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Nonconforming Grids for the Simulation of Fluid-Structure Interaction

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1. Introduction

Fluid structure interaction phenomena occur in a large number of applications and the literature on the subject is quite important, both from the practical and implementation point of view. Nevertheless, most of the applications are focused on a particular range of situations in which the domain that is occupied by the fluid is essentially assumed to be independent on time. Recently, a lot of effort has been made on the numerical simulations of fluid structure interactions in the case where this assumption is no more true and, in particular, in situations where the shape of the domain occupied by the fluid is among the unknowns of the problem. We refer for instance to the works [**9, 10, 11**] and also to some web pages[1] where medical and engineering applications are displayed. We refer also to [**6**] for an analysis of the mathematical problem.

This new range of applications is made possible thanks to the increase in computing power available and the recent advances in CSD and CFD. Indeed, the current implementations for the simulation of the coupled phenomena are mostly based on the effective coupling of codes devoted to fluid simulations for the ones, and structure simulations for the others. Such a coupling procedure allows for flexibility in the choice of the separate constitutive laws and modelisations of the fluid and structure separately and allows also for the rapid development of the simulation of the interaction phenomenon. This flexibility is however at the price of the definition of correct *decoupling* algorithms of the different codes that lead to a resolution of the coupled situation. In this direction, some attention has to be given for the time *decoupling* and we refer to [**11, 10, 8**] for numerical analysis of this part. Another problem has to be faced which consists in the coupling of the spatial discretizations. This difficulty, already present in the former works (where the shape of the fluid part is fixed), is certainly enhanced now that the time dependency has increased by one order of magnitude the size of the computations. Indeed, it is mostly impossible to afford the same mesh size on the fluid and on the structure computational domains, especially in three dimensional situations.

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[1]`www.crs4.it, www.science.gmu.edu/~rlohner/pages/lohner.html`

Another reason for this difficulty appears when different definitions of finite elements are naturally introduced in the structure (hermitian for plates) and the fluid (lagrangian for fluid).

The problem of coupling different discretizations occurs not only in the case of the interaction of different phenomena, but also in cases where, taking benefit of a domain decomposition, one wants to use different discretizations on different subdomains so as to optimise the discretization parameters and the final CPU time. In cases where, a priori, (exact) continuity should be imposed on the unknown solutions we have to face to the same difficulty as before for similar reasons. The mortar element method [**1**] has been proposed in this frame to produce an optimal approximation in case of variational approximations of elliptic and parabolic partial differential equations.

In this paper we state the main results concerning different ways of imposing these different discrete continuities with a particular interest to the coupling conditions that lead to the optimality of the global approximation.

The modelization that we shall consider here is the two or three dimensional incompressible Navier Stokes equations, for the fluid, and a linear elasticity for the structure. In addition, the structure will be assumed to be of small thickness in one dimension so that a one or two dimensional behaviour of beam or plate type will be used.

2. Formulation of the continuous and discrete problems

Let $\Omega_F(t)$ be the (unknown) domain occupied by the fluid at any time t during the simulation, we consider that the boundary $\partial\Omega_F(t)$ is decomposed into two open parts :

$$\partial\Omega_F(t) = \overline{\Gamma}_0 \cup \overline{\Gamma}(t)$$

where Γ_0 is independent of time and the (unknown) part $\Gamma(t)$ is the interface between the solid and the fluid. Note that $\Gamma(t)$ coincides with the position of the structure at time t. In this fluid domain we want to solve the Navier Stokes equations : Find $\mathbf{u}$ and p such that

$$(1) \qquad \left\{ \begin{array}{rcll} \dfrac{\partial \mathbf{u}}{\partial t} - \nu\Delta\mathbf{u} + \nabla p + \mathbf{u}.\nabla\mathbf{u} &=& \mathbf{f} & \text{in } \Omega_F(t), \\ \operatorname{div} \mathbf{u} &=& 0 & \text{in } \Omega_F(t), \end{array} \right.$$

these equations are complemented with appropriate boundary conditions over Γ_0 (that we shall take here as being of homogeneous Dirichlet type on the velocity) and of coupling type over $\Gamma(t)$ (that we shall explicit in a while). We consider now the structure part, that is assumed to be set in a Lagrangian formulation, i.e. the unknowns will be the displacement of the structure points with respect to a reference configuration. Under the assumptions we have done on the structure, we have a reference set Ω_S^0 (that can be the position of the structure at rest) and the position of the structure in time is parametrised by the mapping

$$\mathbf{x} \mapsto \mathbf{x} + \mathbf{d}(\mathbf{x}, t)$$

from Ω_S^0 onto $\Omega_S(t)$ that coincides with $\Gamma(t)$. The equations on $\mathbf{b}$ is here implicitly given in an abstract variational framework : find $\mathbf{d}$ such that $\mathbf{d}(.,t) \in Y$ and for any $\mathbf{b} \in Y$

$$\int_{\Omega_S^0} \frac{\partial^2 \mathbf{d}}{\partial t^2}(\mathbf{x},t)\mathbf{b}(\mathbf{x})d\mathbf{x} + a(\mathbf{d}(.,t),\mathbf{b}) = G(\mathbf{b})(t) \tag{2}$$

Here Y is some appropriate Hilbert space and G is the outside forcing term that is applied to the structure. We shall assume in what follows that this forcing term only results from the interaction with the fluid and is equal to the fluid stresses on the interface $\Gamma(t)$. The bilinear form a is assumed to take into account the elastic behaviour of the structure and is assumed to be elliptic over Y. The remaining constraint is the coupling between the fluid velocity and the displacement. Actually we want to express the fact that the fluid sticks to the boundary $\Gamma(t)$, and thus

$$\forall \mathbf{x} \in \Omega_S^0, \quad \mathbf{u}(\mathbf{x}+\mathbf{d}(\mathbf{x},t),t) = \frac{\partial \mathbf{d}}{\partial t}(\mathbf{x},t)$$

Assuming that we are able to give a proper definition to the space $L^2(\Omega_F(t))$ of all measurable functions defined over $\Omega_F(t)$ with square integrable and $H^1(\Omega_F(t))$ its subspace of all elements the gradient of which belongs to $L^2(\Omega_F(t))$, we first set $H^1_{0,\Gamma_0}(\Omega_F(t))$ as the subspace of $H^1(\Omega_F(t))$ of all functions that vanish over Γ_0, then $X(t) = (H^1_{0,\Gamma_0}(\Omega_F(t)))^2$, and we propose a global variational formulation of the coupled problem : find $(\mathbf{u}, p, \mathbf{d})$ with

$$\begin{aligned} &\forall t, \quad \mathbf{u}(.,t) \in X(t) \\ &\forall t, \quad p(.,t) \in L^2(\Omega_F(t)) \\ &\forall t, \quad \mathbf{d}(.,t) \in Y \\ &\forall t,\ \forall \mathbf{x} \in \Omega_S^0, \quad \mathbf{u}(\mathbf{x}+\mathbf{d}(\mathbf{x},t),t) = \frac{\partial \mathbf{d}}{\partial t}(\mathbf{x},t) \end{aligned} \tag{3}$$

such that, for any $(\mathbf{v}, \mathbf{b})$ in the coupled space V defined as

$$V = \left\{(\mathbf{v},\mathbf{b}) \in X(t) \times Y \ / \ \mathbf{v}(\mathbf{x}+\mathbf{d}(\mathbf{x},t)) = \mathbf{b}(\mathbf{x}),\ \forall \mathbf{x} \in \Omega_S^0\right\},$$

the following equation holds

$$\begin{aligned} &\int_{\Omega_F(t)} \frac{\partial \mathbf{u}}{\partial t}\mathbf{v} + \nu \int_{\Omega_F(t)} \nabla\mathbf{u}\nabla\mathbf{v} + \int_{\Omega_F(t)} \mathbf{u}.\nabla\mathbf{u}.\mathbf{v} + \int_{\Omega_F(t)} p\nabla.\mathbf{v} \\ &\qquad + \int_{\Omega_S^0} \frac{\partial^2 \mathbf{d}}{\partial t^2}\mathbf{b} + a(\mathbf{d},\mathbf{b}) = \int_{\Omega_F(t)} \mathbf{f}\mathbf{v}, \ \forall (\mathbf{v},\mathbf{b}) \in V, \\ &\int_{\Omega_F(t)} \nabla.\mathbf{u}\, q = 0, \ \forall q \in L^2(\Omega_F(t)). \end{aligned} \tag{4}$$

It has already been noticed (see eg [**3**] and [**6**]) that, provided that a solution exists to this system, it is stable in the following sense

$$\|\mathbf{u}\|_{L^\infty(0,T;L^2(\Omega_F(t)))\cap L^2(0,T;H^1(\Omega_F(t)))} \leq c(\mathbf{f}) \tag{5}$$

and

$$\|\mathbf{d}\|_{W^{1,\infty}(0,T;L^2(\Omega_S^0))\cap L^\infty(0,T;Y)} \leq c(\mathbf{f}) \tag{6}$$

Actually it is the kind of stability that we want to preserve in the spatial discretization. To start with, we discretize the reference configuration Ω_S^0 with an appropriate finite element mesh of size h and associate an appropriate finite element space Y_h. We have now to determine a discretization associated to the fluid part. There are many ways to proceed. Here we shall view the domain $\Omega_F(t)$ as the range, through some (time dependent) one to one mapping $\Phi(t)$, of some domain $\hat{\Omega}_F$ (e.g. the initial domain $\Omega_F(0)$ and we shall take this example hereafter). We assume also that the boundary of the domain $\hat{\Omega}_F$ is composed of the structure reference domain $\Omega_S(0)$ and a part $\hat{\Gamma}_0$ associated to the fixed portion Γ_0. We shall mesh $\hat{\Omega}_F$ (with a triangulation of size H) and define over this (fluid) reference domain an acceptable couple $(\hat{X}_H,\ \hat{M}_H)$ of finite element spaces for the approximation of the Stokes problem (we refer to [**5**] or [**2**] for more about this question). We then use the mapping $\Phi(t)$ to define the mesh and the appropriate spaces over $\Omega_F(t)$. The major question is then : how is defined the mapping from $\hat{\Omega}_F$ onto $\Omega_F(t)$?

Of course, it has to coincide in some sense with $\mathbf{x} + \mathbf{d}_h(\mathbf{x}, t)$. Hence we define an operator π_H^* that will allow to associate to each $\mathbf{x} + \mathbf{d}_h(\mathbf{x}, t)$ a discrete position of the interface $\pi_H^*(\mathbf{x} + \mathbf{d}_h(\mathbf{x}, t))$ adapted to the mesh (of size H) that exists on the side $\partial\hat{\Omega}_F \setminus \hat{\Gamma}_0$. From this position $\pi_H^*(\mathbf{x} + \mathbf{d}_h(\mathbf{x}, t))$ extended to $\hat{\Gamma}_0$ by the identity (that is thus only given over the boundary of $\hat{\Omega}_F$) we define, by prolongation, a mapping $\Phi_H(t)$ that is a finite element function over the mesh of $\hat{\Omega}_F(0)$. This mapping provides, at the same time, the domain $\Omega_{H,F}(t)$, the mesh on this domain and the spaces of discretization ($X_H(t)$ and $M_H(t)$) and finally the velocity of the mesh $\mathbf{u}_H^*$ that verifies

$$\mathbf{u}_H^*(\pi_H^*(\mathbf{x} + \mathbf{d}_h(\mathbf{x}, t)), t) = \pi_H^*(\frac{\partial \mathbf{d}_h}{\partial t}(\mathbf{x}, t)) \tag{7}$$

In order to define the discrete problem associated to (4), we first give the proper interpretation at the discrete level of the equality between the velocity of the fluid and the velocity of the structure. Of course, the equality all over the interface cannot be exactly satisfied, this is why we have to introduce again a projection operator π_H from the h mesh onto the H one. We introduce the discrete equivalent of V as follows

$$V_{h,H}(t) = \{(\mathbf{v}_H, \mathbf{b}_h) \in X_H(t) \times Y_h, \quad \mathbf{v}_H(\pi_h^*(\mathbf{x} + \mathbf{d}(\mathbf{x}, t)), t) = \pi_H(\mathbf{b}_h(\mathbf{x}, t))$$

and we look for a solution $(\mathbf{u}_H, p_H, \mathbf{d}_h) \in X_H(t) \times M_H(t) \times Y_h$ such that

$$\begin{aligned}
&\int_{\Omega_{H,F}(t)} \frac{\partial \mathbf{u}_H}{\partial t}\mathbf{v}_H + \nu \int_{\Omega_{H,F}(t)} \nabla \mathbf{u}_H \nabla \mathbf{v}_H + \frac{1}{2}\int_{\Omega_{H,F}(t)} \mathbf{u}_H.\nabla \mathbf{u}_H.\mathbf{v}_H - \\
&\frac{1}{2}\int_{\Omega_{H,F}(t)} \mathbf{u}_H.\nabla \mathbf{v}_H.\mathbf{u}_H + \int_{\Gamma_H(t)} \frac{\mathbf{u}_H(t)\mathbf{v}_H(t)}{2}\mathbf{u}_H^*.\mathbf{n} + \int_{\Omega_{H,F}(t)} p_H \nabla.\mathbf{v}_H + \\
&\int_{\Omega_S^0} \frac{\partial^2 \mathbf{d}_h}{\partial t^2}\mathbf{b}_h + a(\mathbf{d}_h, \mathbf{b}_h) = \int_{\Omega_{H,F}(t)} \mathbf{f}\mathbf{v}_H, \ \ \forall (\mathbf{v}_H, \mathbf{b}_h) \in V_{h,H}(t), \\
&\int_{\Omega_{H,F}(t)} \nabla.\mathbf{u}_H \ \mu_H = 0, \ \forall \mu_H \in M_H(t),
\end{aligned} \tag{8}$$

and of course complemented with the coupling condition

$$\mathbf{u}_H(\pi_h^*(\mathbf{x} + \mathbf{d}(\mathbf{x}, t)), t) = \pi_H(\frac{\partial \mathbf{d}_h(\mathbf{x}, t)}{\partial t}) \tag{9}$$

Note that we have chosen here, as is often the case, to treat the nonlinear convection terms in a skew symmetric way.

It is an easy matter to note that $(\mathbf{u}_H, \frac{\partial \mathbf{d}_h}{\partial t})$ is an admissible test function since from (9), it satisfies the coupling condition on the interface. By plugging this choice of test functions in equation (8) and using the discrete incompressibility condition in (8) we first get

$$\int_{\Omega_{H,F}(t)} \frac{1}{2}\frac{\partial \mathbf{u}_H^2}{\partial t} + \nu \int_{\Omega_{H,F}(t)} (\nabla \mathbf{u}_H)^2 + \int_{\Gamma_H(t)} \frac{\mathbf{u}_H^2(t)}{2}\mathbf{u}_H^*.\mathbf{n}$$
$$+ \int_{\Omega_S^0} \frac{\partial^2 \mathbf{d}_h}{\partial t^2}\frac{\partial \mathbf{d}_h}{\partial t} + a(\mathbf{d}_h, \frac{\partial \mathbf{d}_h}{\partial t}) = \int_{\Omega_{H,F}(t)} \mathbf{f}\mathbf{u}_H,$$

Reminding the Taylor derivation theorem about integral derivatives, and recalling that the velocity of the interface is $\mathbf{u}_H^*$, we end at

$$\frac{1}{2}\frac{d}{dt}\int_{\Omega_{H,F}(t)} \mathbf{u}_H^2 \ + \nu \int_{\Omega_{H,F}(t)} (\nabla \mathbf{u}_H)^2$$
$$+\frac{1}{2}\frac{d}{dt}\int_{\Omega_S^0}\left(\frac{\partial \mathbf{d}_h}{\partial t}\right)^2 + \frac{1}{2}\frac{d}{dt}a(\mathbf{d}_h, \mathbf{d}_h) \le \|\mathbf{f}\|_{L^2(\Omega_{H,F}(t))}\|\mathbf{u}_H\|_{L^2(\Omega_{H,F}(t))}$$

which, similarly as in the continuous case (5),(6) leads to a stability result (and thus to an existence result on any fixed time) of the discrete solution

$$\begin{aligned}(10)\qquad &\|\mathbf{u}_H\|_{L^\infty(0,T;H^1(\Omega_{H,F}(t)))\cap H^1(0,T;L^2(\Omega_{H,F}(t)))}\\ &\quad +\|\mathbf{d}_h\|_{L^\infty(0,T;H^1(\Omega_S^0)\cap W^{1,\infty}(0,T;L^2(\Omega_S^0)} \le C(\mathbf{f})\end{aligned}$$

Now that this problem is set, we want to understand how to define the operator π_H so as to obtain an optimal error that could read

(11)

$$\begin{aligned}&\|\mathbf{u}-\mathbf{u}_H\|_{L^2(0,T;H^1)} + \|p-p_H\|_{L^2(0,T;L^2)} + \|\mathbf{d}-\mathbf{d}_h\|_{L^\infty(0,T;H^1)}\\ &\quad \le c\inf_{\mathbf{v}_H}\|\mathbf{u}-\mathbf{v}_H\|_{L^2(0,T;H^1)} + \inf_{q_H}\|p-q_H\|_{L^2(0,T;L^2)} + \inf_{\mathbf{b}_h}\|\mathbf{d}-\mathbf{b}_h\|_{L^\infty(0,T;H^1)}.\end{aligned}$$

This error analysis is far out of hand since currently, as far as we know, no general existence result is available on the continuous coupled problem in the general case (see however [**6**]). Nevertheless, since this discretization question is mainly related to the spatial discretization, we present in the following section the numerical analysis of a simplified steady problem in which we believe that all the main features for the definition of the coupling operator π_H are present.

3. Steady State Study

We shall degenerate here the original problem as follows:

- the problem is steady
- the "fluid equations" are replaced by a Laplace equation
- the structure has only a normal displacement that is modelled through a fourth order equation

We consider the problem where a Laplace equation is set on an unknown domain that is delimited over an edge, the equation of which is determined through a fourth order equation in the right hand side of which stands the normal derivative of the solution to the Laplace equation. We denote by $\hat{\Omega} =]0,1[^2$ the unit square of $\mathbb{R}^2$. We take two given functions $\mathbf{f}$ and g respectively in $(L^2(\mathbb{R}^2))^2$ and $H^{-2}(0,1)$. We are looking for $\mathbf{v} = (v_1, v_2) \in (H_0^1(\varphi(d)(\hat{\Omega})))^2$ such that :

$$(12)\qquad \left\{\begin{array}{rcll} -\Delta \mathbf{v} &=& \mathbf{f} & \text{in } \varphi(d)(\hat{\Omega}),\\ \mathbf{v} &=& 0 & \text{over } \partial\varphi(d)(\hat{\Omega}),\end{array}\right.$$

and $d \in H_0^2(0,1)$ such that

$$(13) \quad \begin{cases} \dfrac{d^4 d}{dx^4} & = \; g - ((\nabla \mathbf{v}) \circ \varphi(d) \text{ cof } \nabla\varphi(d).\mathbf{n})_2 \quad \text{sur } (0,1), \\ \dfrac{d}{dx}d(1) & = \; \dfrac{d}{dx}d(0) = d(1) = d(0) = 0, \end{cases}$$

where $\varphi(d)$ maps $\hat{\Omega}$ onto $\varphi(d)(\hat{\Omega})$ and is one to one and satisfies on the interface

$$(14) \qquad \varphi(d)(x,1) = (x, 1 + d(x)).$$

A simple choice for $\varphi(d)$ is the following

$$(15) \qquad \varphi(d)(x,y) = (x, y + yd(x)).$$

We can rewrite this strong formulation in a variational formulation. We introduce the space of test functions

$$V_d = \quad \Big\{(\mathbf{w}, b) \in H_0^1(\varphi(d)(\hat{\Omega})) \times H_{0,\Gamma_0}^1(\varphi(d)(\hat{\Omega})) \times H_0^2(0,1)/ \\ w_2 \circ \varphi(d) = b \text{ sur }]0,1[\times\{1\}\Big\}.$$

The problem is then the following :
find $(\mathbf{v}, d) \in (H_0^1(\varphi(d)(\hat{\Omega})))^2 \times H_0^2(0,1)$ *such that*

$$(16) \quad \begin{cases} \displaystyle\int_{\varphi(d)(\hat{\Omega})} \nabla\mathbf{v}\nabla\mathbf{w} + \int_0^1 \frac{d^2 d}{dx^2}\frac{d^2 b}{dx^2} = \int_{\varphi(d)(\hat{\Omega})} \mathbf{f}\mathbf{w} + < g, b >_{H^{-2}, H_0^2} \\ \forall(\mathbf{w}, b) \in V_d. \end{cases}$$

Let ε and α be two real numbers, $0 < \varepsilon < 1$ and $0 < \alpha < 1/2$. We search the displacement in the set defined by

$$B_\varepsilon^\alpha = \Big\{z \in H_0^{2-\alpha}(0,1) \; / \; \|z\|_{H_0^{2-\alpha}(0,1)} \le M^{-1}(1-\varepsilon)\Big\},$$

where M is the continuity constant of the injection of $H^{2-\alpha}(0,1)$ in $C^1([0,1])$. The problem (16) has at least a solution, for small enough exterior forces. We have

THEOREM 1. *Assume that* $\mathbf{f}$ *and* g *satisfy*

$$\|g\|_{H^{-2}(0,1)} + C(\varepsilon)\|\mathbf{f}\|_{(L^2(\mathbb{R}^2))^2} \le M^{-1}(1-\varepsilon).$$

Then there exists a solution of (16), $(\mathbf{v}, d) \in (H_0^1(\varphi(d)(\hat{\Omega}))^2 \times (H_0^2(0,1) \cap B_\varepsilon^\alpha$. *If we suppose, moreover, that the function* $\mathbf{f}$ *is lipschitz with an* L^2*-norm and a lipschitz constant small enough, then the solution is unique.*

The proof of this theorem is based on a fixed point theorem. For a given deformation γ of the interface Γ, we have the existence of $(\mathbf{v}(\gamma), d(\gamma))$ and we prove that the application T defined by :

$$T : \begin{array}{ccc} B_\varepsilon^\alpha & \longrightarrow & T(B_\varepsilon^\alpha) \\ \gamma & \longmapsto & d(\gamma), \end{array}$$

satisfies the hypothesis of Schauder theorem.

Next, we want to discretize the problem. For the fluid part, we consider a P_k finite element discretization, with $k \ge 1$, we denote by H the associated space step and X_H the associated finite element space. This discretization of the reference domain $\hat{\Omega}$ is mapped on the deformed configuration as was explained in section 2. For the structure part, we consider P_3- Hermitian finite element, since the displacement is solution of a fourth-order equation. The space step is denoted by h and Y_h^0 is the associated finite element space.

To discretize the problem we are going to work with the variational formulation but written on the reference domain $\hat{\Omega}$. With the help of the mapping $\varphi(d)$ we can change of variables in (16). We obtain

$$\begin{aligned}(17)\qquad &\int_{\hat{\Omega}} \nabla\varphi(d)^{-t}\nabla\varphi(d)^{-1}\,\det(\nabla\varphi(d))\nabla(\mathbf{u})\nabla\mathbf{w} + \int_0^1 \frac{d^2d}{dx^2}\frac{d^2b}{dx^2} = \\ &\int_{\hat{\Omega}} \mathbf{f}\circ\varphi(d)\,\det(\nabla\varphi(d))\mathbf{w} + < g, b >_{H^{-2},H_0^2}, \quad \forall(\mathbf{w}, b) \in V^*\end{aligned}$$

where

$$V^* = \left\{(\mathbf{w}, b) \in H_0^1(\hat{\Omega}) \times H^1_{0,\Gamma_0}(\hat{\Omega}) \times H_0^2(0,1)/\mathbf{w}_2 = b \text{ sur } \Gamma\right\},$$

and $\mathbf{v}\circ\varphi(d) = \mathbf{u}$. We set $F(\gamma) = \nabla\varphi(\gamma)^{-t}\nabla\varphi(\gamma)^{-1}\,\det(\nabla\varphi(\gamma))$.
The discrete variational formulation is the following find $\mathbf{u}_H \in (X_H^0)^2$ and $d_h \in Y_h^0$ such that

$$(18)\qquad \left\{\begin{array}{l}\displaystyle\int_{\hat{\Omega}} \mathbf{F}(d_h)\nabla\mathbf{u}_H\nabla\mathbf{w}_H + \int_0^1 \frac{d^2d_h}{dx^2}\frac{d^2b_h}{dx^2} = \\ < g, b >_{H^{-2},H_0^2} + \int_{\hat{\Omega}}(1+d_h)\mathbf{f}\circ\varphi(d_h)\mathbf{w}_H, \quad \forall(\mathbf{w}_H, b_h) \in V_{H,h},\end{array}\right.$$

with

$$\begin{aligned}X_H &\overset{def}{=} \{\mathbf{v} \in C^0(\overline{\hat{\Omega}})\ /\ \mathbf{v}|_T \in P_k(T),\ \forall T \in \tau_H\},\\ Y_h^0 &\overset{def}{=} \{b \in C^1([0,1])\ /\ b|_S \in P_3(S),\ \forall S \in \tau_h\} \cap H_0^2(0,1),\\ X_H^0 &\overset{def}{=} \{\mathbf{v} \in C^0(\overline{\hat{\Omega}})\ /\ \mathbf{v}|_T \in P_k(T),\ \forall T \in \tau_H\} \cap H_0^1(\hat{\Omega}),\\ V_{H,h} &\overset{def}{=} \{(\mathbf{w}_H, b_h) \in (X_H)^2 \times Y_h^0\ /\ \mathbf{w}_H|_{\Gamma_0} = 0,\ (\mathbf{w}_H)_2|_\Gamma = \Pi_H(b_h),\\ &\qquad\qquad (\mathbf{w}_H)_1 \in X_H^0\}\end{aligned}$$

and τ_H (resp. τ_h) denotes the triangulation associated to the fluid part (resp. to the structure) and Π_H represents the matching operator of the test functions. As was explained in the previous section, the nonconforming grids prevent the discrete test functions to satisfy the continuity condition at the interface. The discrete space of test functions $V_{H,h}$ is thus not included in the continuous space V. We are going to study different type of matching : a pointwise matching and an integral matching. On one hand, we will consider for Π_H the finite element interpolation operator associated to the fluid part, and in the other hand the mortar finite element operator [**1**]. So, we have the two cases

$$\begin{aligned}V_{H,h} &= \{(\mathbf{w}_H, b_h) \in (X_H)^2 \times Y_h^0\ /\ \mathbf{w}_H|_{\Gamma_0} = 0,\ (\mathbf{w}_H)_2|_\Gamma = I_H(b_h),\\ (19)\qquad &\qquad\qquad (\mathbf{w}_H)_1 \in X_H^0\},\end{aligned}$$

where I_H denotes the finite element interpolation operator, and

$$\begin{aligned}V_{H,h} &= \Big\{(\mathbf{w}_H, b_h) \in (X_H)^2 \times Y_h^0\ /\ \mathbf{w}_H|_{\Gamma_0} = 0,\ (\mathbf{w}_H)_1|_\Gamma = 0,\\ (20)\qquad &\qquad \int_0^1 (b_h - (\mathbf{w}_H)_2)\psi = 0\ \forall\psi \in \tilde{X}_H(\Gamma)\Big\},\end{aligned}$$

where $\tilde{X}_H(\Gamma)$ is a subspace of codimension 2 in $X_H(\Gamma)$, space of trace on Γ of X_H and define by

$$\tilde{X}_H(\Gamma) \overset{def}{=} \{w_h \in X_H(\Gamma)/\forall T\in\tau_H,\ \text{if } (0,1)\in T \text{ or if } (1,1)\in T, w_H|_{\bar{\Gamma}\cap T} \in P_{k-1}(T)\}.$$

THEOREM 2. *Let* $\mathbf{f} \in (W^{2,\infty}(\mathbb{R}^2))^2$ *and* $g \in H^{-2}(0,1)$ *there exists* $(\mathbf{u}_H, d_h) \in X_H^0 \times Y_h^0$ *solution of (18) such that*
For the pointwise matching through the interpolation operator I_H*, we have*

- *if* $k \leq 2$

$$\begin{aligned} \|d - d_h\|_{H_0^2(0,1)} &\leq C(\mathbf{f}, \lambda)\left[H^k + \|d - d_h^*\|_{H_0^2(0,1)}\right], \\ \|\mathbf{u} - \mathbf{u}_H\|_{(H_0^1(\hat{\Omega}))^2} &\leq C(\mathbf{f}, \lambda)\left[H^k + \|d - d_h^*\|_{H_0^2(0,1)}\right]. \end{aligned}$$

- *if* $k > 2$

$$\begin{aligned} \|d - d_h\|_{H_0^2(0,1)} &\leq C(\mathbf{f}, \lambda)\left[H^2 + \|d - d_h^*\|_{H_0^2(0,1)}\right], \\ \|\mathbf{u} - \mathbf{u}_H\|_{(H_0^1(\hat{\Omega}))^2} &\leq C(\mathbf{f}, \lambda)\left[H^2 + \|d - d_h^*\|_{H_0^2(0,1)}\right]. \end{aligned}$$

For the matching through the mortar operator, we have

$$\begin{aligned} \|d - d_h\|_{H_0^2(0,1)} &\leq C(\mathbf{f}, \lambda)\left[H^k + \|d - d_h^*\|_{H_0^2(0,1)}\right], \\ \|\mathbf{u} - \mathbf{u}_H\|_{(H_0^1(\hat{\Omega}))^2} &\leq C(\mathbf{f}, \lambda)\left[H^k + \|d - d_h^*\|_{H_0^2(0,1)}\right] \end{aligned}$$

where d_h^* *denotes the projection of* d *on* Y_h^0 *in semi norm* $H^2(0,1)$.

We remark that for the finite element interpolation operator we obtain optimal error estimates when the degree of the fluid polynomial is less or equal to 2. These estimates are no more optimal when $k \geq 2$. This is due to the fact that the displacement is solution of a fourth order equation. When the weak matching is imposed through the mortar operator then the error estimates are optimal in all the cases.
The proof of this result is based on a discrete fixed point theorem due to Brezzi Rappaz Raviart in a modified version due to Crouzeix [**4**]. This theorem gives us the existence of the discrete solution together with the error between this discrete solution and the exact solution. We want to underline the reason why the pointwise matching yields optimal error estimates in some (interesting) cases. When we consider the nonconforming discretization of a second order equation using nonoverlapping domain decomposition for the pointwise matching the error estimates are never optimals. In fact in both situations, the error analysis involves a best fit error and a consistency error which measures the effect of the nonconforming discretization. Classically, for the Laplace equation (c.f. Strang's lemma) this term can be written as follows

$$\sup_{w_h \in V_\delta} \frac{\displaystyle\int_\Gamma \frac{\partial u}{\partial n}[w_\delta]}{\|w_\delta\|_{H^{1/2}(\Gamma)}},$$

where $[w_\delta]$ represents the jump at the interface of the functions belonging to the discrete space V_δ, and u denotes the exact solution. In the fluid structure interaction, even if we deal with a non linear problem a similar term appears (not exactly under this form) in the proof and affects the final estimate. The jump of the test functions at the interface is equal to $\mathbf{w}_H - b_h = \Pi_H(b_h) - b_h$. Since b_h is $H^2(0,1)$, for $\Pi_H = I_H$ we have

$$\|I_H(b_h) - b_h\|_{L^2(0,1)} \leq CH^2 \|b_h\|_{H^2(0,1)}.$$

TABLE 1

	"1D Structure " second order operator	"1D Structure " fourth order operator	"2D Structure " (Laplace equation)
P_k 2D"Fluid" + interpolation	$k = 1$ optimal $k > 1$ non optimal	$k \leq 2$ optimal $k > 2$ non optimal	non optimal
P_k 2D"Fluid" + Mortar Method	optimal $\forall k$	optimal $\forall k$	optimal $\forall k$

That explains why for $k \leq 2$ the estimates is optimal. On the opposite the integral matching (mortar element method) gives for all value of k optimal error estimates. We have also studied a linear problem in the case where the displacement is solution of a second order equation on the interface (this is the case when the longitudinal displacements are taken into account). We can summarise the results in Table 1. For more details see [**7**].

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