http://dx.doi.org/10.1090/conm/218/03033

Contemporary Mathematics
Volume **218**, 1998
B 0-8218-0988-1-03033-8

A Semi-dual Mode Synthesis Method for Plate Bending Vibrations

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1. Introduction

Once a structure is decomposed into substructures, mode synthesis is the Rayleigh-Ritz approximation of the global eigenvalue problem on the space spanned by a few eigenmodes of each substructure and some *coupling modes* which aim at describing the restriction to the interface of the global eigenmodes [**7**]. These *coupling modes* are defined here at the continuous level as the eigenfunctions of an *ad hoc* preconditionner of the Poincaré-Steklov operator associated with the interface as in [**2**]. The definition of this preconditionner of Neumann-Neumann type relies on suitable extension operators from the boundary of the subdomains to the whole interface. This paper concentrates on the definition of such extension operators in the case of plate bending and for general domain decompositions with cross-points and extends [**2**].

The plate bending problem is posed over a domain $\omega \subset \mathbb{R}^2$ which is splitted in p substructures ω_i separated by an interface γ. Let D, ν, and ρ denote the non-necessarily constant stiffness, Poisson's ratio and mass density respectively. Greek indices take their value in $\{1,2\}$ and summation of repeated indices is assumed. For u , $v \in H^2(\omega)$, define :

$$(1) \qquad \begin{cases} (u,v)_i = \displaystyle\int_{\omega_i} \rho u v, \\ a_i(u,v) = \displaystyle\int_{\omega_i} D((1-\nu)\partial_{\alpha\beta} u \partial_{\alpha\beta} v + \nu \Delta u \Delta v) + d(u,v)_i, \\ (u,v) = \displaystyle\sum_{i=1}^{p} (u,v)_i \ , \ a(u,v) = \sum_{i=1}^{p} a_i(u,v), \end{cases}$$

where d stands for an arbitrary positive constant and Δ for the Laplacian. Let V denote the space of admissible displacements, *i.e.* the subspace of $H^2(\omega)$ satisfying the Dirichlet boundary conditions along $\partial\omega$ of the problem, if any.

1991 *Mathematics Subject Classification.* Primary 65N55; Secondary 65N25, 73K10.

The authors express their warmest thanks to J-J. Brioist and the whole team in charge of developing CESAR-LCPC for their help during the implementation of this method.

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It is well-known that the global eigenvalue problem :

$$\text{Find } (\lambda, u) \in \mathbb{R} \times V \text{ s.t. } a(u,v) = \lambda(u,v) \ \forall v \in V$$

possesses a family $(\lambda_k, u_k)_{k=1}^{+\infty}$ of solutions. In the same way, for $1 \leq i \leq p$ set :

$$\gamma_i = \gamma \bigcap \partial\omega_i, \ \ V_i = \{\mathrm{w}_{|\omega_i}; \mathrm{w} \in V\} \quad \text{and} \quad V_i^0 = \{v \in V_i; v = 0 \text{ and } \nabla v = 0 \ on \ \gamma_i\}.$$

Let $(\lambda_{ij}, u_{ij})_{j=1}^{+\infty} \in \mathbb{R}^+ \times V_i^0$ denote the family of solutions of the problem :

$$\begin{aligned} &\text{Find } \ (\lambda, u) \in (R) \times V_i^0 \ \text{ s.t.} \\ &a_i(u,v) = \lambda(u,v)_i \ \forall v \in V_i^0. \end{aligned}$$

Mode synthesis uses as test functions the *fixed interface modes* u_{ij} and *coupling modes* that do not identically vanish on γ.

2. Definition of the coupling modes

2.1. Basic trace and extension properties. The admissible displacements $\mathrm{w} \in H^2(\omega)$ possess two independent traces along γ, $\mathrm{w}_{|\gamma}$ and $\theta_n = (\frac{\partial \mathrm{w}}{\partial \vec{n}})_{|\gamma}$, where $\vec{n}$ denotes a unit normal vector along γ, that is defined on each edge independently. Let us recall a characterization of the space $V_\gamma = \left\{\mathrm{w}_{|\gamma}, \ (\frac{\partial \mathrm{w}}{\partial \vec{n}})_{|\gamma}; \ \mathrm{w} \in V\right\}$: along each edge Γ_i, $(\mathrm{w}_{|\gamma}, \ (\frac{\partial \mathrm{w}}{\partial \vec{n}})_{|\gamma}) \ \in H^{3/2}(\Gamma_i) \times H^{1/2}(\Gamma_i)$. Moreover, compatibility conditions hold at every vertex $\mathcal{O}$ of the interface. Assume that two edges Γ_1 and Γ_2 share a common vertex $\mathcal{O}$, as in Figure 1, and denote by $(\mathrm{w}_1, \theta_{n,1})$ (resp. $(\mathrm{w}_2, \theta_{n,2})$) the traces of w along Γ_1 (resp. Γ_2). The continuity of w and the $H^{1/2}$-continuity of $\nabla \mathrm{w}$ must be ensured at $\mathcal{O}$. Since $\nabla \mathrm{w}_{|\Gamma_i} = \frac{\partial \mathrm{w}_i}{\partial s_i} \ \vec{\tau}_i + \theta_{n,i} \ \vec{n}_i$ if $\vec{\tau}_i$ denotes a unit tangential vector along Γ_i, s_i an associated curvilinear abscissa on Γ_i, and $\vec{n}_i = \vec{n}_{|\Gamma_i}$, the compatibility conditions write :

$$\left\{ \begin{array}{rcl} \mathrm{w}_1(\mathcal{O}) & = & \mathrm{w}_2(\mathcal{O}) \\ \dfrac{\partial \mathrm{w}_1}{\partial s_1}\vec{\tau}_1 + \theta_{n,1}\vec{n}_1 & = & \dfrac{\partial \mathrm{w}_2}{\partial s_2}\vec{\tau}_2 + \theta_{n,2}\vec{n}_2 \quad at \ \mathcal{O} \end{array} \right. \tag{2}$$

for every vertex $\mathcal{O}$ and every set of edges that cross at $\mathcal{O}$. If N_e denotes the number of edges, it turns out that V_γ is isomorphic to the subspace of $\prod_{i=1}^{N_e} H^{3/2}(\Gamma_i) \times H^{1/2}(\Gamma_i)$ of pairs satisfying above compatibility conditions as well as the Dirichlet boundary conditions along $\partial\omega$, if any.

Let $\mathcal{R} \ : \ V_\gamma \longrightarrow V$ denote the *biharmonic* extension operator defined by

$$\left\{ \begin{array}{l} a(\mathcal{R}(v,\theta), z) = 0 \ \ \forall z \in V^0 = \bigcup_{i=1}^{p} V_i^0, \\ \mathcal{R}(v,\theta) = v \ , \ \dfrac{\partial \mathcal{R}(v,\theta)}{\partial \vec{n}} = \theta \ \ along \ \gamma. \end{array} \right. \tag{3}$$

This problem splits into p independent plate problems. Now, let $(\mathrm{w}_l, \theta_{n,l})_{l=1}^{+\infty}$ denote a given dense family in V_γ. Then *coupling modes* will be defined as $\mathcal{R}(\mathrm{w}_l, \theta_{n,l}) \in V$. The question of chosing a suitable family is addressed in the next section.

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2.2. A generalized Neumann-Neumann preconditioner. Let us set $V_{\gamma_i} = \{(v,\theta)_{|\gamma_i};\ (v,\theta) \in V_\gamma\}$, and let $P_i\ :\ V_{\gamma_i} \longrightarrow V_\gamma$ denote a continuous extension operator, that is also defined as a continuous operator from $L^2(\gamma_i) \times L^2(\gamma_i)$ to $L^2(\gamma) \times L^2(\gamma)$. For any pair of functions (T, M) defined on γ_i, let $\mathrm{w} \in V_i$ stand for the solution of the well-posed Neumann problem

$$a_i(\mathrm{w}, z) = \int_{\gamma_i} Tz + M\frac{\partial z}{\partial \vec{n}}\ \ \forall z \in V_i,$$

and define the mapping S_i by $S_i(T, M) = (\mathrm{w}_{|\gamma_i}\ ,\ \frac{\partial \mathrm{w}}{\partial \vec{n}}_{|\gamma_i})$. This local dual Schur complement $S_i\ :\ V'_{\gamma_i} \longrightarrow V_{\gamma_i}$ is continuous.

Then the operator $S\ :\ V'_\gamma \longrightarrow V_\gamma$, $S = \sum_{i=1}^p P_i S_i P_i^*$ is continuous. It is compact over $L^2(\gamma) \times L^2(\gamma)$ and symmetric, hence it possesses a family of finite-dimensional eigenspaces associated with positive decreasing eigenvalues $(\mu_{\gamma\ell})_{\ell=1}^{+\infty}$ and also a possibly infinite-dimensional kernel, $\ker(S)$. Since by construction the operators S_i are isomorphisms, $\ker(S) = \ker(\sum_{i=1}^p P_i P_i^*)$. Therefore, the family chosen of *coupling modes* naturally splits in two subfamilies :

- The first one is made of the *biharmonic* extensions $\mathcal{R}(\mathrm{w}_\ell, \theta_\ell)$ of a given number N_γ of independent eigenfunctions $(\mathrm{w}_\ell, \theta_\ell)$ associated with the largest eigenvalues $\mu_{\gamma\ell}$.

- As for the second one, noticing that

$$\ker(\sum_{i=1}^p P_i P_i^*) = \bigcap_{i=1}^p \ker(P_i P_i^*) \subset \bigcup_{i=1}^p \ker(P_i P_i^*)$$

we decide to retain the low frequency content of each subspace $\ker(P_i P_i^*)$, namely the M_i first solutions of the auxiliary eigenvalue problem

$$\int_{\gamma_i} \frac{\partial^2 \mathrm{w}}{\partial \tau_i^2}\frac{\partial^2 v}{\partial \tau_i^2} + \int_{\gamma_i} \frac{\partial \theta}{\partial \tau_i}\frac{\partial \psi}{\partial \tau_i} + \frac{1}{\epsilon}\int_{\gamma_i} P_i^*(\mathrm{w},\theta) P_i^*(v,\psi) = \xi \int_{\gamma_i} \mathrm{w}v + \theta\psi \tag{4}$$

$\forall\ (v, \psi) \in H^2(\gamma) \times H^1(\gamma)$, where ϵ is a small parameter.

The *biharmonic* extensions $\mathcal{R}(\tilde{\mathrm{w}}_{ij}, \tilde{\theta}_{ij})$ of these $M = \sum_{i=1}^p M_i$ modes $(\tilde{\mathrm{w}}_{ij}, \tilde{\theta}_{ij})$ form the second family of *coupling modes*. The resulting mode synthesis method amounts to define the finite-dimensional space

$$V_N = span\left\{\bigcup_{i=1}^p (u_{ij})_{j=1}^{N_i}\ \bigcup (\mathcal{R}(\mathrm{w}_\ell, \theta_\ell))_{\ell=1}^{N_\Gamma}\ \bigcup_{i=1}^p (\mathcal{R}(\tilde{\mathrm{w}}_{ij}, \tilde{\theta}_{ij}))_{j=1}^{M_i}\right\},$$

for some numbers N_i, M_i, N_γ, $1 \le i \le p$, and to perform the Galerkin approximation of the global eigenvalue problem on this space.

REMARK 1. it is possible to further filter the kernels $\ker(P_i P_i^*)$ by just solving the eigenvalue problem (4) again after projection over the solutions of the first solve and with $\epsilon = +\infty$

REMARK 2. If the extension operator preserves some locality, this eigenvalue problem is posed over a limited set of edges, not on the whole interface. Moreover, since P_i only depends on the geometry of the interface and, at the discrete level, on the mesh, the auxiliary eigenvalue problem (4) does not require any subdomain solve, and, in practice, proves very cheap.

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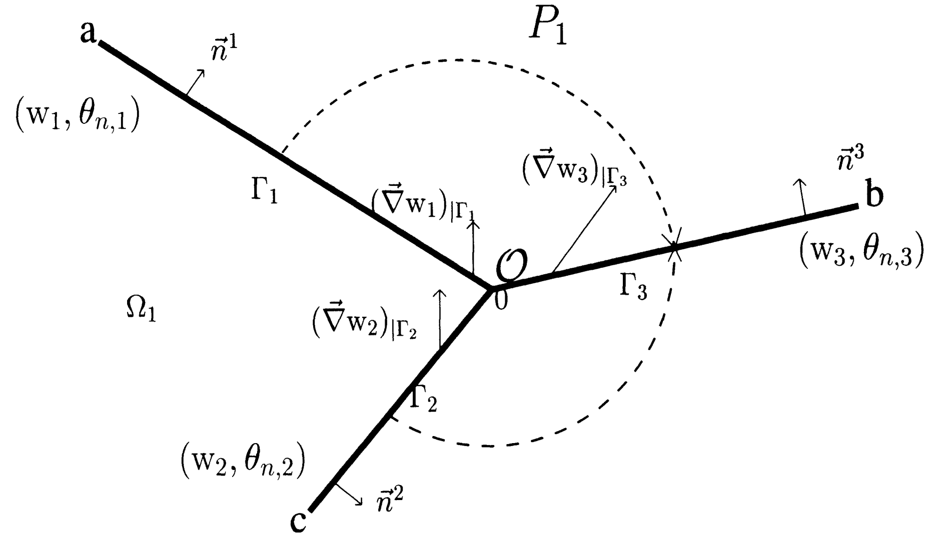

FIGURE 1. The extension operator by reflection for H^2 functions.

REMARK 3. Mode synthesis appears as a non iterative domain decomposition method contrary to [**5**], [**6**]. The proposed method differs from [**1**] where the Schur complement is used instead of a preconditioner. It also differs from [**4**] since the preconditioner is not used to compute the spectrum of the Schur complement.

The next section is devoted to the construction of the extension operators P_i.

3. The extension operators $\mathbf{P_i} : \mathbf{V}_{\gamma_i} \to \mathbf{V}_\gamma$

Let $\mathrm{w} \in V_i$ denote some given function. Its traces $(\mathrm{w}_{|\gamma_i}, \dfrac{\partial \mathrm{w}}{\partial \vec{n}}_{|\gamma_i})$ are to be extended onto the adjacent edges of the interface. First pick out two edges Γ_1 and Γ_2 of γ_i that share a common vertex $\mathcal{O}$, and choose an adjacent edge $\Gamma_3 \not\subset \gamma_i$, as in Figure 1. Parametrize Γ_1 (resp. Γ_2, Γ_3) by the curvilinear abscissa $s_1 \in [a, 0]$ (resp. $s_2 \in [c, 0]$, $s_3 \in [0, b]$), starting from the vertex $\mathcal{O}$. As in section 2, let $(\mathrm{w}_1, \theta_{n,1})$ (resp. $(\mathrm{w}_2, \theta_{n,2})$) stand for the traces of w along Γ_1 (resp. Γ_2). These traces satisfy the compatibility conditions (2). A pair of traces $(\mathrm{w}_3, \theta_{n,3})$ is sought on Γ_3 in such a way that the compatibility conditions

$$
(5) \qquad \left\{ \begin{array}{rcl} \mathrm{w}_3(\mathcal{O}) & = & \mathrm{w}_1(\mathcal{O}) \\ \dfrac{\partial \mathrm{w}_3}{\partial s_3} \vec{\tau}_3 + \theta_{n,3} \vec{n}_3 & = & \dfrac{\partial \mathrm{w}_1}{\partial s_1} \vec{\tau}_1 + \theta_{n,1} \vec{n}_1 \quad \text{at } \mathcal{O} \end{array} \right.
$$

hold for every possible value of $\mathrm{w}_1(\mathcal{O})$, $\dfrac{\partial \mathrm{w}_1}{\partial s_1}(\mathcal{O})$ and $\theta_{n,1}(\mathcal{O})$. The resulting traces will then coincide with the traces of a function in $H^2(\omega)$.

The compatibility conditions (5) lead to 5 scalar equations. This is why the proposed extension operator P_i involves 5 parameters to be identified :

$$
\left(P_i \left(\mathrm{w}_{|\gamma_i}, \frac{\partial \mathrm{w}}{\partial \vec{n}}_{|\gamma_i} \right) \right)_{|\Gamma_3} = (\mathrm{w}_3, \theta_{n,3}) \quad \text{with}
$$

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$$(6) \quad \begin{cases} \mathrm{w}_3(s) &= \varphi(s)\left\{\frac{\alpha_{13}}{2}\mathrm{w}_1(\frac{a}{b}s) + \beta_{13}\mathrm{w}_1(\frac{2a}{b}s) + \frac{\alpha_{13}}{2}\mathrm{w}_2(\frac{c}{b}s) + \beta_{23}\mathrm{w}_2(\frac{2c}{b}s)\right\} \\ \theta_{n,3}(s) &= \varphi(s)\left\{\eta_{13}\theta_{n,1}(\frac{a}{b}s) + \eta_{23}\theta_{n,2}(\frac{c}{b}s)\right\}, \end{cases}$$

and where φ denotes some cut-off function whose support forms a neightborhood of $\mathcal{O}$.

Expressing the compatibility conditions leads to the linear system $A_3X_3 = F_3$, where :

$$X_3 = \begin{bmatrix} \alpha_{13} \\ \beta_{13} \\ \beta_{23} \\ \eta_{13} \\ \eta_{23} \end{bmatrix}$$

$$(7) \qquad A_3 = \begin{bmatrix} 1 & 1 & 1 & 0 & 0 \\ \frac{a}{2b}\tau_{3,x} & \frac{2a}{b}\tau_{3,x} & 0 & C_1 n_{3,x} & n_{3,x}[C_1\frac{n_{1,x}}{n_{2,x}} + \frac{\tau_{1,x}}{n_{2,x}}] \\ \frac{c}{2b}\tau_{3,x} & 0 & \frac{2c}{b}\tau_{3,x} & C_2 n_{3,x} & n_{3,x}[C_2\frac{n_{1,x}}{n_{2,x}} - \frac{\tau_{2,x}}{n_{2,x}}] \\ \frac{a}{2b}\tau_{3,y} & \frac{2a}{b}\tau_{3,y} & 0 & C_1 n_{3,y} & n_{3,y}[C_1\frac{n_{1,x}}{n_{2,x}} + \frac{\tau_{1,x}}{n_{2,x}}] \\ \frac{c}{2b}\tau_{3,y} & 0 & \frac{2c}{b}\tau_{3,y} & C_2 n_{3,y} & n_{3,y}[C_2\frac{n_{1,x}}{n_{2,x}} - \frac{\tau_{2,x}}{n_{2,x}}] \end{bmatrix}$$

$$F_3 = \begin{bmatrix} 1 \\ \tau_{1,x} + C_1 n_{1,x} \\ C_2 n_{1,x} \\ \tau_{1,y} + C_1 n_{1,y} \\ C_2 n_{1,y} \end{bmatrix}$$

with

$$C_1 = \frac{\tau_{1,x}n_{2,y} - \tau_{1,y}n_{2,x}}{n_{1,y}n_{2,x} - n_{1,x}n_{2,y}} \ , \ C_2 = \frac{\tau_{2,y}n_{2,x} - \tau_{2,x}n_{2,y}}{n_{1,y}n_{2,x} - n_{1,x}n_{2,y}}$$

and

$$\tau_{\ell,x} = \vec{\tau}_\ell.\vec{x}\ , \quad \tau_{\ell,y} = \vec{\tau}_\ell.\vec{y}\ , \quad n_{\ell,x} = \vec{n}_\ell.\vec{x}\ , \quad n_{\ell,y} = \vec{n}_\ell.\vec{y}\ , \quad 1 \le \ell \le 3.$$

From symbolic calculus the determinant is equal to $2\dfrac{ac}{b}$. This system is solved once and for all, thus yielding the parameters of the extension operator.

This process is repeated for all adjacent edges containing $\mathcal{O}$, and for all vertices. It follows from (6) that compatibility will also hold among all edges onto which the original traces are extended. Therefore $P_i : V_{\gamma_i} \to V_\gamma$ is continuous. Moreover, it is clear from its definition that $P_i : L^2(\gamma_i) \times L^2(\gamma_i) \to L^2(\gamma) \times L^2(\gamma)$ is also continuous see [**3**] for details.

4. Numerical tests

A square plate is decomposed into 9 subdomains and discretized with 15000 dof. Since we focuss on the coupling strategy, a large number of *fixed interface modes* is used. Notice that only 17 modes of the dual Schur complement and less than 3 modes of each kernel $\ker(P_iP_i^*)$ are sufficient to yield a 1% accuracy on the first 20 eigenfrequencies (Fig. 2).

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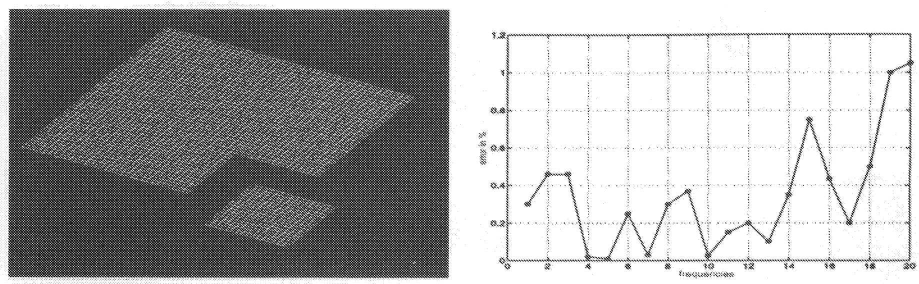
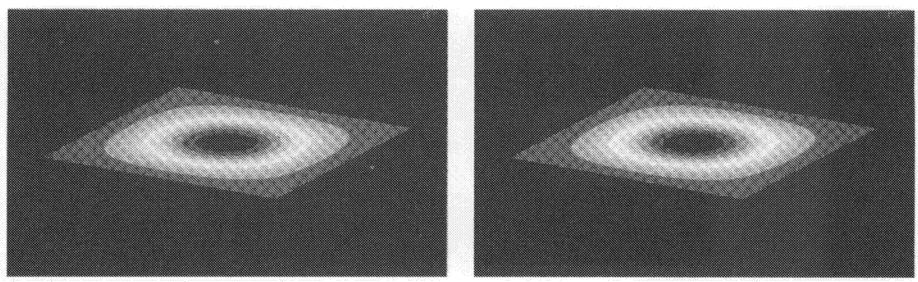


(a) (b)

FIGURE 2. (a): The mesh and a typical subdomain, (b): the accuracy for $N_\gamma = 17$ and $M = \sum_{i=1}^{p} M_i = 23$



(a) (b)

FIGURE 3. The first mode shape computed with : (a) A global F.E.M., and (b) Mode Synthesis for $N_\gamma = 17$ and $M = 23$

The mode shapes are also very accurate and smoothness is achieved, mainly because the extension operators are defined at the continuous level (Fig. 3, 4, and 5).

5. Concluding remarks

A new mode synthesis method is proposed. It can be formulated at the continuous level and at the discrete level. It is based on a *generalized* Neumann-Neumann preconditioner. It yields accurate frequencies and smooth mode shapes with a small number of *coupling modes* even when the interface exhibits cross-points. Its numerical analysis remains fairly open. See [**3**] for details.

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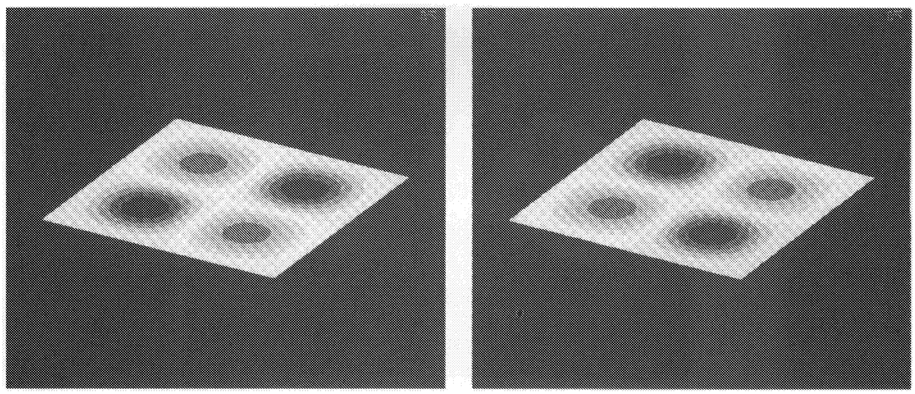
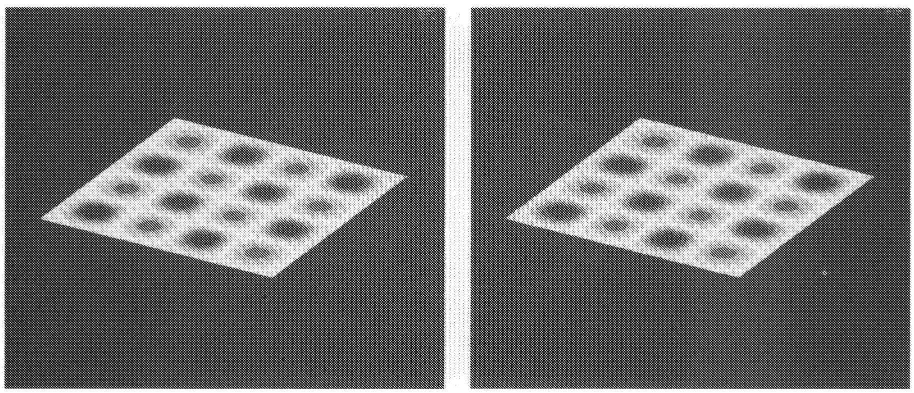


(a) (b)

FIGURE 4. The $20th$ mode shape computed with : (a) A global F.E.M., and (b) Mode Synthesis for $N_\gamma = 17$ and $M = 23$



(a) *(b)*

FIGURE 5. The $20th$ mode shape computed with : *(a)* A global F.E.M., and *(b)* Mode Synthesis for $N_\gamma = 17$ and $M = 23$

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