Overlapping Nonmatching Grids Method: Some Preliminary Studies

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1. Introduction

In this paper, we report some preliminary studies of a finite difference method on overlapping nonmatching grids for a two-dimensional Poisson problem. The method can be regarded as an extension of the Generalised Additive Schwarz Method (GASM). GASM was originally developed as a preconditioning technique that uses special transmission boundary conditions at the subdomain interfaces. By involving a nonmatching grids interpolation operator in the subdomain boundary conditions, we show that the method can also be used as a discretisation scheme. We focus only on the error issues.

2. Generalised Additive Schwarz Method

We first recall briefly the GASM. Suppose we wish to solve $Au = f$ where A represents the discretisation of a PDE defined on a domain which is partitioned into nonoverlapping subdomains. Let $R_i \colon \Omega \mapsto \Omega_i$ denote the linear restriction operator that maps onto subdomain i by selecting the components corresponding to this subdomain. The matrix $M_i = R_i A R_i^T$ denotes the principal submatrix of the matrix A associated with subdomain Ω_i. The result of applying the GASM can be written as a sum of the solutions of independent subdomain problems, which can be solved in parallel: $M^{-1} = \sum_{i=1}^{p} R_i^T M_i^{-1} R_i$.

We describe this GASM for the case of two subdomains separated by the interface Γ. A more detailed description has been given by Tan [**12**] and Goossens et al. [**2**]. At the heart of the GASM lies an extension of the subdomains to slightly overlapping grids. With a proper definition of the overlap, the restrictions R_i can be defined in such a way that the original discretisation is distributed across the subdomain operators M_i. Figure 1 illustrates the extension process. In case the classical five-point star stencil is used, an overlap of one mesh width is sufficient. After extension towards overlap, and thus duplication of Ω_l and Ω_r into $\Omega_{\bar{l}}$ and $\Omega_{\bar{r}}$ respectively, we obtain an enhanced system of equations $Au = f$ in which we still
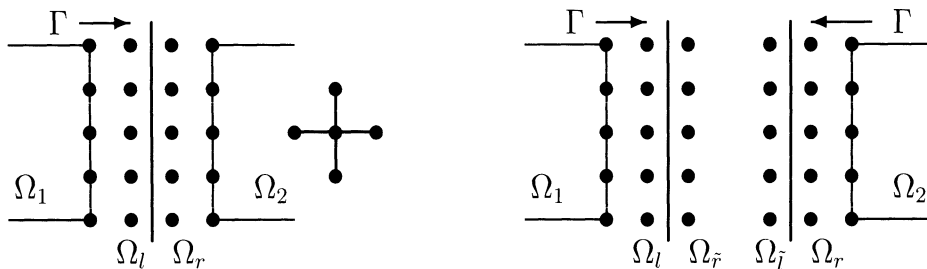
FIGURE 1. Extension of the subdomains to slightly overlapping grids.

have to specify the relation between the overlapping unknowns. The obvious way is just to state that the values in the duplicated subdomains $\Omega_{\tilde{l}}$ and $\Omega_{\tilde{r}}$ should be copied from the values in the original subdomains Ω_l and Ω_r respectively. This is known as the Dirichlet-Dirichlet coupling.

Tang [15] has shown that fast convergence can be obtained by choosing a good splitting, instead of increasing the overlap when a Schwarz enhanced matrix is used. Tan [12] has shown that the spectral properties of the preconditioned operator AM^{-1} and thus the convergence properties of a Krylov subspace method preconditioned by a GASM, are improved by pre-multiplying the enhanced linear system $Au = f$ with a properly chosen nonsingular matrix P. This has been exploited by Goossens et al. [2] to accelerate the solution of the Shallow Water Equations.

This pre-multiplication with P boils down to imposing more general conditions at the subdomain interfaces. This approach has originally been introduced by Lions [5] and subsequently been used by several authors. Hagstrom et. al. [3] advocate the use of nonlocal transmission conditions. Tan and Borsboom [13] have applied the Generalised Schwarz Coupling to advection-dominated problems. Nataf and Rogier [8, 9] have shown that the rate of convergence of the Schwarz algorithm is significantly higher when operators arising from the factorisation of the convection-diffusion operator are used as transmission conditions. Based on these results, Japhet [4] has developed the so-called optimised order 2 (OO2) conditions which result in even faster convergence.

The submatrices C_{lr}, C_{ll}, C_{rr} and C_{rl} represent the discretisation of the transmission conditions and can be chosen freely subject to the condition that the matrix $C = \begin{pmatrix} C_{lr} & -C_{ll} \\ -C_{rr} & C_{rl} \end{pmatrix}$ remains nonsingular. This gives rise to the Generalised Additive Schwarz Preconditioners which are thus based on the enhanced system of equations $Au = f$:

$$
(1) \quad
\begin{pmatrix}
A_{11} & A_{1l} & 0 & 0 & 0 & 0 \\
A_{l1} & A_{ll} & A_{lr} & 0 & 0 & 0 \\
0 & C_{ll} & C_{lr} & -C_{ll} & -C_{lr} & 0 \\
0 & -C_{rl} & -C_{rr} & C_{rl} & C_{rr} & 0 \\
0 & 0 & 0 & A_{rl} & A_{rr} & A_{r2} \\
0 & 0 & 0 & 0 & A_{2r} & A_{22}
\end{pmatrix}
\begin{pmatrix}
u_1 \\ u_l \\ \tilde{u}_r \\ \tilde{u}_l \\ u_r \\ u_2
\end{pmatrix}
=
\begin{pmatrix}
f_1 \\ f_l \\ 0 \\ 0 \\ f_r \\ f_2
\end{pmatrix}.
$$

The GASM differs from the classical Additive Schwarz Preconditioner introduced by Dryja and Widlund [1] in that the transmission conditions at the interfaces, i.e. the boundary conditions for the subdomain problems, can be changed in
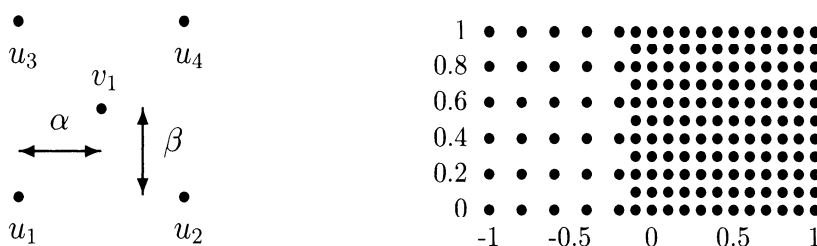
FIGURE 2. Stencil for interpolation and nonmatching grid.

order to improve the spectral properties of the preconditioned operator. An excellent description of the classical Additive Schwarz Preconditioner is given by Smith et al. [**10**].

3. Nonmatching Grids

The main topic addressed in this paper is a technique aiming at expanding the applicability of this GASM to nonmatching grids. Tan and Borsboom [**14**] have already shown how to apply the GASM on patched subgrids. The domain they are using, consists of a set of naturally ordered parallelograms, all of which have the same mesh width tangent to the interface. The mesh widths normal to the interface can be different on opposite sides of the interface. We want to alleviate this restriction and present a technique which also allows the GASM to be used when the mesh widths tangent to the interface are different on the opposite sides of the interface. The fact that nonmatching grids are being used implies that interpolation is necessary to transfer information from one grid to the other grid.

3.1. Consistence of grid interpolations. The following definition encapsulates an important concept in the nonmatching grids case.

DEFINITION 1 (Consistent Interpolation). Let $I_{h_j \to h_i}$ be the interpolation operator from Ω_j to Ω_i with mesh parameters h_j and h_i. Suppose D is the differential operator to be approximated by a finite difference operator $D_i(L_{h_i}, I_{h_j \to h_i})$, which depends on the usual finite difference operator L_{h_i} and on $I_{h_j \to h_i}$. We claim that the interpolation operator $I_{h_j \to h_i}$ is consistent on Ω_i if

$$(2) \qquad \left(D - D_i \left(L_{h_i}, I_{h_j \to h_i}\right)\right) u(x) = O(h_i)$$

for all $x \in \Omega_l$, the part of Ω_i that is overlapped.

The rest of this section is devoted to consistency. Bilinear interpolation is not sufficient for the interpolation operator $I_{h_j \to h_i}$. Figure 2 shows the stencil. The value of v_1 is given by $v_1 = (1 - \alpha)(1 - \beta)u_1 + \alpha(1 - \beta)u_2 + (1 - \alpha)\beta u_3 + \alpha\beta u_4$ where $\alpha = (x_{v_1} - x_{u_1})/(x_{u_2} - x_{u_1})$ and $\beta = (y_{v_1} - y_{u_1})/(y_{u_3} - y_{u_1})$. If in the discretisation of $-\nabla^2 u$ in Ω_l, the point in $\Omega_{\bar{r}}$ does not match with a point in Ω_r and its value is computed by bilinear interpolation from points in Ω_r and Ω_2, then this discretisation is not consistent. Hence higher order interpolation is required.

In [**11**], a fourth order interpolation formula was constructed with a small interpolation constant for smooth functions satisfying an elliptic equation of the form $-(Au_x)_x - (Bu_y)_y + au = f$, where $A, B > 0$ and $a \geq 0$. This interpolation

formula uses a 4 by 4 stencil. Consequently, the GASM constructed with this interpolation formula requires an extension of at least two grid lines.

Instead of using bilinear interpolation to compute v_1 from u_1, u_2, u_3 and u_4, which may result in an inconsistent discretisation, we discretise the partial differential equation on the stencil formed by u_1, u_2, u_3, u_4 and v_1. We seek the coefficients γ_0, γ_1, γ_2, γ_3 and γ_4 in

$$
\begin{aligned}
L(\alpha,\beta) \;=\;& \gamma_0 u\,(0,0) + \gamma_1 u\,((1-\alpha)h,(1-\beta)h) + \gamma_2 u\,(-\alpha h,(1-\beta)h) \\
& + \;\gamma_3 u\,(-\alpha h,-\beta h) + \gamma_4 u\,((1-\alpha)h,-\beta h)
\end{aligned}
\tag{3}
$$

so that a consistent approximation to $(u_{xx}+u_{yy})h^2/2$ at $v_1 = u(0,0)$ is obtained. This can be done using the Taylor expansion for $u(x,y)$ about the origin: $u(x,y) = u+u_x x+u_y y+u_{xx}x^2/2+u_{xy}xy+u_{yy}y^2/2+\mathcal{O}(h^3)$. We assume $|x| \le h$ and $|y| \le h$ so that the remainder term can be bounded by Ch^3. The requirements that the coefficients of u, u_x, u_y and u_{xy} vanish together with the requirements that the coefficients of $u_{xx}h^2/2$ and $u_{yy}h^2/2$ equal 1 in the Taylor expansion of (3), yield an overdetermined system $Cg = c$:

$$
\begin{pmatrix}
1 & 1 & 1 & 1 & 1 \\
0 & (1-\beta) & (1-\beta) & -\beta & -\beta \\
0 & (1-\alpha) & -\alpha & -\alpha & (1-\alpha) \\
0 & (1-\beta)^2 & (1-\beta)^2 & \beta^2 & \beta^2 \\
0 & (1-\alpha)^2 & \alpha^2 & \alpha^2 & (1-\alpha)^2 \\
0 & (1-\alpha)(1-\beta) & -\alpha(1-\beta) & \alpha\beta & -(1-\alpha)\beta
\end{pmatrix}
\begin{pmatrix}
\gamma_0 \\ \gamma_1 \\ \gamma_2 \\ \gamma_3 \\ \gamma_4
\end{pmatrix}
=
\begin{pmatrix}
0 \\ 0 \\ 0 \\ 1 \\ 1 \\ 0
\end{pmatrix}.
\tag{4}
$$

This overdetermined system $Cg = c$ can only have solutions if the determinant of the enhanced matrix $\tilde{C} = (C \mid c)$ is zero:

$$
\det \tilde{C} = \det (C \mid c) = (\beta - \alpha)(\alpha + \beta - 1).
\tag{5}
$$

Hence a solution can only exist when $\alpha = \beta$ or $\alpha+\beta = 1$, i.e. when the point v_1 lies on one of the diagonals of the square, formed by u_1, u_2, u_3 and u_4. The solution is

$$
g = \begin{pmatrix} -2 - (1-\alpha)/\alpha - \alpha/(1-\alpha) & \alpha/(1-\alpha) & 1 & (1-\alpha)/\alpha & 1 \end{pmatrix}^T
\tag{6}
$$

when $\alpha = \beta$ and when $\alpha + \beta = 1$ it is

$$
g = \begin{pmatrix} -2 - (1-\alpha)/\alpha - \alpha/(1-\alpha) & 1 & (1-\alpha)/\alpha & 1 & \alpha/(1-\alpha) \end{pmatrix}^T.
\tag{7}
$$

The truncation error is determined by substitution of this solution in (3):

$$
L(\alpha,\beta) = h^2 (u_{xx} + u_{yy})/2 + C_\alpha h^3 (u_{xxx} + u_{yyy})/6 + \mathcal{O}(h^4),
\tag{8}
$$

where $C_\alpha = 1 - 2\alpha$ in case $\alpha = \beta$ and $C_\alpha = 2\alpha - 1$ when $\alpha+\beta = 1$. Hence an $\mathcal{O}(h)$ approximation to $u_{xx} + u_{yy}$ can be obtained, an $\mathcal{O}(h^2)$ approximation can only be obtained when $\alpha = \beta = 1/2$:

$$
L(\alpha,\beta) = h^2 (u_{xx} + u_{yy})/2 + h^4 (u_{xxxx} + 6u_{xxyy} + u_{yyyy})/96 + \mathcal{O}(h^6).
\tag{9}
$$

In summary, a consistent discretisation exists only if v_1 is in the center or on one of the diagonals of the square formed by u_1, u_2, u_3 and u_4. The truncation error is $O(h^2)$ when v_1 is in the center and is $O(h)$ when v_1 is on one of the diagonals.

3.2. Error Analysis. Miller [6] has proven the convergence of the Schwarz algorithm based on a maximum principle. We restrict ourselves here to showing that second order accuracy in the L_∞ norm is obtained when a consistent interpolation is used. The convergence of the GASM will be studied elsewhere.

We denote by $p = (i, j)$ an index pair. J_Ω is the set of index pairs of grid points in the domain Ω. We make the following assumptions.

1. For all $p \in J_\Omega$: $\mathcal{L}_h$ has the form: $\mathcal{L}_h u_p = -c_p u_p + \sum_k c_k u_k$ where the coefficients are positive and the sum over k is taken over mesh points which are neighbours of p.

2. For all $p \in J_\Omega$: $c_p \geq \sum_k c_k$.

3. The set J_Ω is *connected*. By definition a point is connected to each of its neighbours occurring in (1) with a nonzero coefficient. By definition a set is connected if, given any two points p and q in J_Ω, there is a sequence of points $p = p_0, p_1, \ldots, p_{m+1} = q$, such that each point p_i is connected to p_{i-1} and p_{i+1}, for $i = 1, 2, \ldots, m$.

4. At least one of the equations must involve a Dirichlet boundary condition.

The maximum principle as given by Morton and Meyers [7] can briefly be stated as follows.

LEMMA 2 (Maximum Principle [7]). *Suppose that $\mathcal{L}_h$, J_Ω and $J_{\partial\Omega}$ satisfy all the assumptions mentioned above and that a mesh function u_p satisfies $\mathcal{L}_h u_p \geq 0$ for all $p \in J_\Omega$. Then u_p cannot attain a nonnegative maximum at an interior point:*

$$(10) \qquad\qquad \max_{p \in J_\Omega} u_p \leq \max \left\{ \max_{a \in J_{\partial\Omega}} u_a, 0 \right\}.$$

THEOREM 3. *Suppose a nonnegative mesh function Φ_p is defined on $J_\Omega \cup J_{\partial\Omega}$ such that $\mathcal{L}_h \Phi_p \geq 1$ for all $p \in J_\Omega$ and that all the assumptions mentioned above are satisfied. Then the error in the approximation is bounded by*

$$(11) \qquad\qquad |e_p| \leq \max_{a \in J_{\partial\Omega}} \Phi_a \, \max_{p \in J_\Omega} |T_p|$$

where T_p is the truncation error.

To prove second order accuracy in the L_∞ norm, we show that the discretisation of $\nabla^2(-u) = f$ and the coupling equations satisfy the assumptions (1) and (2) for the maximum principle. The comparison function is chosen as $\Phi(x, y) = \left((x - \mu)^2 + (y - \nu)^2 \right) / 4$, resulting in $\mathcal{L}_h \Phi_p = 1$ for all $p \in J_\Omega$. The scalars μ and ν are chosen to minimise the maximum value of this function $\Phi(x, y)$ on the boundary $\partial\Omega$. The classical five-point discretisation of $\nabla^2 u$

$$(12) \qquad \mathcal{L}_{h_i} u_p = \left(u_{i-1,j} + u_{i,j-1} - 4 u_{i,j} + u_{i+1,j} + u_{i,j+1} \right) / h_i^2$$

satisfies the assumptions for the maximum principle. In case (9) is used to obtain an equation for v_1, the assumptions (1) and (2) for the maximum principle are satisfied since this equation has $c_k = 1/(2h_i^2)$ and $c_p = 4/(2h_i^2)$. For problems with at least one Dirichlet boundary condition, the standard error analysis using the maximum principle yields second order accuracy in the L_∞ norm. The proof is essentially the same as the one given by Morton and Meyers [7].

If the point v_1 is not in the center of the square formed by u_1, u_2, u_3 and u_4, we have to use (8) to obtain an equation from which v_1 can be determined. In this case we still have second order accuracy, but a different comparison function must be defined in Ω_l. This is analogous to the classical result that second order accuracy is obtained with a second order discretisation of the partial differential equation and only first order discretisation of the boundary conditions.

TABLE 1. Results for $u_1(x, y) = \exp(-x^2 - y^2)$.

Results for $h_0 = 2h_1$.					
n_0	n_1	L_∞ in block 0	ratio	L_∞ in block 1	ratio
6	11	0.00173386		0.00160317	
11	21	0.000492551	3.52016	0.000439111	3.65094
21	41	0.000128595	3.83025	0.000113528	3.86787
41	81	3.28033e-05	3.92018	2.87286e-05	3.95174
81	161	8.28181e-06	3.96089	7.21997e-06	3.97905
n_0	n_1	L_2 in block 0	ratio	L_2 in block 1	ratio
6	11	0.0006622		0.00051287	
11	21	0.000194007	3.41328	0.000137398	3.73273
21	41	5.20312e-05	3.72867	3.51015e-05	3.91431
41	81	1.34408e-05	3.87114	8.8388e-06	3.9713
81	161	3.41357e-06	3.93746	2.2156e-06	3.98935
Reference results for $h_0 = h_1$.					
n_0	n_1	L_∞ in block 0	L_2 in block 0	L_∞ in block 1	L_2 in block 1
6	6	0.00288425	0.0010893	0.00288425	0.00117384
11	11	0.000736578	0.000280968	0.000736578	0.000299917
21	21	0.000185341	7.12195e-05	0.000185341	7.41716e-05
41	41	4.63929e-05	1.79235e-05	4.63929e-05	1.83303e-05
81	81	1.16036e-05	4.49551e-06	1.16036e-05	4.54876e-06
161	161	2.90103e-06	1.1257e-06	2.90103e-06	1.1325e-06

4. Numerical Examples

The testcases are concerned with the solution of

(13) $$-\nabla^2 u = f \text{ on } \Omega \text{ and } u = g \text{ on } \partial\Omega.$$

The domain $\Omega = \Omega_0 \cup \Omega_1$ consists of two subdomains $\Omega_0 = (-1, 0) \times (0, 1)$ and $\Omega_1 = (-h_1, 1) \times (0, 1)$. The coordinates of the grid points are (x_i, y_j), where $x_i = x_{\text{ref}} + ih$ and $y_j = y_{\text{ref}} + jh$, for $i = 0, 1, \ldots, (n_0 - 1)$ for block 0; $i = 0, 1, \ldots, n_1$ for block 1 and $j = 0, 1, \ldots, (n - 1)$. The reference point for block 0 is $(-1, 0)$ and for block 1 it is $(-h_1, 0)$. The grid sizes are $h_0 = 1/(n_0 - 1)$ and $h_1 = 1/(n_1 - 1)$. The interface Γ is defined by $x = -h_1/2$. The right-hand side f and the boundary conditions g are chosen such that the exact solution is u_1 resp. u_2 in the testcases, where $u_1(x, y) = \exp(-x^2 - y^2)$ and $u_2(x, y) = \exp(\alpha x) \sin(\beta y)$, where $\alpha = 2$ and $\beta = 8\pi$.

By definition the error is $e_{i,j} = u(x_i, y_j) - u_{i,j}$ where $u(x_i, y_j)$ is the exact solution and $u_{i,j}$ is the computed approximation. In Tables 1–3 we list both the L_∞ norm and L_2 norm of the error, defined by $L_\infty(e) = \max_{i,j} |e_{i,j}|$ and $L_2(e) = \sqrt{\frac{1}{n^2} \sum_{i=0}^{n-1} \sum_{j=0}^{n-1} e_{i,j}^2}$.

The results for $-\nabla^2 u_1 = f_1$ on a nonmatching grid are given in Table 1. We also give the results for the same problem on matching grids. This allows us to verify the accuracy of the results. In Table 2 we give the results for $-\nabla^2 u_2 = f_2$. The ratios in the fourth and sixth columns approach 4 as the mesh widths are divided by 2, showing that the method is second order accurate.

To emphasize the importance of consistent interpolation, we give in Table 3 the results for $-\nabla^2 u_2 = f_2$ when bilinear interpolation is used. In this case (2) is not

TABLE 2. Results for $u_2(x, y) = \exp(2x) \sin(8\pi y)$.

Results for $h_0 = 2h_1$.					
n_0	n_1	L_∞ in block 0	ratio	L_∞ in block 1	ratio
6	11	7.23983		4.21601	
11	21	0.48669	14.8756	0.718661	5.86648
21	41	0.102748	4.73673	0.17356	4.14071
41	81	0.0249472	4.11862	0.0453624	3.82608
81	161	0.00656458	3.80027	0.011314	4.0094
n_0	n_1	L_2 in block 0	ratio	L_2 in block 1	ratio
6	11	3.20061		1.66157	
11	21	0.204226	15.6719	0.303296	5.47838
21	41	0.0435961	4.6845	0.0748621	4.0514
41	81	0.01074	4.05923	0.0188233	3.9771
81	161	0.00269816	3.98049	0.00473253	3.97743
Reference results for $h_0 = h_1$.					
n_0	n_1	L_∞ in block 0	L_2 in block 0	L_∞ in block 1	L_2 in block 1
6	6	18.0965	6.69002	47.1923	21.3855
11	11	0.717338	0.264132	3.3735	1.40949
21	21	0.136687	0.0502781	0.718661	0.301632
41	41	0.0321275	0.011811	0.17356	0.074284
81	81	0.00831887	0.00290772	0.0453624	0.0186726
161	161	0.00207194	0.000724103	0.011314	0.00469548

TABLE 3. Results for $u_2(x, y) = \exp(2x) \sin(8\pi y)$ when bilinear interpolation is used.

n_0	n_1	L_∞ in block 0	L_∞ in block 1	L_2 in block 0	L_2 in block 1
7	12	9.63784	7.65477	4.32959	2.7709
12	22	0.847174	0.995454	0.262247	0.366744
22	42	0.142845	0.220307	0.048968	0.082186
42	82	0.0315985	0.0496966	0.011165	0.0197278
82	162	0.00710074	0.0118877	0.00271561	0.00485607

satisfied. The results are for the same problem but solved on shifted grids. The reason for using shifted grids is that an interpolation is required for every point in $\Omega_{\tilde{r}}$, while for grids as in Fig. 2 only half of the points in $\Omega_{\tilde{r}}$ require an interpolation since the other points match some point in Ω_r. The coordinates of the grid points are now given by (x_i, y_j) where $x_i = x_{\text{ref}} + (i - \frac{1}{2})h$ and $y_j = y_{\text{ref}} + (j - \frac{1}{2})h$ for $i = 0, 1, \ldots, (n - 1)$ and $j = 0, 1, \ldots, (n - 1)$. The reference point for block 0 is $(-1, 0)$ and for block 1 it is $(0, 0)$. The grid sizes $h_0 = 1/(n_0 - 2)$ and $h_1 = 1/(n_1 - 2)$ are the same as in the previous case. Since an inconsistent interpolation is used, the error is larger.

5. Concluding Remarks

We studied an overlapping nonmatching grids finite difference method. A consistency condition is introduced for the nonmatching grids interpolation operator, and under the consistency condition we proved second order global accuracy of the discretisation scheme.

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