A Semi-dual Mode Synthesis Method for Plate Bending Vibrations

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1. Introduction

Once a structure is decomposed into substructures, mode synthesis is the Rayleigh-Ritz approximation of the global eigenvalue problem on the space spanned by a few eigenmodes of each substructure and some coupling modes which aim at describing the restriction to the interface of the global eigenmodes [7]. These coupling modes are defined here at the continuous level as the eigenfunctions of an ad hoc preconditioner of the Poincaré-Steklov operator associated with the interface as in [2]. The definition of this preconditioner of Neumann-Neumann type relies on suitable extension operators from the boundary of the subdomains to the whole interface. This paper concentrates on the definition of such extension operators in the case of plate bending and for general domain decompositions with cross-points and extends [2].

The plate bending problem is posed over a domain \( \omega \subset \mathbb{R}^2 \) which is split in \( p \) substructures \( \omega_i \) separated by an interface \( \gamma \). Let \( D, \nu, \) and \( \rho \) denote the non-necessarily constant stiffness, Poisson’s ratio and mass density respectively. Greek indices take their value in \( \{1, 2\} \) and summation of repeated indices is assumed. For \( u, v \in H^2(\omega) \), define:

\[
\begin{aligned}
(u, v)_i &= \int_{\omega_i} \rho uv, \\
a_i(u, v) &= \int_{\omega_i} D((1 - \nu)\partial_{\alpha\beta}u\partial_{\alpha\beta}v + \nu\Delta u\Delta v) + d(u, v)_i, \\
(u, v) &= \sum_{i=1}^{p} (u, v)_i, \quad a(u, v) = \sum_{i=1}^{p} a_i(u, v),
\end{aligned}
\]

where \( d \) stands for an arbitrary positive constant and \( \Delta \) for the Laplacian. Let \( V \) denote the space of admissible displacement, i.e. the subspace of \( H^2(\omega) \) satisfying the Dirichlet boundary conditions along \( \partial \omega \) of the problem, if any.

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It is well-known that the global eigenvalue problem:

Find \((\lambda, u) \in \mathbb{R} \times V\) s.t. \(a(u, v) = \lambda(u, v) \quad \forall v \in V\)

possesses a family \((\lambda_k, u_k)_{k=1}^{+\infty}\) of solutions. In the same way, for \(1 \leq i \leq p\) set:

\(\gamma_i = \gamma \cap \partial \omega_i, \quad V_i = \{w|\gamma_i; w \in V\}\) and \(V_i^0 = \{v \in V_i; v = 0\) on \(\gamma_i\}\).

Let \((\lambda_{ij}, u_{ij})_{i=1}^{+\infty} \in \mathbb{R}^+ \times V_i^0\) denote the family of solutions of the problem:

Find \((\lambda, u) \in (R) \times V_i^0\) s.t. \(a_i(u, v) = \lambda(u, v); \forall v \in V_i^0\).

Mode synthesis uses as test functions the fixed interface modes \(u_{i,j}\) and coupling modes that do not identically vanish on \(\gamma\).

2. Definition of the coupling modes

2.1. Basic trace and extension properties. The admissible displacements \(w \in H^2(\omega)\) possess two independent traces along \(\gamma\), \(w|\gamma\) and \(\theta_n = (\frac{\partial w}{\partial n})|\gamma\), where \(\vec{n}\) denotes a unit normal vector along \(\gamma\), that is defined on each edge independently. Let us recall a characterization of the space \(V_\gamma = \{w|\gamma, (\frac{\partial w}{\partial n})|\gamma; w \in V\}\): along each edge \(\Gamma_i, (w|\gamma, (\frac{\partial w}{\partial n})|\gamma) \in H^{3/2}(\Gamma_i) \times H^{1/2}(\Gamma_i)\). Moreover, compatibility conditions hold at every vertex \(O\) of the interface. Assume that two edges \(\Gamma_1\) and \(\Gamma_2\) share a common vertex \(O\), as in Figure 1, and denote by \((w_1, \theta_{n,1})\) (resp. \((w_2, \theta_{n,2})\)) the traces of \(w\) along \(\Gamma_1\) (resp. \(\Gamma_2\)). The continuity of \(w\) and the \(H^{1/2}\) continuity of \(\nabla w\) must be ensured at \(O\). Since \(\nabla w|\Gamma_i = \frac{\partial w_i}{\partial s_i} \vec{n}_i + \theta_{n,i} \vec{n}_i\), \(\vec{n}_i\) if \(\vec{n}\) denotes a unit tangential vector along \(\Gamma_i, s_i\) an associated curvilinear abscissa on \(\Gamma_i\), and \(\vec{n}_i = \vec{n}|\Gamma_i\), the compatibility conditions write:

\[
\begin{align*}
\left\{ \begin{array}{c}
w_1(O) = w_2(O) \\
\frac{\partial w_1}{\partial s_1} \vec{n}_1 + \theta_{n,1} \vec{n}_1 = \frac{\partial w_2}{\partial s_2} \vec{n}_2 + \theta_{n,2} \vec{n}_2 
\end{array} \right. \quad \text{at } O
\end{align*}
\]

for every vertex \(O\) and every set of edges that cross at \(O\). If \(N_e\) denotes the number of edges, it turns out that \(V_\gamma\) is isomorphic to the subspace of \(\prod_{i=1}^{N_e} H^{3/2}(\Gamma_i) \times H^{1/2}(\Gamma_i)\) of pairs satisfying above compatibility conditions as well as the Dirichlet boundary conditions along \(\partial \omega_i\), if any.

Let \(R : V_\gamma \longrightarrow V\) denote the biharmonic extension operator defined by

\[
\begin{align*}
&\left\{ \begin{array}{c}
a(R(v, \theta), z) = 0 \quad \forall z \in V^0 = \bigcup_{i=1}^{\nu} V_i^0, \\
R(v, \theta) = v, \quad \frac{\partial R(v, \theta)}{\partial \vec{n}} = \theta \quad \text{along } \gamma
\end{array} \right.
\end{align*}
\]

This problem splits into \(p\) independent plate problems. Now, let \((w_1, \theta_{n,1})_{i=1}^{+\infty}\) denote a given dense family in \(V_\gamma\). Then coupling modes will be defined as \(R(w_1, \theta_{n,1}) \in V\).

The question of choosing a suitable family is addressed in the next section.
2.2. A generalized Neumann-Neumann preconditioner. Let us set 
\( V_{\gamma} = \{ (v, \theta) \}_{\gamma_i}; (v, \theta) \in V_i \}_{i=1}^p \), and let \( P_i: V_{\gamma} \rightarrow V_i \) denote a continuous extension operator, that is also defined as a continuous operator from \( L^2(\gamma_i) \times L^2(\gamma_i) \) 
to \( L^2(\gamma) \times L^2(\gamma) \). For any pair of functions \( (T, M) \) defined on \( \gamma_i \), let \( w \in V_i \) stand 
for the solution of the well-posed Neumann problem

\[
a_i(w, z) = \int_{\gamma_i} Tz + M \frac{\partial z}{\partial n} \forall z \in V_i,
\]

and define the mapping \( S_i \) by \( S_i(T, M) = (w|_{\gamma_i}, \frac{\partial w}{\partial n}|_{\gamma_i}) \). This local dual Schur complement \( S_i: V_{\gamma} \rightarrow V_{\gamma} \) is continuous.

Then the operator \( S: V_{\gamma} \rightarrow V_{\gamma} \), \( S = \sum_{i=1}^p P_i S_i P_i^* \) is continuous. It is 
compact over \( L^2(\gamma) \times L^2(\gamma) \) and symmetric, hence it possesses a family of finite-dimensional eigenspaces associated with positive decreasing eigenvalues \( (\mu_{\gamma\ell})_{\ell=1}^{+\infty} \) and also a possibly infinite-dimensional kernel, \( \ker(S) \). Since by construction the 
operators \( S_i \) are isomorphisms, \( \ker(S) = \ker(\sum_{i=1}^p P_i P_i^*) \). Therefore, the family 
chosen of coupling modes naturally splits in two subfamilies:

- The first one is made of the biharmonic extensions \( \mathcal{R}(w_\ell, \theta_\ell) \) of a given number \( N_\gamma \), of independent eigenfunctions \( (w_\ell, \theta_\ell) \) associated with the largest eigenvalues \( \mu_{\gamma\ell} \).

- As for the second one, noticing that

\[
\ker(\sum_{i=1}^p P_i P_i^*) = \bigcap_{i=1}^p \ker(P_i P_i^*) \subset \bigcup_{i=1}^p \ker(P_i P_i^*)
\]

we decide to retain the low frequency content of each subspace \( \ker(P_i P_i^*) \), namely the \( M_i \) first solutions of the auxiliary eigenvalue problem

\[
(4) \quad \int_{\gamma_i} \frac{\partial^2 v}{\partial \tau_i^2} \frac{\partial^2 v}{\partial \tau_i^2} + \int_{\gamma_i} \frac{\partial \theta}{\partial \tau_i} \frac{\partial \psi}{\partial \tau_i} + \frac{1}{\epsilon} \int_{\gamma_i} P_i^*(w, \theta) P_i^*(v, \psi) = \xi \int_{\gamma_i} wv + \theta \psi
\]

\( \forall (v, \psi) \in H^2(\gamma) \times H^1(\gamma) \), where \( \epsilon \) is a small parameter.

The biharmonic extensions \( \mathcal{R}(\tilde{w}_{ij}, \tilde{\theta}_{ij}) \) of these \( M = \sum_{i=1}^p M_i \) modes \( (\tilde{w}_{ij}, \tilde{\theta}_{ij}) \),

form the second family of coupling modes. The resulting mode synthesis method amounts to define the finite-dimensional space

\[
V_N = \text{span}\left\{ \bigcup_{i=1}^p (u_{ij})_{j=1}^{N_i} \cup \bigcup_{\ell=1}^{N_\ell} \mathcal{R}(w_\ell, \theta_\ell)_{\ell=1}^{N_\ell} \cup \bigcup_{i=1}^p (\mathcal{R}(\tilde{w}_{ij}, \tilde{\theta}_{ij}))_{j=1}^{M_i} \right\},
\]

for some numbers \( N_i, M_i, N_\ell, 1 \leq i \leq p \), and to perform the Galerkin approximation of the global eigenvalue problem on this space.

**Remark 1.** It is possible to further filter the kernels \( \ker(P_i P_i^*) \) by just solving the eigenvalue problem (4) again after projection over the solutions of the first solve and with \( \epsilon = +\infty \).

**Remark 2.** If the extension operator preserves some locality, this eigenvalue problem is posed over a limited set of edges, not on the whole interface. Moreover, since \( P_i \) only depends on the geometry of the interface and, at the discrete level, on the mesh, the auxiliary eigenvalue problem (4) does not require any subdomain solve, and, in practice, proves very cheap.
Remark 3. Mode synthesis appears as a non iterative domain decomposition method contrary to [5], [6]. The proposed method differs from [1] where the Schur complement is used instead of a preconditioner. It also differs from [4] since the preconditioner is not used to compute the spectrum of the Schur complement.

The next section is devoted to the construction of the extension operators $P_i$.

3. The extension operators $P_i : V_{\sigma_i} \rightarrow V_{\gamma_i}$

Let $w \in V_{\sigma_i}$ denote some given function. Its traces $(w|_{\gamma_i}, \frac{\partial w}{\partial \vec{n}^i}|_{\gamma_i})$ are to be extended onto the adjacent edges of the interface. First pick out two edges $\Gamma_1$ and $\Gamma_2$ of $\gamma_i$ that share a common vertex $\mathcal{O}$, and choose an adjacent edge $\Gamma_3 \subset \gamma_i$, as in Figure 1. Parametrize $\Gamma_1$ (resp. $\Gamma_2$, $\Gamma_3$) by the curvilinear abscissa $s_1 \in [a, 0]$ (resp. $s_2 \in [c, 0]$, $s_3 \in [0, b]$), starting from the vertex $\mathcal{O}$. As in section 2, let $(w_1, \theta_{n1})$ (resp. $(w_2, \theta_{n2})$) stand for the traces of $w$ along $\Gamma_1$ (resp. $\Gamma_2$). These traces satisfy the compatibility conditions (2). A pair of traces $(w_3, \theta_{n3})$ is sought on $\Gamma_3$ in such a way that the compatibility conditions

\[
\begin{align*}
\left\{ \begin{array}{c}
\frac{\partial w_3}{\partial s_3} \vec{t}_3 + \theta_{n3} \vec{n}_3 &= \frac{\partial w_1}{\partial s_1} \vec{t}_1 + \theta_{n1} \vec{n}_1 \quad \text{at } \mathcal{O} \\
\frac{\partial w_3}{\partial s_3} \vec{t}_3 + \theta_{n3} \vec{n}_3 &= \frac{\partial w_2}{\partial s_2} \vec{t}_2 + \theta_{n2} \vec{n}_2 \quad \text{at } \mathcal{O}
\end{array} \right.
\end{align*}
\]

hold for every possible value of $w_1(\mathcal{O}), \frac{\partial w_1}{\partial s_1}(\mathcal{O})$ and $\theta_{n1}(\mathcal{O})$. The resulting traces will then coincide with the traces of a function in $H^2(\omega)$.

The compatibility conditions (5) lead to 5 scalar equations. This is why the proposed extension operator $P_i$ involves 5 parameters to be identified:

\[
\left( P_i \left( w_{|\gamma_i}, \frac{\partial w}{\partial \vec{n}^i}_{|\gamma_i} \right) \right)_{|\Gamma_3} = (w_3, \theta_{n3}) \quad \text{with}
\]
\[
\begin{align*}
\begin{cases}
    w_3(s) &= \varphi(s) \left\{ \frac{\alpha_{12}}{2} w_1(\frac{\sigma}{b} s) + \beta_{13} w_1(\frac{2\sigma}{b} s) + \frac{\alpha_{12}}{2} w_2(\frac{\sigma}{b} s) + \beta_{23} w_2(\frac{2\sigma}{b} s) \right\} \\
    \theta_3(s) &= \varphi(s) \left\{ \eta_{13} \theta_{h,1}(\frac{\sigma}{b} s) + \eta_{23} \theta_{h,2}(\frac{\sigma}{b} s) \right\},
\end{cases}
\end{align*}
\]

and where \( \varphi \) denotes some cut-off function whose support forms a neighborhood of \( \mathcal{O} \).

Expressing the compatibility conditions leads to the linear system \( A_3 X_3 = F_3 \), where:

\[
X_3 = \begin{bmatrix}
\alpha_{13} \\
\beta_{13} \\
\beta_{23} \\
\eta_{13} \\
\eta_{23}
\end{bmatrix}
\]

\[
A_3 = \begin{bmatrix}
1 & 1 & 1 & 0 & 0 \\
\frac{a}{2b} \tau_{3,x} & \frac{a}{b} \tau_{3,x} & 0 & C_{1n_{3,x}} & n_{3,x} \left[ C_{1n_{1,x}} + \tau_{1,x} \right] \\
0 & \frac{a}{b} \tau_{3,x} & C_{2n_{3,x}} & n_{3,x} \left[ C_{2n_{2,x}} - \tau_{2,x} \right] \\
\frac{a}{2b} \tau_{3,y} & \frac{a}{b} \tau_{3,y} & 0 & C_{1n_{3,y}} & n_{3,y} \left[ C_{1n_{1,y}} + \tau_{1,y} \right] \\
0 & \frac{a}{b} \tau_{3,y} & C_{2n_{3,y}} & n_{3,y} \left[ C_{2n_{2,y}} - \tau_{2,y} \right]
\end{bmatrix}
\]

\[
F_3 = \begin{bmatrix}
\tau_{1,x} + C_{1n_{1,x}} \\
C_{2n_{1,x}} \\
\tau_{1,y} + C_{1n_{1,y}} \\
C_{2n_{1,y}}
\end{bmatrix}
\]

with

\[
C_1 = \frac{\tau_{1,x} n_{2,y} - \tau_{1,y} n_{2,x}}{n_{1,y} n_{2,x} - n_{1,x} n_{2,y}}, \quad C_2 = \frac{\tau_{2,y} n_{2,x} - \tau_{2,x} n_{2,y}}{n_{1,y} n_{2,x} - n_{1,x} n_{2,y}}.
\]

and

\[
\tau_{\ell,x} = \tilde{\tau}_{\ell,x}, \quad \tau_{\ell,y} = \tilde{\tau}_{\ell,y}, \quad n_{\ell,x} = \tilde{n}_{\ell,x}, \quad n_{\ell,y} = \tilde{n}_{\ell,y}, \quad 1 \leq \ell \leq 3.
\]

From symbolic calculus the determinant is equal to \( 2^{\frac{16u_c}{b}} \). This system is solved once and for all, thus yielding the parameters of the extension operator.

This process is repeated for all adjacent edges containing \( \mathcal{O} \), and for all vertices. It follows from (6) that compatibility will also hold among all edges on which the original traces are extended. Therefore \( P_1 : V_{\gamma_1} \rightarrow V_{\gamma} \) is continuous. Moreover, it is clear from its definition that \( P_1 : L^2(\gamma_i) \times L^2(\gamma_i) \rightarrow L^2(\gamma) \times L^2(\gamma) \) is also continuous see [3] for details.

4. Numerical tests

A square plate is decomposed into 9 subdomains and discretized with 15000 dof. Since we focus on the coupling strategy, a large number of fixed interface modes is used. Notice that only 17 modes of the dual Schur complement and less than 3 modes of each kernel \( \ker(P_i^*) \) are sufficient to yield a 1% accuracy on the first 20 eigenfrequencies (Fig. 2).
Figure 2. (a): The mesh and a typical subdomain, (b): the accuracy for $N_\gamma = 17$ and $M = \sum_{i=1}^{p} M_i = 23$

Figure 3. The first mode shape computed with: (a) A global F.E.M., and (b) Mode Synthesis for $N_\gamma = 17$ and $M = 23$

The mode shapes are also very accurate and smoothness is achieved, mainly because the extension operators are defined at the continuous level (Fig. 3, 4, and 5).

5. Concluding remarks

A new mode synthesis method is proposed. It can be formulated at the continuous level and at the discrete level. It is based on a generalized Neumann-Neumann preconditioner. It yields accurate frequencies and smooth mode shapes with a small number of coupling modes even when the interface exhibits cross-points. Its numerical analysis remains fairly open. See [3] for details.

References

1. F. Bourquin and F. d’Hennezel, Intrinsic component mode synthesis and plate vibrations, Comp. and Str. 44 (1992), no. 1, 315–324
Figure 4. The 20th mode shape computed with: (a) A global F.E.M., and (b) Mode Synthesis for $N_s = 17$ and $M = 23$

Figure 5. The 20th mode shape computed with: (a) A global F.E.M., and (b) Mode Synthesis for $N_s = 17$ and $M = 23$


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