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Non-overlapping Schwarz Method for Systems of First Order Equations

Sébastien Clerc

1. Introduction

Implicit time-stepping is often necessary for the simulation of compressible fluid dynamics equations. For slow transient or steady-state computations, the CFL stability condition of explicit schemes is indeed too stringent. However, one must solve at each implicit time step a large linear system, which is generally unsymmetric and ill-conditioned. In this context, domain decomposition can be used to build efficient preconditioners suited for parallel computers. This goal was achieved by [**7, 9**], among others.

Still, important theoretical questions remain open to our knowledge, such as: estimate of the condition number, optimality of the preconditioner. This aspect contrasts with the existing results in structural mechanics, where the subspace correction framework allows a complete analysis (see for instance [**15**] and the references therein). This work is a preliminar step towards a better understanding of domain decomposition for systems of equations in the context of fluid dynamics.

To this purpose, we study the steady linearized equations, following the ideas of [**6**] and [**11**] for instance. In section 2, we present the derivation of these equations from the time-dependent non-linear hyperbolic systems of conservation laws. Symmetrization is also addressed, as well as the nature of the resulting equations.

A classical well-posedness result is recalled in section 3 for the boundary value problem. An energy estimate allows one to prove the convergence of the Schwarz iterative method, using the same arguments as [**4**] for the Helmholtz problem. This result generalizes those of [**6**] for scalar transport equations, and [**12**] for one-dimensional systems.

In section 4, we emphasize the difference between the purely hyperbolic case and the elliptic case, as far as the convergence of the Schwarz method is concerned. We also mention the influence of the transmission condition on the convergence for a model problem.

Section 5 deals with the space-discretization of the problem with a finite volume method and a first order upwind scheme. We propose two different implementations: a direct one which can be interpreted as a block-Jacobi preconditioner, and a Schur complement formulation. The latter is all the more useful when used in

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combination with the GMRES [**14**] algorithm. Numerical results are given for the Cauchy-Riemann equations and Saint-Venant's equations of shallow water flow.

Finally, conclusions are drawn in section 6.

2. Derivation of the equations

The equations of compressible fluid dynamics are generally formulated as a non-linear system of conservation laws:

$$\partial_t u + \sum_{\alpha=1}^{d} \partial_\alpha F^\alpha(u) = 0. \tag{1}$$

Here the solution u is a vector of $\mathbb{R}^p$ and the fluxes F^α are non-linear vector-valued functions of u. The equations are posed in $\mathbb{R}^d$ and the index α refers to the direction in the physical space.

Suppose that the solution $u^n(x)$ at time t^n is known. We seek a solution u^{n+1} at time $t^{n+1} = t^n + \delta t$. If the increment $\delta u = u^{n+1} - u^n$ is small enough, we can write the following linearized implicit equation:

$$\delta u/\delta t + \sum \partial_\alpha (D_u F^\alpha(u^n)\,\delta u) = -\sum \partial_\alpha F^\alpha(u^n),$$

$D_u F^\alpha$ being the Jacobian matrix of the flux F^α.

The next step consists in writing the non-conservative form, which is valid if u^n is smooth:

$$\left[(1/\delta t)Id + \sum \partial_\alpha D_u F^\alpha(u^n)\right] \delta u + \sum D_u F^\alpha\, \partial_\alpha \delta u = -\sum \partial_\alpha F^\alpha(u^n). \tag{2}$$

Finally, we recall that these equations can be symmetrized if the system of conservation laws (1) admits a mathematical entropy S (see for instance [**8**]). We multiply system (2) by the Hessian matrix $D_{uu}S$ of the entropy. The resulting system takes the following form:

$$A^0 u + \sum_{\alpha=1}^{d} A^\alpha \partial_\alpha u = f, \tag{3}$$

with:

$$A^0 = D_{uu}S(u^n)\left[(1/\delta t)Id + \sum \partial_\alpha D_u F^\alpha(u^n)\right], \tag{4}$$

$$A^\alpha = D_{uu}S(u^n) \cdot D_u F^\alpha(u^n).$$

The matrices A^α are symmetric, but A^0 may be any matrix for the moment.

In the sequel, we will consider boundary value problems for general symmetric systems of first order equations of type (3).

When going from the time-dependent system (1) to the steady system (3), the equations may not remain hyperbolic. The scalar case studied by [**6, 17**] and the one-dimensional case studied by [**12**] are two examples of hyperbolic steady equations.

The linearized steady Euler equations in 2-D are also hyperbolic in the supersonic regime, but become partially elliptic in the subsonic regime (see for instance

[16]). This aspect dramatically changes the nature of the boundary value problem. The simplest model exhibiting such behaviour is the time-dependent Cauchy-Riemann system:

$$(5)\qquad \begin{cases} \partial_t u - \partial_x u + \partial_y v &= 0 \\ \partial_t v + \partial_x v + \partial_y u &= 0, \end{cases}$$

which becomes purely elliptic after an implicit time-discretization. This system therefore retains some of the difficulties involved in the computation of subsonic flows.

3. A well-posed boundary value problem

Let Ω be a domain of $\mathbb{R}^d$ with a smooth boundary $\partial\Omega$. If $n = (n_1, \ldots, n_d)$ is the outward normal vector at $x \in \partial\Omega$, we denote by $A_n = \sum A^\alpha n_\alpha$ the matrix of the flux in the direction n. This matrix is real and symmetric, and therefore admits a complete set of real eigenvectors. We define $A_n^+ = P^{-1}\Lambda^+ P$ with $\Lambda^+ = \text{diag}(\max\{\lambda_i, 0\})$. Similarly, $A_n^- = P^{-1}\Lambda^- P$ with $\Lambda^- = \text{diag}(\min\{\lambda_i, 0\})$, so that $A_n = A_n^+ + A_n^-$.

Next, we introduce a minimal rank positive-negative decomposition of the matrix A_n. Let A_n^{pos}(respectively A_n^{neg}) be a symmetric positive (resp. negative) matrix such that $A_n = A_n^{pos} + A_n^{neg}$ and $\text{rank}(A_n^{pos}) = \text{rank}(A_n^+)$, $\text{rank}(A_n^{neg}) = \text{rank}(A_n^-)$. The simplest choice is of course $A_n^{pos} = A_n^+$ and $A_n^{neg} = A_n^-$: it corresponds to a decomposition with local characteristic variables, which is also the first order absorbing boundary condition (cf. [5]). In the scalar case ($p = 1$), $A_n^{neg} = A_n^-$ is the only possible choice.

With these notations, we can define a dissipative boundary condition of the form:

$$(6)\qquad A_n^{neg} u = A_n^{neg} g \qquad \text{on } \partial\Omega.$$

In the scalar case, this condition amounts to prescribing the boundary data only on the inflow boundary.

With some regularity and positivity assumptions, one can prove the following theorem:

THEOREM 1. *Let $f \in L^2(\Omega)^p$ and g such that $\int_{\partial\Omega} A_n^{neg} g \cdot g < \infty$ be given. There exists a unique solution $u \in L^2(\Omega)^p$, with $\sum A^\alpha \partial_\alpha u \in L^2(\Omega)^p$, such that*

$$(7)\qquad \begin{cases} A^0 u + \sum A^\alpha \partial_\alpha u &= f \quad \text{in } \Omega \\ A_n^{neg} u &= A_n^{neg} g \quad \text{on } \partial\Omega. \end{cases}$$

The solution satisfies the following estimate:

$$(8)\qquad C_0 \|U_k\|_{L^2}^2 + \int_{\partial\Omega} A_n^{pos} U_k \cdot U_k \leq \frac{1}{C_0} \|f\|_{L^2}^2 - \int_{\partial\Omega} A_n^{neg} g \cdot g.$$

Proof: We refer to [10] and [1] for the proof in the case where $g = 0$. See also [6] for the scalar case. The general case ($g \neq 0$) is addressed in [3].

4. The Schwarz algorithm

For simplicity, we consider a non-overlapping decomposition of the domain Ω: $\bigcup_{1 \leq i \leq N} \Omega_i = \Omega$. We will denote by $\Gamma_{i,j} = \partial\Omega_i \cap \partial\Omega_j$ the interface between two subdomains, when it exists. If n is the normal vector to $\Gamma_{i,j}$, oriented from Ω_i to

Ω_j, we set $A_n = A_{i,j}$. Hence, $A_{i,j} = -A_{j,i}$. We can decompose the global problem in Ω in a set of local problems:

$$\begin{cases} A^0 u_i + \sum A^\alpha \partial_\alpha u_i &= f \quad \text{in } \Omega_i \\ A_n^{neg} u_i &= A_n^{neg} g \quad \text{on } \partial\Omega \cup \partial\Omega_i, \end{cases}$$

which we supplement with the transmission conditions:

$$A_{i,j}^{neg} u_i = A_{i,j}^{neg} u_j \quad A_{j,i}^{neg} u_i = A_{j,i}^{neg} u_j \quad \text{on } \Gamma_{i,j}.$$

We now describe the classical Schwarz algorithm for the solution of these transmission conditions. We make use a vector $U^k = (u_1^k, \dots, u_N^k)$ of local solutions. Let U^0 be given. If U^k is known, U^{k+1} is defined by:

$$\text{(9)} \qquad \begin{cases} A^0 u_i^{k+1} + \sum A^\alpha \partial_\alpha u_i^{k+1} &= f \quad \text{in } \Omega_i \\ A_n^{neg} u_i^{k+1} &= A_n^{neg} g \quad \text{on } \partial\Omega \cap \partial\Omega_i, \\ A_{i,j}^{neg} u_i^{k+1} &= A_{i,j}^{neg} u_j^k \quad \text{on } \Gamma_{i,j}. \end{cases}$$

Inequality (8) shows that the trace of the solution u_j^k satisfies

$$\sum_{\partial\Omega_j} A_n^{pos} u_j \cdot u_j < \infty.$$

The trace of u_j^k on $\Gamma_{i,j}$ can therefore be used as a boundary condition for u_i^{k+1}, which ensures that the algorithm is well defined.

The Schwarz method converges if each u_i^k tends to the restriction of u to Ω_i as k tends to infinity. More precisely, we can prove the following theorem:

THEOREM 2. *The algorithm (9) converges in the following sense:*

$$\|e_i^k\|_{L^2} \to 0, \quad \|\sum A^\alpha \partial_\alpha e_i^k\|_{L^2} \to 0,$$

where $e_i^k = u - u_i^k$ *is the error in subdomain* Ω_i.

Proof: The proof is similar to [**4**] (see also [**11**]). See [**3**] for more details.

5. Examples

5.1. Example 1: the scalar case. In the case of decomposition in successive slabs following the flow (see Fig. 1), it is easily seen that the Schwarz method converges in a finite number of steps. The Schwarz method is thus optimal in this case but the parallelization is useless. Indeed, the residual does not decrease until the last iteration: the sequential multiplicative algorithm would be as efficient in this case. This peculiar behaviour is due to the hyperbolic nature of the equations and would also occur for the linearized Euler equations in one dimension and for 2-D supersonic flows.

5.2. Example 2: the Cauchy-Riemann equations. Here the convergence is not optimal but the error is reduced at each iteration. In the case where the domain $\mathbb{R}^2$ is decomposed in two half-planes, the convergence of the Schwarz method can be investigated with a Fourier transform in the direction of the interface, see [**3**]. This computation shows that the Schwarz method behaves similarly for this first order elliptic system as for a usual scalar second order elliptic equation. It is possible to define an analogue of the Steklov-Poincare operator in this case, which is naturally non-local.

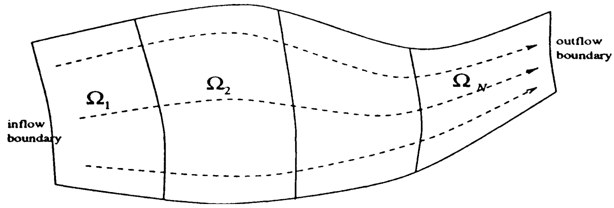

FIGURE 1. Decomposition of the domain for a transport equation. In this case, convergence is reached in exactly N steps, where N is the number of subdomains.

As mentionned in section 3, any decomposition $A = A^{neg} + A^{pos}$ with minimal rank leads to a dissipative boundary condition $A^{neg}u = A^{neg}g$. For the Cauchy-Riemann equations (5), there is a one-parameter family of such decompositions. In the case where $n = (1,0)$ for instance, we may take:

$$A^{neg} = \begin{pmatrix} -\cosh^2\varphi & -\sinh\varphi\cosh\varphi \\ -\sinh\varphi\cosh\varphi & -\sinh^2\varphi \end{pmatrix},$$

for any real number φ. The computation of the spectrum of the Schwarz method in this case shows that the convergence is optimal when $\varphi = 0$, i.e. when $A^{neg} = A^-$. This case corresponds to the first order absorbing boundary condition [**5**]. The optimality of this kind of transmission conditions has been first recognized by [**11**] in the context of convection-diffusion equations.

6. Numerical results with a first order implicit finite volume scheme

6.1. Finite volume discretization. We consider a triangulation of Ω. For simplicity, we will assume that $\Omega \subset \mathbb{R}^2$, but the extension to $\mathbb{R}^3$ is straightforward. If K is a cell of the triangulation, the set of its edges e will be denoted by ∂K. $|K|$ is the total area of the cell and $|e|$ the length of edge e.

We seek a piecewise constant approximation to (3), u_K being the cell-average of u in cell K. The finite volume scheme reads:

$$|K|A_0u_K + \sum_{e\in\partial K} |e|\,\Phi_e^K = |K|f_K. \tag{10}$$

In problems arising from a non-linear system of conservation laws, we use the implicit version of Roe's scheme [**13**], written in the usual non-conservative form:

$$\Phi_e^K(u_K,u_J) = A_n^-[u_J - u_K], \qquad \Phi_e^J(u_K,u_J) = A_n^+[u_J - u_K], \tag{11}$$

if e is the common edge between K and J, with a normal vector n oriented from K to J. The averaged Jacobian matrix A_n is computed from the preceding time-step and must satisfy Roe's condition:

$$A_n(u_K,u_J)[u_J - u_K] = [F_n(u_J) - F_n(u_K)].$$

When this condition holds, the implicit scheme (11) is conservative at steady-state.

Note that in the linear case with constant coefficients, (11) is equivalent to the classical first order upwind scheme:

$$\Phi_e^K(u_K,u_J) = -\Phi_e^J = A_n^-u_J + A_n^+u_K.$$

6.2. Boundary conditions. The boundary conditions defined by local characteristics decomposition, namely $A_n^- u = A_n^- g$ are naturaly discretized by:

$$\Phi_e = \Phi(u_K, g_e) = A_n^- [g_e - u_K],$$

if e is an edge of cell K lying on the boundary. For general dissipative boundary conditions of the type $A_n^{neg} u = A_n^{neg} g$, one can use a different scheme at the boundary: this modification is linked to the so-called preconditioning technique, see [**16, 3**] for more details. An alternative will be described in the sequel.

6.3. The Schwarz method. It is easily seen that the Schwarz method, discretized by a first order upwind scheme can be interpreted as a block-Jacobi solver for the global linear system. The local problems in each subdomain leads to a local sub-system, which can be solved with a LU factorization of the corresponding matrix block.

Alternatively, a substructuring approach is possible. In finite element applications, this approach consists in eliminating all interior unknowns and writing a condensed system involving only the unknowns at the interfaces. The resulting matrix is the so-called Schur complement. The dimension of the condensed linear system is much lower than that of the initial system. This is especially interesting with GMRES, as all intermediate vectors of the Krylov subspace have to be stored.

However, the unknowns at the interfaces do not appear explicitly in the finite volume discretization. Rather, they are reconstructed from the interior values. To bypass this difficulty, we propose to introduce redundant unknowns at the interface. Namely, if e is an edge lying on an interface, we define u_e by:

$$A_n^- u_e = A_n^- u_K, \qquad A_n^+ u_e = A_n^+ u_J. \tag{12}$$

The flux at the interface is thus:

$$\Phi_e^K = A_n^- [u_e - u_K], \qquad \Phi_e^J = A_n^+ [u_J - u_e].$$

Thanks to this formulation, we can solve all the interior unknowns u_K in terms of the redundant unknowns u_e.

With the latter formulation, one can easily implement transmission conditions of the type $A_n^{neg} u = A_n^{neg} g$: the definition of the interface-based unknowns simply becomes:

$$A_n^{neg} u_e = A_n^{neg} u_K, \qquad A_n^{pos} u_e = A_n^{pos} u_J.$$

This formulation has been used for the Cauchy-Riemann equations to verify the results of the preceding section.

7. Numerical results

7.1. Cauchy-Riemann equations. We first present some numerical results for the Cauchy-Riemann equations (5). The computationnal domain is the unit square $[0,1] \times [0,1]$, with homogeneous boundary conditions $A_n^- U = 0$ on the boundaries $x = 0$ et $x = 1$, and periodic boundary conditions in the y direction. The subdomains consist of parallel slabs $[l_i, l_{i+1}] \times [0,1]$ of constant width. Note that a "box" decomposition ($[l_i, l_{i+1}] \times [L_j, L_{j+1}]$) is possible and would lead to similar results.

The value of δt is 10^2, and the left hand side is $(\sin(4\pi x)\cos(4\pi y), 0)$. The meshing is 40×40. The stopping criterion for the iterative procedure is an overall residual lower than 10^{-10}. The corresponding number of iterations is given in Table 1. This result shows that the Schwarz method is indeed convergent. The linear

TABLE 1. Schwarz method for the Cauchy-Riemann equations.

Number of Subdomains	Iterations
2	61
4	63
5	65
8	70
10	76

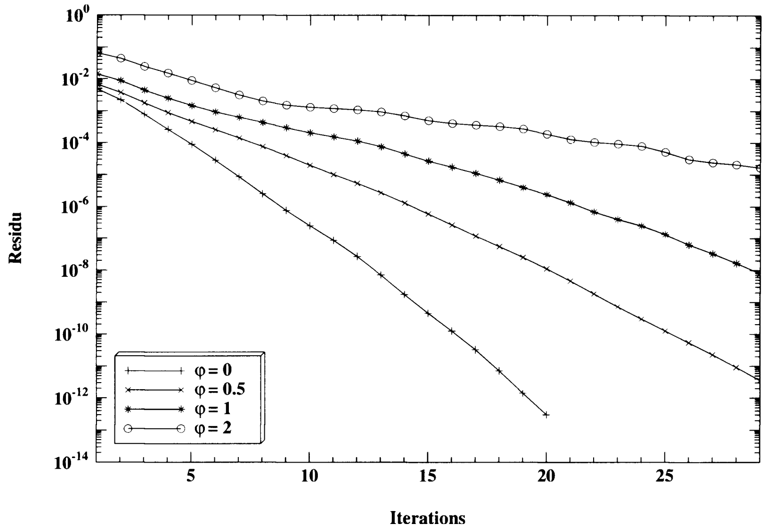

FIGURE 2. History of the residual, Schwarz method for the Cauchy-Riemann equations. The case $\varphi = 0$ corresponding to the absorbing boundary condition yields the best convergence.

growth with the number of subdomains is a usual feature of one-level Schwarz methods.

For practical applications, it is preferable to use the Schwarz method as a preconditioner for a Krylov subspace method.

7.2. Acceleration via GMRES. From now on, we use the Schwarz method as a preconditioner for GMRES [**14**]. With this approach, the number of iterations required to reach convergence for the Cauchy-Riemann system is typically of the order of 20 (see Fig. 2). This contrasts with the 60 iterations needed for the Schwarz method alone.

The main issue now becomes the condition number of the method. As explained in the preceding section, the transmission conditions have a significant impact on the behaviour of the method. Figure 2 shows the history of the residual for several transmission conditions of the type $A^{neg}u_i = A^{neg}u_j$ with two subdomains. The best behaviour is clearly obtained with $\varphi = 0$, i.e. with the first order absorbing boundary condition.

7.3. Saint-Venant's equations. We now apply the preceding ideas to the computation of smooth, non-linear, steady-state flow. The proposed problem is a two dimensional shallow water flow over a bump.

TABLE 2. Schwarz method, Shallow water problem: iterations for several decompositions.

Decomposition	Iterations
1×2	23
1×3	29
1×4	35
2×2	25
2×3	38
3×3	37

The Saint-Venant equations read:

$$\begin{aligned} \partial_t h + \partial_x(hu) + \partial_y(hv) &= 0 \\ \partial_t hu + \partial_x(hu^2 + gh^2/2) + \partial_y(huv) &= -gh\partial_x q \\ \partial_t hv + \partial_x(huv) + \partial_y(hv^2 + gh^2/2) &= -gh\partial_y q. \end{aligned}$$

Here h is the water depth, $(hu, hv)^T$ isthe momentum vector, and $q(x, y)$ is the given height of the sea bottom. The equation of the free surface is therefore $h + q$. The construction of Roe's matrix for this problem is classical. The treatment of the source term follows the work of [**2**]: at the discrete level, we simply end up with a right hand side in the linear system.

The test case is a subsonic flow over a circular bump, with a Froude number of approximately .42. The computational domain is the unit square and a first order absorbing boundary condition is imposed at the boundary. The mesh size is 60×60 and the time step is such that $\delta t/\delta x = 10^4$. A steady-state solution is reached within 5 time steps.

The Schwarz method is used with GMRES. Table 2 gives the number of iterations required to decrease the residual by a factor of 10^{-10} for several decompositions of the computational domain. The decomposition referred to as "$i \times j$" consists of i subdomains in the x direction and j subdomains in the y direction. For instance, 1×4 and 2×2 refer to a decomposition in 4 slices or boxes respectively.

These results show the applicability of the method to the solution of compressible flows. The growth of the iterations with the number of subdomains seems reasonable.

8. Conclusion

We have considered linear systems of first order equations. A Schwarz iterative method has been defined and the convergence of the algorithm has been studied.

Numerically, a first order finite volume discretization of the equations has been considered. A Schur complement formulation has been proposed. The influence of the number of subdomains and of the transmission condition has been investigated.

Finally, we have shown the applicability of the algorithm to a non-linear flow computation. We have used the linearly implicit version of Roe's scheme for a two-dimensional shallow water problem.

A better preconditioning might however be necessary for problems in 3-D or involving a great number of subdomains. For this purpose, a more complete numerical analysis of domain decomposition methods for first order systems is needed.

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SERMA - LETR, BAT 470, COMMISSARIAT À L'ÉNERGIE ATOMIQUE, 91191 GIF-SUR-YVETTE FRANCE

E-mail address: Sebastien.Clerc@cea.fr