum_{i=1}^{m} v_i - u* = \sum_{i=1}^{m} w_i - mu^*, \tag{9}$$

where $w_i = u^* + I^h(\theta_i(v - u^*))$. It is easy to see that $w_i \in K_i^*$. Hence we have, by (7), that

$$a_h(u^*, \sum_{i=1}^{m} w_i - mu^*) \geq f_h(\sum_{i=1}^{m} w_i - mu^*),$$

which combined with (9) yields (8). The lemma is proved.

Let $J_h(v) = a_h(v,v)/2 - f_h(v)$. Then $u_h, u^{n,i}$ are, respectively, the solutions of the following problems:

$$u^h \in K^h, \quad J^h(u_h) = \min_{v \in K^h} J_h(v),$$
$$u^{n,i} \in K_i^n, \quad J_h(u^{n,i}) = \min_{v \in K_i^n} J_h(v).$$

By using the lemma and the strict convexity of $J(v)$, we obtain the following convergence theorem.

**Theorem 1.** *The sequence $\{u^n\}$ produced by Algorithm I converges to $u_h$ in $V^h$. Moreover we have $u^{n,i} \to u_h, (n \to \infty), i = 1, \cdots, m$.*

Assume that $k = 1$ and that $V^h$ is a Lagrange finite element space. Then Problem (4), (5) is equivalent to a linear complementary problem which has the following matrix form:

$$U \geq \Phi,\ AU - F \geq 0,\ (U - \Phi)^T(AU - F) = 0, \tag{10}$$

where $U, \Phi \in R^N$, their components $\{U_j\}, \{\Phi_j\}$ are respectively the values of $u_h, \varphi$ at the interior nodes, $A = (a_{ij})_{i,j=1}^N, a_{ij} = a_h(\varphi_i, \varphi_j), F = (F_j)_{j=1}^N, F_j = f(\varphi_j)$, and $\{\varphi_j\}$ is the basis of $V^h$. Then $K^h$ corresponds to

$$C = \{V \in R^N : V_j \geq \Phi_j; j = 1 \cdots, N\}.$$

Let $I = \{1, \cdots, N\}$. Denote the index set of the nodes in $\Omega_i$ by $I_i$. The matrix form of Algorithm I is as follow.

**Algorithm $I^*$**

Step 1 Given $\omega_i > 0, 1, \cdots, m$ with $\sum_{i=1}^m \omega_i = 1$. Take $U^0 \in C. n := 0$;

Step 2 For $i = 1, \cdots, m$, solve the subproblems : find $U^{n,i} \in R^N$ such that $U^{n,i} \geq \Phi$ and

$$U_j^{n,i} = U_j^n \text{ for } j \in I/I_i,$$
$$(AU^{n,i} - F)_j \geq 0, (U_j^{n,i} - \Phi_j)(AU^{n,i} - F)_j = 0 \text{ for } j \in I_i;$$

Step 3 $U^{n+1} = \sum_{i=1}^m \omega_i U^{n,i}$;

Step 4 $n := n + 1$, go to Step 2.

Assume that $U$ is the solution of (10) and let $J = \{j \in I : U_j = \Phi_j\}$. We call $J$ the coincident set and $U_j$ the coincident component if $j \in J$. We say that problem

(10) is nondegenerate if $j \in J$ implies $(AU - F)_j > 0$. By Theorem 1 we know that $U^n \to U(n \to \infty)$ and $U^{n,i} \to U(n \to \infty), i = 1, \cdots, m$. It is not difficult to show that the coincident components are reached by the corresponding components of $U^{n,i}$ within a finite number of iteration steps, just as the following theorem states.

**Theorem 2.** *If (10) is nondegenerate, then there exists a positive integer $n_0$ such that for $n \geq n_0$*

$$U_j^{n,i} = \Phi_j, \forall j \in J \cap I_i,$$
$$U_j^{n,i} > \Phi_j, \forall j \in (I \backslash J) \cap I_i.$$

A counterexample shows that $U^n$ does not have this property of $U^{n,i}$. Now we consider the so-called average with variable weights (ref. [4]). Assume $Q \in \Omega$. We say that $Q$ is a $k$-point if $Q$ belongs to at most $k$ subregions.

Let

$$\Omega(i_1, \cdots, i_k) = \Omega_{i_1} \bigcap \cdots \bigcap \Omega_{i_k}.$$

We construct a function $\bar{u}^n$ as follows: for a k-point $Q \in \Omega(i_1, \cdots, i_k)$ define

$$\bar{u}^{n+1}(Q) = \frac{1}{k} \sum_{j=1}^{k} u^{n,i_j}(Q),$$

where $u^{n,i}$ is defined by Algorithm $I$ with $\omega_i = \frac{1}{m}, i = 1, \cdots, m$. Then it is easy to see that the following relationship between $u^n$ and $\bar{u}^n$ holds:

$$u^{n+1}(Q) = \frac{k}{m} \bar{u}^{n+1}(Q) + \frac{m-k}{m} u^n(Q) \text{ for } Q \in \Omega(i_1, \cdots, i_k).$$

Therefore we know that $\bar{u}^n \to u(n \to \infty)$ by Theorem 1. Moreover we can show

**Theorem 3.** *If (10) is nondegenerate and $\bar{U}^n$ is the vector corresponding to $\bar{u}^n$ then there exists a positive integer $n_0$ such that for $n \geq n_0$*

$$\bar{U}_j^n = \Phi_j, \forall j \in J,$$
$$\bar{U}_j^n > \Phi_j, \forall j \in I \backslash J.$$

For Problem (4), (6) we can establish similar definitions and conclusions.

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