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# An Iterative Substructuring Method for Elliptic Mortar Finite Element Problems with Discontinuous Coefficients

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## 1. Introduction

In this paper, we discuss a domain decomposition method for solving linear systems of algebraic equations arising from the discretization of elliptic problem in the 3-D by the mortar element method, see [**4, 5**] and the literature given theirin. The elliptic problem is second-order with piecewise constant coefficients and the Dirichlet boundary condition. Using the framework of the mortar method, the problem is approximated by a finite element method with piecewise linear functions on nonmatching meshes.

Our domain decomposition method is an iterative substructuring one with a new coarse space. It is described as an additive Schwarz method (ASM) using the general framework of ASMs; see [**11, 10**]. The method is applied to the Schur complement of our discrete problem, i.e. we assume that interior variables of all subregions are first eliminated using a direct method.

In this paper, the method is considered for the mortar elements in the geometrically conforming case, i.e. the original region $\Omega$, which for simplicity of presentation is a polygonal region, is partitioned into polygonal subregions (substructures) $\Omega_i$ that form a coarse finite element triangulation.

The described ASM uses a coarse space spanned by special functions associated with the substructures $\Omega_i$. The remaining spaces are local and are associated with the mortar faces of the substructures and the nodal points of the wire basket of the substructures. The problems in these subspaces are independent so the method is well suited for parallel computations. The described method is almost optimal and its rate of convergence is independent of the jumps of coefficients.

The described method is a generalization of the method presented in [**8**] to second order elliptic problems with discontinuous coefficients. Other iterative substructuring methods for the mortar finite elements have been described and analyzed in several papers, see [**1, 2, 6, 12, 13**] and the literature given theirin. Most of them are devoted to elliptic problems with regular coefficients and the 2-D case.

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The outline of the paper is as follows: In Section 2, the discrete problem obtained from the mortar element technique in the geometrically conforming case is described. In Section 3, the method is described in terms of an ASM and Theorem 1 is formulated as the main result of the paper. A proof of this theorem is given in Section 5 after that certain auxiliary results, which are needed for that proof, are given in Section 4.

## 2. Mortar discrete problem

We solve the following differential problem: Find $u^* \in H_0^1(\Omega)$ such that

$$a(u^*, v) = f(v), \quad v \in H_0^1(\Omega), \tag{1}$$

where

$$a(u,v) = \sum_{i=1}^{N} \rho_i (\nabla u, \nabla v)_{L^2(\Omega_i)}, \quad f(v) = (f, v)_{L^2(\Omega)},$$

$\bar{\Omega} = \cup_{i=1}^{N} \bar{\Omega}_i$ and $\rho_i$ is a positive constant.

Here $\Omega$ is a polygonal region in the 3-D and the $\Omega_i$ are polygonal subregions of diameter $H_i$. They form a coarse triangulation with a mesh parameter $H = \max_i H_i$. In each $\Omega_i$ triangulation is introduced with triangular elements $e_j^{(i)}$ and a parameter $h_i = \max_j h_i^{(j)}$ where $h_i^{(j)}$ is a diameter of $e_i^{(j)}$. The resulting triangulation of $\Omega$ can be nonmatching. We assume that the coarse triangulation and the $h_i$-triangulation in each $\Omega_i$ are shape-regular in the sense of [**7**]. Let $X_i(\Omega_i)$ be the finite element space of piecewise linear continuous functions defined on the triangulation of $\Omega_i$ and vanishing on $\partial\Omega_i \cap \partial\Omega$, and let

$$X^h(\Omega) = X_1(\Omega_1) \times \cdots \times X_N(\Omega_N).$$

To define the mortar finite element method, we introduce some notation and spaces. Let

$$\Gamma = (\cup_i \partial\Omega_i) \backslash \partial\Omega$$

and let $F_{ij}$ and $E_{ij}$ denote the faces and edges of $\Omega_i$. The union of $\bar{E}_{ij}$ forms the wire basket $W_i$ of $\Omega_i$. We now select open faces $\gamma_m$ of $\Gamma$, called mortars (masters), such that

$$\bar{\Gamma} = \cup_m \bar{\gamma}_m \quad \text{and} \quad \gamma_m \cap \gamma_n = \emptyset \quad \text{if} \quad m \neq n.$$

We denote the face of $\Omega_i$ by $\gamma_{m(i)}$. Let $\gamma_{m(i)} = F_{ij}$ be a face common to $\Omega_i$ and $\Omega_j$. $F_{ij}$ as a face of $\Omega_j$ is denoted by $\delta_{m(j)}$ and it is called nonmortar (slave). The rule for selecting $\gamma_{m(i)} = F_{ij}$ as mortar is that $\rho_i \geq \rho_j$. Let $W^{h_i}(F_{ij})$ be the restriction of $X_i(\Omega_i)$ to $F_{ij}$. Note that on $F_{ij} = \gamma_{m(i)} = \delta_{m(j)}$ we have two triangulation and two different face spaces $W^{h_i}(\gamma_{m(i)})$ and $W^{h_j}(\delta_{m(j)})$.

Let $M^{h_j}(\delta_{m(j)})$ denote a subspace of $W^{h_j}(\delta_{m(j)})$ defined as follows: The values at interior nodes of $\delta_{m(j)}$ are arbitrary, while those at nodes on $\partial\delta_{m(j)}$ are a convex combination values at interior neighboring nodes:

$$v(x_k) = \sum_{i=1}^{n_k} \alpha_i v(x_{i(k)}) \varphi_{i(k)}, \quad \sum_{i=1}^{n_k} \alpha_i = 1.$$

Here $\alpha_i \geq 0$, $x_k \in \partial\delta_{m(j)}$ and the sum is taken over interior nodal points $x_{i(k)}$ of $\delta_{m(j)}$ such that an interval $(x_k, x_{i(k)})$ is an edge of the triangulation and their number is equal to $n_k$; $\varphi_{i(k)}$ is a nodal basis function associated with $x_{i(k)}$, for details see [**4**].

We say that $u_{i(m)}$ and $u_{j(m)}$, the restrictions of $u_i \in X_i(\Omega_i)$ and $u_j \in X_j(\Omega_j)$ to $\delta_m$, a face common to $\Omega_i$ and $\Omega_j$, satisfy the mortar condition if

$$\int_{\delta_m} (u_{i(m)} - u_{j(m)}) w ds = 0, \quad w \in M^{h_j}(\delta_m). \tag{2}$$

This condition can be rewritten as follows: Let $\Pi_m(u_{i(m)}, v_{j(m)})$ denote a projection from $L^2(\delta_m)$ on $W^{h_j}(\delta_m)$ defined by

$$\int_{\delta_m} \Pi_m(u_{i(m)}, v_{j(m)}) w ds = \int_{\delta_m} u_{i(m)} w ds, \quad w \in M^{h_j}(\delta_m) \tag{3}$$

and

$$\Pi_m(u_{i(m)}, v_{j(m)})_{|\partial\delta_m} = v_{j(m)}. \tag{4}$$

Thus $u_{j(m)} = \Pi_m(u_{i(m)}, v_{j(m)})$ if $v_{j(m)} = u_{j(m)}$ on $\partial\delta_m$.

By $V^h$ we denote a space of $v \in X^h$ which satisfy the mortar condition for each $\delta_m \subset \Gamma$. The discrete problem for (1) in $V^h$ is defined as follows: Find $u_h^* \in V^h$ such that

$$a(u_h^*, v_h) = f(v_h), \quad v_h \in V^h, \tag{5}$$

where

$$a(u_h, v_h) = \sum_{i=1}^{N} a_i(u_{ih}, v_{ih}) = \sum_{i=1}^{N} \rho_i (\nabla u_{ih}, \nabla v_{ih})_{L^2(\Omega_i)}$$

and $v_h = \{v_{ih}\}_{i=1}^N \in V^h$. $V^h$ is a Hilbert space with an inner product defined by $a(u, v)$. This problem has an unique solution and an estimate of the error is known, see [**4**].

We now give a matrix form of (5). Let

$$V^h = span\{\Phi_k\}$$

where $\{\Phi_k\}$ are mortar basis functions associated with interior nodal points of the substructures $\Omega_i$ and the mortars $\gamma_{m(i)}$, and with nodal points of $\partial\gamma_{m(i)}$ and $\partial\delta_{m(i)}$, except those on $\partial\Omega$. These sets of nodal points are denoted by adding the index $h$. The functions $\Phi_k$ are defined as follows. For $x_k \in \Omega_{ih}$, $\Phi_k(x) = \varphi_k(x)$, the standard nodal basis function associated with $x_k$. For $x_k \in \gamma_{m(i)h}$, $\Phi_k = \varphi_k$ on $\gamma_{m(i)} \subset \partial\Omega_i$ and $\Pi_m(\varphi_k, 0)$ on $\delta_{m(j)} = \gamma_{m(i)} \subset \partial\Omega_j$, see (3) and (4), and $\Phi_k = 0$ at the remaining nodal points. If $x_k$ is a nodal point common to two or more boundaries of mortars $\gamma_{m(i)}$, then $\Phi_k(x) = \varphi_k$ on these mortars and extended on the nonmortars $\delta_{m(j)}$ by $\Pi_m(\varphi_k, 0)$, and set to zero at the remaining nodal points. Let $x_k$ be a common nodal point to two or more boundaries of nonmortars $\delta_{m(j)}$, then $\Phi_k = \Pi_m(0, \varphi_k)$ on these nonmortars and zero at the remaining nodal points. In the case when $x_k$ is a common nodal point to boundaries of mortars and nonmortars faces, $\Phi_k$ is defined on these faces as above. Note that there are no basis functions associated with interior nodal points of the nonmortar faces.

Using these basis functions, the problem (5) can be rewritten as

$$A\mathbf{u}_\mathbf{h}^* = \mathbf{f} \tag{6}$$

where $\mathbf{u}_\mathbf{h}^*$ is a vector of nodal values of $u_h^*$. The matrix is symmetric and positive definite, and its condition number is similar to that of a conforming finite element method provided that the $h_i$ are all of the same order.

## 3. The additive Schwarz method

In this section, we describe an iterative substructuring method in terms of an additive Schwarz method for solving (5). It will be done for the Schur complement system. For that we first eliminate all interior unknowns of $\Omega_i$ using for $u_i \in X_i(\Omega_i)$ the decomposition $u_i = Pu_i + Hu_i$. Here and below, we drop the index $h$ for functions. $Hu_i$ is discrete harmonic in $\Omega_i$ in the sense of $(\nabla u_i, \nabla v_i)_{L^2(\Omega_i)}$ with $Hu_i = u_i$ on $\partial\Omega_i$. We obtain

$$s(u^*, v) = f(v), \quad v \in V^h \tag{7}$$

where from now on $V^h$ denote the space of piecewise discrete harmonic functions and

$$s(u, v) = a(u, v), \quad u, v \in V^h.$$

An additive Schwarz method for (7) is designed and analyzed using the general ASM framework, see [**11**], [**10**]. Thus, the method is designed in terms of a decomposition of $V^h$, certain bilinear forms given on these subspaces, and the projections onto these subspaces in the sense of these bilinear forms.

The decomposition of $V^h$ is taken as

$$V^h(\Omega) = V_0(\Omega) + \sum_{\gamma_m \subset \Gamma} V_m^{(F)}(\Omega) + \sum_{i=1}^{N} \sum_{x_k \in W_{ih}} V_k^{(W_i)}(\Omega). \tag{8}$$

The space $V_m^{(F)}(\Omega)$ is a subspace of $V^h$ associated with the master face $\gamma_m$. Any function of $V_m^{(F)}$ differs from zero only on $\gamma_m$ and $\delta_m$. $W_{ih}$ is the set of nodal points of $W_i$ and $V_k^{(W_i)}$ is an one-dimensional space associated with $x_k \in W_{ih}$ and spanned by $\Phi_k$.

The coarse space $V_0$ is spanned by discrete harmonic functions $\Psi_i$ defined as follows. Let the set of substructures $\Omega_i$ be partitioned into two sets $N_I$ and $N_B$. The boundary of a substructure in $N_B$ intersects $\partial\Omega$ in at least one point, while those of the interior set $N_I$, do not. For simplicity of presentation, we assume that $\partial\Omega_i \cap \partial\Omega$ for $i \in N_B$ are faces. The general case when $\partial\Omega_i \cap \partial\Omega$ for $i \in N_B$ are also edges and vertices, can be analyzed as in [**10**]. The function $\Psi_i$ is associated with $\Omega_i$ for $i \in N_I$ and it is defined by its values on boundaries of substructures as follows: $\Psi_i = 1$ on $\bar{\gamma}_{m(i)} \subset \partial\Omega_i$, the mortar faces of $\Omega_i$, and $\Psi_i = \Pi_m(1, 0)$ on $\delta_{m(j)} = \gamma_{m(i)}$, the face common to $\Omega_i$ and $\Omega_j$; see (3) and (4). On the nonmortar faces $\bar{\delta}_{m(i)} \subset \partial\Omega_i$, $\Psi_i = \Pi_m(0, 1)$. It is zero on the remaining mortar and nonmortar faces. We set

$$V_0 = span\{\Psi_i\}_{i \in N_I}. \tag{9}$$

Let us now introduce bilinear forms defined on the introduced spaces. $b_m^{(F)}$ associated with $V_m^{(F)} \times V_m^{(F)} \to R$ is of the form

$$b_m^{(F)}(u_{m(i)}, v_{m(i)}) = \rho_i(\nabla u_{m(i)}, \nabla v_{m(i)})_{L^2(\Omega_i)}, \tag{10}$$

where $u_{m(i)}$ is the discrete harmonic function in $\Omega_i$ with data $u_{m(i)}$ on the mortar face $\gamma_{m(i)}$ of $\Omega_i$, which is common to $\Omega_j$, and zero on the remaining faces of $\Omega_i$.

We set $b_k^{(W_i)} : V_k^{(W_i)} \times V_k^{(W_i)} \to R$ equal to $a(u, v)$.

A bilinear form $b_0(u,v) : V_0 \times V_0 \to R$ is of the form

$$b_0(u,v) = \sum_{i \in N_I} (1 + \log \frac{H_i}{h_i}) H_i \rho_i \sum_{\delta_{m(i)} \subset \partial\Omega_i} (\alpha_j \bar{u}_j - \bar{u}_i)(\alpha_j \bar{v}_j - \bar{v}_i) +$$

$$+ \sum_{i \in N_B} (1 + \log \frac{H_i}{h_i}) H_i \rho_i \sum_{\delta_{m(i)} \subset \partial\Omega_i} \bar{u}_j \bar{v}_j. \tag{11}$$

Here $\delta_{m(i)} = \gamma_{m(j)}$ is the face common to $\Omega_i$ and $\Omega_j$, $\alpha_j = 0$ if $\delta_{m(i)} = \gamma_{m(j)} \subset \partial\Omega_j$ and $j \in N_B$, otherwise $\alpha_j = 1$,

$$u = \sum_{i \in N_I} \bar{u}_i \Psi_i, \quad v = \sum_{i \in N_I} \bar{v}_i \Psi_i \tag{12}$$

and $\bar{u}_i$ is the discrete average value of $u_i$ over $\partial\Omega_{ih}$, i.e.

$$\bar{u}_i = (\sum_{x \in \partial\Omega_{ih}} u_i(x))/m_i, \tag{13}$$

and $m_i$ is the number of nodal points of $\partial\Omega_{ih}$.

Let us now introduce operators $T_m^{(F)}$, $T_k^{(W_i)}$ and $T_0$ by the bilinear forms $b_m^{(F)}$, $b_k^{(W_i)}$ and $b_0$, respectively, in the standard way. For example, $T_m^{(F)} : V^h \to V_m^{(F)}$ is the solution of

$$b_m^{(F)}(T_m^{(F)} u, v) = a(u,v), \quad v \in V_m^{(F)}. \tag{14}$$

Let

$$T = T_0 + \sum_{\gamma_m \subset \Gamma} T_m^{(F)} + \sum_{i=1}^{N} \sum_{x_k \in W_{ih}} T_k^{(W_i)}.$$

The problem (5) is replaced by

$$Tu^* = g \tag{15}$$

with the appropriate right-hand side.

THEOREM 1. *For all* $u \in V^h$

$$C_0 (1 + \log \frac{H}{h})^{-2} a(u,u) \leq a(Tu,u) \leq C_1 a(u,u) \tag{16}$$

*where* $C_i$ *are positive constants independent of* $H = \max_i H_i$, $h = \min_i h_i$ *and the jumps of* $\rho_i$.

## 4. Auxiliary results

In this section, we formulate some auxiliary results which we need to prove Theorem 1.

Let for $u \in V^h$, $u_0 \in V_0$ be defined as

$$u_0 = \sum_{i \in N_I} \bar{u}_i \Psi_i \tag{17}$$

where the $\bar{u}_i$ are defined in (13).

LEMMA 2. *For* $u_0 \in V_0$ *defined in (17)*

$$a(u_0,u_0) \leq C b_0(u_0,u_0) \tag{18}$$

*where* $b_0(.,.)$ *is given in (11) and* $C$ *is a positive constant independent of the* $H_i$, $h_i$ *and the jumps of* $\rho_i$.

PROOF. Note that $u_0$ on $\partial\Omega_i$, $i \in N_I$, is of the form

$$u_0 = \bar{u}_i \Psi_i + \sum_j \bar{u}_j \Psi_j \tag{19}$$

where the sum is taken over the nonmortars $\delta_{m(i)} = \gamma_{m(j)}$ of $\Omega_i$ and $\gamma_{m(j)}$ is the face common to $\Omega_i$ and $\Omega_j$. In this formula $\Psi_j = 0$ if $j \in N_B$. Let us first discuss the case when all $j \in N_I$ in (19). Note that $\Psi_i + \sum_j \Psi_j = 1$ on $\partial\Omega_i$. Using this, we have

$$\rho_i |u_0|^2_{H^1(\Omega_i)} = \rho_i |u_0 - \bar{u}_i|^2_{H^1(\Omega_i)} \leq C \sum_{\delta_{m(i)} \subset \partial\Omega_i} \rho_i (\bar{u}_j - \bar{u}_i)^2 ||\Psi_j||^2_{H^{\frac{1}{2}}_{00}(\delta_{m(i)})}.$$

It can be shown that

$$||\Psi_j||^2_{H^{\frac{1}{2}}_{00}(\delta_{m(i)})} \leq C H_i (1 + \log \frac{H_i}{h_i}). \tag{20}$$

For that note that $\Psi_j = \Pi_m(1,0)$ on $\delta_{m(i)}$ and use the properties of $\Pi_m$; for details see the proof of Lemma 4.5 in [**8**]. Thus

$$\rho_i |u_0|^2_{H^1(\Omega_i)} \leq C H_i \sum_{\delta_{m(i)} \subset \partial\Omega_i} \rho_i (1 + \log \frac{H_i}{h_i})(\bar{u}_j - \bar{u}_i)^2. \tag{21}$$

For $i \in N_I$ with $j \in N_B$, we have

$$\rho_i |u_0|^2_{H^1(\Omega_i)} \leq C H_i \sum_{\delta_{m(i)} \subset \partial\Omega_i} \rho_i (1 + \log \frac{H_i}{h_i})(\alpha_j \bar{u}_j - \bar{u}_i)^2$$

where $\alpha_j = 0$ if $\delta_{m(i)} = \gamma_{m(j)} \subset \partial\Omega_j$ and $j \in N_B$, otherwise $\alpha_j = 1$. For $i \in N_B$

$$\rho_i |u_0|^2_{H^1(\Omega_i)} \leq C H_i \sum_{\delta_{m(i)} \subset \partial\Omega_i} \rho_i (1 + \log \frac{H_i}{h_i}) \bar{u}_j^2$$

Summing these inequalities with respect to $i$, we get

$$a(u_0, u_0) \leq C \{ \sum_{i \in N_I} (1 + \log \frac{H_i}{h_i}) H_i \rho_i \sum_{\delta_{m(i)} \subset \partial\Omega_i} (\alpha_j \bar{u}_j - \bar{u}_i)^2 + \\ + \sum_{i \in N_B} (1 + \log \frac{H_i}{h_i}) H_i \rho_i \sum_{\delta_{m(i)} \subset \partial\Omega_i} \bar{u}_j^2 \},$$

which proves (18). □

LEMMA 3. *Let $\gamma_{m(i)} = \delta_{m(j)}$ be the face common to $\Omega_i$ and $\Omega_j$, and let $u_{i(m)}$ and $u_{j(m)}$ be the restrictions of $u_i \in X_i(\Omega_i)$ and $u_j \in X_j(\Omega_j)$ to $\gamma_{m(i)}$ and $\delta_{m(j)}$, respectively. Let $u_{i(m)}$ and $u_{j(m)}$ satisfy the mortar condition (2) on $\delta_{m(j)}$. If $u_{i(m)}$ and $u_{j(m)}$ vanish on $\partial\gamma_{m(i)}$ and $\partial\delta_{m(j)}$, respectively, then*

$$||u_{j(m)}||^2_{H^{\frac{1}{2}}_{00}(\delta_{m(j)})} \leq C ||u_{i(m)}||^2_{H^{\frac{1}{2}}_{00}(\gamma_{m(i)})}$$

*where $C$ is independent of $h_i$ and $h_j$.*

This lemma follows from Lemma 1 in [**3**]. A short proof for our case is given in Lemma 4.2 of [**8**].

LEMMA 4. *Let $\Phi_k$ be a function defined in Section 2 and associated with a nodal point $x_k \in W_i \subset \partial\Omega_i$. Then*

$$a(\Phi_k, \Phi_k) \leq Ch_i\rho_i \sum_{\gamma_{m(i)} \subset \partial\Omega_i} (1 + \log \frac{h_i}{h_j})$$

*where $C$ is independent of $h_i$ and $\rho_i$, and $\gamma_{m(i)} = \delta_{m(j)}$.*

The proof of this lemma differs slightly from that of Lemma 4.3 in [**8**], therefore it is omitted here.

## 5. Proof of Theorem 1

Using the general theorem of ASMs, we need to check three key assumptions; see [**11**] and [**10**].

*Assumption (iii)* For each $x \in \Omega$ the number of substructures with common $x$ is fixed, therefore $\rho(\varepsilon) \leq C$.

*Assumption (ii)* Of course $\omega = 1$ for $b_k^{(W_i)}(u, u)$, $u \in V_k^{(W_i)}$. The estimate

$$a(u, u) \leq \omega b_0(u, u), \quad u \in V_0$$

follows from Lemma 2 with $\omega = C$.

We now show that for $u \in V_m^{(F)}$, see (10),

$$a(u, u) \leq Cb_m^{(F)}(u, u). \tag{22}$$

Let $\gamma_{i(m)} = \delta_{j(m)}$ be the mortar and nonmortar sides of $\Omega_i$ and $\Omega_j$, respectively. For $u \in V_m^{(F)}$, we have

$$a(u, u) = a_i(u_i, u_i) + a_j(u_j, u_j) \leq C\big(\rho_i||u_i||^2_{H_{00}^{\frac{1}{2}}(\gamma_{i(m)})} + \rho_j||u_j||^2_{H_{00}^{\frac{1}{2}}(\delta_{j(m)})}\big).$$

Using now Lemma 3 and the fact that $\rho_i \geq \rho_j$ since $\gamma_{m(i)}$ is the mortar, we get (22), i.e. $\omega = C$.

*Assumption (i)* We show that for $u \in V^h$, there exists a decomposition

$$u = u_0 + \sum_{\gamma_m \subset \Gamma} u_m^{(F)} + \sum_{i=1}^{N} \sum_{x_k \in W_{ih}} u_k^{(W_i)}, \tag{23}$$

where $u_0 \in V_0$, $u_m^{(F)} \in V_m^{(F)}$ and $u_k^{(W_i)} \in V_k^{(W_i)}$, such that

$$b_0(u_0, u_0) + \sum_{\gamma_m \subset \Gamma} b_m^{(F)}(u_m^{(F)}, u_m^{(F)}) + \sum_{i=1}^{N} \sum_{x_k \in W_{ih}} b_k^{(W_i)}(u_k^{(W_i)}, u_k^{(W_i)})$$
$$\leq C(1 + \log \frac{H}{h})^2 a(u, u). \tag{24}$$

Let $u_0$ be defined by (17), and let $w_i$ be the restriction of $w = u - u_0$ to $\bar{\Omega}_i$. It is decomposed on $\partial\Omega_i$ as

$$w_i = \sum_{F_{ij} \subset \partial\Omega_i} w_i^{(F_{ij})} + w_i^{(W_i)}, \quad w_i^{(W_i)} = \sum_{x_k \in W_{ih}} w_i(x_k)\Phi_k \tag{25}$$

where $w_i^{(F_{ij})}$ is the restriction of $w_i - w_i^{(W_i)}$ to $F_{ij}$, the face of $\Omega_i$, and zero on $\partial\Omega_i \backslash F_{ij}$.

To define $u_m^{(F)}$, let $F_{ij} = \gamma_{m(i)} = \delta_{m(j)}$ be a face common to $\Omega_i$ and $\Omega_j$. We set

$$u_m^{(F)} = \{w_i^{(F_{ij})} \text{ on } \partial\Omega_i \text{ and } w_j^{(F_{ij})} \text{ on } \partial\Omega_j\}$$

and set it to zero at the remaining nodal points of $\Gamma$. The function $u_k^{(W_i)}$ is defined as

$$u_k^{(W_i)}(x) = w_i(x_k)\Phi_k(x). \tag{26}$$

It is easy to see that these functions satisfy (23).

To prove (24), we first show that

$$b_0(u_0, u_0) \le C(1 + \log\frac{H}{h})a(u,u). \tag{27}$$

Note that, see (11), for $\delta_{m(i)} \subset \delta\Omega_i$, $i \in N_I$ with $j \in N_I$ when $\delta_{m(i)} = F_{ij}$ is a face common to $\Omega_i$ and $\Omega_j$,

$$H_i\rho_i(\bar{u}_j - \bar{u}_i)^2 \le CH_i^{-1}\{\rho_i||u_i||^2_{L^2(\partial\Omega_i)} + \rho_j||u_j||^2_{L^2(\partial\Omega_j)}\}.$$

Using the fact that the average values of $u_j$ and $u_i$ over $\delta_{m(i)} = \gamma_{m(j)} = F_{ij}$ are equal to each other, and using the Poincare inequality, we get

$$H_i\rho_i(\bar{u}_j - \bar{u}_i)^2 \le C\{\rho_i|u_i|^2_{H^1(\Omega_i)} + \rho_j|u_j|^2_{H^1(\Omega_j)}\}.$$

For $i \in N_I$ with $j \in N_B$ we have similar estimates:

$$H_i\rho_i(\alpha_j\bar{u}_j - \bar{u}_i)^2 \le C\{\rho_i|u_i|^2_{H^1(\Omega_i)} + \rho_j|u_j|^2_{H^1(\Omega_j)}\}.$$

Here we have used the Friedrichs inequality in $\Omega_j$. Thus

$$\sum_{i\in N_I}\sum_{\delta_{m(i)}\subset\partial\Omega_i} H_i\rho_i(\alpha_j\bar{u}_j - \bar{u}_i)^2 \le Ca(u,u). \tag{28}$$

In the similar way it is shown that for $i \in N_B$

$$H_i\rho_i\bar{u}_j^2 \le C\{\rho_i|u_i|^2_{H^1(\Omega_i)} + \rho_j|u_j|^2_{H^1(\Omega_j)}\}.$$

Summing this with respect to $i \in N_B$ and adding the resulting inequality to (28), we get (27).

Let us now consider the estimate for $u_m^{(F)} \in V_M^{(F)}$ when $\gamma_{m(i)} = \delta_{m(j)} = F_{ij}$, the face common to $\Omega_i$ and $\Omega_j$. We have, see (10),

$$b_m^{(F)}(u_m^{(F)}, u_m^{(F)}) \le C\rho_i||w_i^{(F_{ij})}||^2_{H_{00}^{\frac{1}{2}}(\gamma_{m(i)})}.$$

Note that on $F_{ij} = \gamma_{m(i)}$

$$w_i^{(F_{ij})} = I_{h_i}(\theta_{F_{ij}}u_i) - I_{h_i}(\theta_{F_{ij}}u_0)$$

where $\theta_{F_{ij}} = 1$ at interior nodal points of the $h_i$-triangulation of $F_{ij}$ and zero on $\partial F_{ij}$, and $I_{h_i}$ is the interpolant. Using Lemma 4.5 from [**9**], we have

$$||I_{h_i}(\theta_{F_{ij}}u_i)||^2_{H_{00}^{\frac{1}{2}}(F_{ij})} \le C(1 + \log\frac{H_i}{h_i})^2||u_i||^2_{H^1(\Omega_i)}.$$

To estimate the second term, note that $u_0 = \bar{u}_i\Psi_i = \bar{u}_i$ on $\bar{F}_{ij}$ since it is the mortar. Using Lemma 4.4 from [**9**], we get

$$||I_h(\theta_{F_{ij}}u_0)||^2_{H_{00}^{\frac{1}{2}}(F_{ij})} = (\bar{u}_i)^2||I_{h_i}\theta_{F_{ij}}||^2_{H_{00}^{\frac{1}{2}}(F_{ij})} \le$$
$$\le CH_i^{-1}(1 + \log\frac{H_i}{h_i})||u_i||^2_{L^2(\partial\Omega_i)}.$$

Thus

$$||w_i^{(F_{ij})}||^2_{H_{00}^{\frac{1}{2}}(\gamma_{m(i)})} \le C\{(1 + \log\frac{H_i}{h_i})^2||u_i||^2_{H^1(\Omega_i)} + H_i^{-1}(1 + \log\frac{H_i}{h_i})||u_i||^2_{L^2(\partial\Omega_i)}\}.$$

Using now a simple trace theorem and the Poincare inequality, we have

$$||w_i^{(F_{ij})}||^2_{H_{00}^{\frac{1}{2}}(\gamma_{m(i)})} \le C(1+\log\frac{H_i}{h_i})^2|u_i|^2_{H^1(\Omega_i)}.$$

Multiplying this by $\rho_i$ and summing with respect to $\gamma_m$, we get

$$\sum_{\gamma_m\subset\Gamma} b_m^{(F)}(u_m^{(F)},u_m^{(F)}) \le C(1+\log\frac{H}{h})^2 a(u,u). \tag{29}$$

We now prove that

$$\sum_{i=1}^{N}\sum_{x_k\in W_{ih}} b_k^{(W_i)}(u_k^{(W_i)},u_k^{(W_i)}) \le C(1+\log\frac{H}{h})^2 a(u,u). \tag{30}$$

We first note that by (26) and Lemma 4

$$b_k^{(W_i)}(u_k^{(W_i)},u_k^{(W_i)}) \le Cw_i^2(x_k)a(\Phi_k,\Phi_k) \le C\rho_i h_i(1+\log\frac{H_i}{h_i})w_i^2(x_k).$$

Summing over the $x_k\in W_{ih}$, we get

$$\begin{aligned}\sum_{x_k\in W_{ih}} b_k^{(W_i)}(u_k^{(W_i)},u_k^{(W_i)}) \le C\rho_i(1+\log\frac{H_i}{h_i})\{||u_i||^2_{L^2(W_i)}\\ +h_i\sum_{x_k\in W_{ih}} u_0^2(x_k)\}.\end{aligned} \tag{31}$$

Using a well known Sobolev-type inequality, see for example Lemma 4.3 in [**9**], we have

$$||u_i||^2_{L^2(W_i)} \le C(1+\log\frac{H_i}{h_i})||u_i||^2_{H^1(\Omega_i)}. \tag{32}$$

To estimate the second term, we note that, see (17),

$$h_i\sum_{x_k\in W_{ih}} u_0^2(x_k) \le CH_i(\bar{u}_i)^2 \le C||u_i||^2_{H^1(\Omega_i)}. \tag{33}$$

Here we have also used a simple trace theorem. Substituting (32) and (33) into (31), and using the Poincare inequality, we get

$$\sum_{x_k\in W_{ih}} b_k^{(W_i)}(u_k^{(W_i)},u_k^{(W_i)}) \le C(1+\log\frac{H_i}{h_i})^2\rho_i|u_i|^2_{H^1(\Omega_i)}.$$

Summing now with respect to $i$, we get (30).

To get (24), we add the inequalities (27), (29) and (30). The proof of Theorem 1 is complete.

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