http://dx.doi.org/10.1090/conm/218/3019

Contemporary Mathematics
Volume **218**, 1998
B 0-8218-0988-1-03019-3

Extension of a Coarse Grid Preconditioner to Non-symmetric Problems

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1. Introduction

The Optimized Order 2 (OO2) method is a non-overlapping domain decomposition method with differential interface conditions of order 2 along the interfaces which approximate the exact artificial boundary conditions [**13, 9**]. The convergence of Schwarz type methods with these interface conditions is proved in [**12**]. There already exists applications of the OO2 method to convection-diffusion equation [**9**] and Helmholtz problem [**3**]. We first recall the OO2 method and present numerical results for the convection-diffusion equation discretized by a finite volume scheme. The aim of this paper is then to provide an extension of a preconditioning technique introduced in [**7, 5**] based upon a global coarse problem to non-symmetric problems like convection-diffusion problems. The goal is to get the independence of the convergence upon the number of subdomains. Numerical results on convection-diffusion equation will illustrate the efficiency of the OO2 algorithm with this coarse grid preconditioner.

2. The Optimized Order 2 Method

We recall the OO2 Method in the case of the convection-diffusion problem:

$$\begin{aligned} L(u) = cu + a(x,y)\frac{\partial u}{\partial x} + b(x,y)\frac{\partial u}{\partial y} - \nu\Delta u &= f \text{ in } \Omega \\ C(u) &= g \text{ on } \partial\Omega \end{aligned} \tag{1}$$

where Ω is a bounded open set of $\mathbb{R}^2$, $\vec{a} = (a,b)$ is the velocity field, ν is the viscosity, C is a linear operator, c is a constant which could be $c = \frac{1}{\Delta t}$ with Δt a time step of a backward-Euler scheme for solving the time dependent convection-diffusion problem. The method could be applied to other PDE's.

The OO2 method is based on an extension of the additive Schwarz algorithm with non-overlapping subdomains : $\bar{\Omega} = \cup_{i=1}^{N} \bar{\Omega}_i$, $\Omega_i \cap \Omega_j = \emptyset$, $i \neq j$. We denote by $\Gamma_{i,j}$ the common interface to Ω_i and Ω_j, $i \neq j$. The outward normal from Ω_i is $\mathbf{n_i}$ and $\boldsymbol{\tau}_\mathbf{i}$ is a tangential unit vector.

1991 *Mathematics Subject Classification.* Primary 65F30; Secondary 65M60, 65Y05.

The additive Schwarz algorithm with non-overlapping subdomains ([**11**]) is :

$$\begin{aligned} L(u_i^{n+1}) &= f, \quad \text{in } \Omega_i \\ B_i(u_i^{n+1}) &= B_i(u_j^n), \quad \text{on } \Gamma_{i,j},\ i \neq j \\ C(u_i^{n+1}) &= g \quad \text{on } \partial\Omega_i \cap \partial\Omega \end{aligned} \tag{2}$$

where B_i is an interface operator. We recall first the OO2 interface operator B_i and then the substructuring formulation of the method.

2.1. OO2 interface conditions. In the case of Schwarz type methods, it has been proved in [**14**] that the optimal interface conditions are the exact artificial boundary conditions [**8**]. Unfortunately, these conditions are pseudo-differential operators. Then, it has been proposed in [**13**] to use low wave number differential approximations to these optimal interface conditions. Numerical tests on a finite difference scheme with overlapping subdomains has shown that the convergence was very fast for a velocity field non tangential to the interface, but very slow, even impossible, for a velocity field tangential to the interface. So, instead of taking low-wave number approximations, it has been proposed in [**9**] to use differential interface conditions of order 2 along the interface which optimize the convergence rate of the Schwarz algorithm. These "Optimized Order 2" interface operators are defined as follows:

$$B_i = \frac{\partial}{\partial \mathbf{n_i}} - \frac{\mathbf{a}.\mathbf{n_i} - \sqrt{(\mathbf{a}.\mathbf{n_i})^2 + 4c\nu}}{2\nu} + c_2 \frac{\partial}{\partial \boldsymbol{\tau}_\mathbf{i}} - c_3 \frac{\partial^2}{\partial \boldsymbol{\tau}_\mathbf{i}^2}$$

where $c_2 = c_2(\mathbf{a}.\mathbf{n_i}, \mathbf{a}.\boldsymbol{\tau}_\mathbf{i})$ and $c_3 = c_3(\mathbf{a}.\mathbf{n_i}, \mathbf{a}.\boldsymbol{\tau}_\mathbf{i})$ minimize the convergence rate of the Schwarz algorithm. The analytic analysis in the case of 2 subdomains and constant coefficients in (1) reduce the minimization problem to a one parameter minimization problem. This technique is extended in the case of variable coefficients and an arbitrary decomposition, that is only one parameter is computed, with a dichotomy algorithm. With this parameter we get c_2 and c_3 (see [**10**]). So the OO2 conditions are easy to use and not costly. The convergence of the Schwarz algorithm with the OO2 interface conditions is proved for a decomposition in N subdomains (strips) using the techniques in [**12**].

2.2. Substructuring formulation. In [**14**], the non-overlapping algorithm (2) is interpreted as a Jacobi algorithm applied to the interface problem

$$D\lambda = b \tag{3}$$

where λ, restricted to Ω_i, represents the discretization of the term $B_i(u_i)$ on the interface $\Gamma_{i,j},\ i \neq j$. The product $D\lambda$, restricted to Ω_i, represents the discretization of the jump $B_i(u_i) - B_i(u_j)$ on the interface $\Gamma_{i,j},\ i \neq j$. To accelerate convergence, the Jacobi algorithm is replaced by a Krylov type algorithm [**16**].

2.3. Numerical results. The method is applied to a finite volume scheme [**1**] (collaboration with MATRA BAe Dynamics France) with a decomposition in N non-overlapping subdomain. The interface problem (3) is solved by a BICG-stab algorithm. This involves solving N independent subproblems which can be done in parallel. Each subproblem is solved by a direct method. We denote by h the mesh size. We compare the results obtained with the OO2 interface conditions and the Taylor order 0 ([**4**],[**2**], [**13**]) or order 2 interface conditions ([**13**]).

TABLE 1. Number of iterations versus the convection velocity's angle: 16×1 subdomains, $\nu = 1.d-2$, $CFL = 1.d9$, $h = \frac{1}{241}$, $\log_{10}(Error) < 1.d-6$

convection velocity	OO2	Taylor order 2	Taylor order 0
normal velocity to the interface $a = y,\ b = 0$	15	123	141
tangential velocity to the interface $a = 0,\ b = y$	20	not convergent	75

TABLE 2. Number of iterations versus the mesh size: 16×1 subdomains, $a = y$, $b = 0$, $\nu = 0.01$, $CFL = 1.d9$, $\log_{10}(Error) < 1.d-6$

grid	65×65	129×129	241×241
OO2	15	15	15
Taylor order 2	49	69	123
Taylor order 0	49	82	141

TABLE 3. Number of iterations versus the mesh size: 16×1 subdomains, rotating velocity, $a = -\sin(\pi(y - \frac{1}{2}))\cos(\pi(x - \frac{1}{2}))$, $b = \cos(\pi(y - \frac{1}{2}))\sin(\pi(x - \frac{1}{2}))$ $\nu = 1.d-2$, $CFL = 1.d9$, $\log_{10}(Error) < 1.d-6$

grid	65×65	129×129	241×241
OO2	49	48	48
Taylor order 0	152	265	568

1. We consider the problem: $L(u) = 0,\quad 0 \le x \le 1,\ 0 \le y \le 1$ with $u(0,y) = \frac{\partial u}{\partial x}(1,y) = 0,\ 0 \le y \le 1,\ \frac{\partial u}{\partial y}(x,1) = 0,\ u(x,0) = 1,\ 0 \le x \le 1$. In order to observe the influence on the convergence both of the convection velocity angle to the interfaces, and of the mesh size, we first take a decomposition in strips. The Table 1 shows that the OO2 interface conditions give a significantly better convergence which is independent of the convection velocity angle to the interfaces. One of the advantages is that for a given number of subdomains, the decomposition of the domain doesn't affect the convergence. We also observe that the convergence for the studied numerical cases is independent of the mesh size (see Table 2 and Table 3).

2. The OO2 method was also tested for a convection velocity field issued from the velocity field of a Navier-Stokes incompressible flow, with Reynolds number $Re = 10000$, around a cylinder. This velocity field is issued from a computation at the aerodynamic department at Matra. The computational domain is defined by $\Omega = \{(x,y) = (r\cos(\theta), r\sin(\theta)),\ 1 \le r \le R,\ 0 \le \theta \le 2\pi\}$ with $R > 0$ given.
We consider the problem $L(u) = 0$ in Ω with $u = 1$ on $\{(x,y) = (\cos(\theta), \sin(\theta)),\ 0 \le \theta \le 2\pi\}$ and $u = 0$ on $\{(x,y) = (R\cos(\theta), R\sin(\theta))\ 0 \le \theta \le 2\pi\}$. The grid is $\{(x,y) = (r_i\cos(\theta_j), r_i\sin(\theta_j))\}$, and is refined around the cylinder and in the direction of the flow. We note N_{max} = (number of points on the boundary of a subdomain) $\times$ (number of subdomains). The OO2 interface conditions give

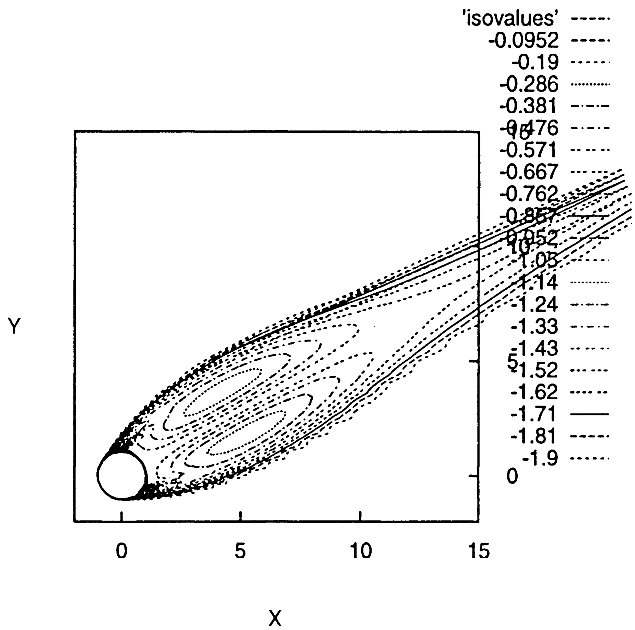

FIGURE 1. Iso-values of the solution u, $\nu = 1.d-4$, $CFL = 1.d9$

TABLE 4. Number of iterations versus the viscosity; 4×2 subdomains, $CFL = 1.d9$, $\log_{10}(Error) < 1.d-6$

	OO2	Taylor order 2	Taylor order 0
$\nu = 1.d-5$	56	41	119
$\nu = 1.d-4$	43	121	374
$\nu = 1.d-3$	32	$N_{max} = 768$ $log_{10}(Error) = -5.52$	$N_{max} = 768$ $log_{10}(Error) = -2.44$

also significantly better convergence in that case. Numerically the convergence is practically independent of the viscosity ν (see Table 4).

3. Extension of a coarse grid preconditioner to non-symmetric problems

Numerically, the convergence ratio of the method is nearly linear upon the number of subdomains in one direction of space. To tackle this problem, the aim of this paper is to extend a coarse grid preconditioner introduced in [**7**], [**5**] to non-symmetric problems like convection-diffusion problems. This preconditioning technique has been introduced for the FETI method, in linear elasticity, when local Neumann problems are used and are ill posed (see [**7**]). It has been extended for plate or shell problems, to tackle the singularities at interface cross-points ([**6**], [**5**], [**15**]). In that case, this preconditioner is a projection for $(D.,.)_2$ on the space orthogonal to a coarse grid space which contain the corner modes. This consists in constraining the Lagrange multiplier to generate local displacement fields which are continuous at interface cross-points. The independence upon the number of subdomains has been proved.

In this paper we extend this preconditioner by considering a $(D.,D.)_2$ projection on the space orthogonal to a coarse grid space. The goal is to filter the low frequency phenomena, in order to get the independence of the convergence upon the number of subdomains. So the coarse grid space, denoted W, is a set of functions called "coarse modes" which are defined on the interfaces by :

- Preconditioner M1 : the "coarse modes" are the fields with unit value on one interface and 0 on the others.
- Preconditioner M2 : the "coarse modes" in a subdomain Ω_i are on one interface the restriction of $K_i u_i$ where $u_i = 1 \in \Omega_i$ and K_i is the stiffness matrix, and 0 on the others.

Then, at each iteration, λ^p satisfies the continuity requirement of associated field u^p at interface :

$$(DW)_i^t(D\lambda^p - b) = 0 \quad \forall i$$

That is, if we introduce the projector P on $W^\perp$ for $(D.,D.)_2$, the projected gradient of the condensed interface problem is:

$$Pg^p = g^p + \sum_i (DW)_i \delta_i \tag{4}$$

and verify

$$(DW)_i^t Pg^p = 0 \quad \forall i \tag{5}$$

With (4), the condition (5) can be written as the coarse problem :

$$(DW)^t(DW)\delta = -(DW)^t g^p$$

So the method has two level : at each iteration of the Krylov method at the fine level, an additional problem has to be solved at the coarse grid level.

3.1. Numerical results. The preconditioned OO2 method is applied to problem (1) discretized by the finite volume scheme with non-overlapping subdomains. The interface problem (3) is solved by a projected GCR algorithm, that is the iterations of GCR are in the $(D.,D.)_2$ orthogonal to the coarse grid space. Each subproblem is solved by a direct method. We compare the results obtained with the preconditioners M1 and M2.

1. We consider the problem: $L(u) = 0,\ \ 0 \le x \le 1,\ 0 \le y \le 1$ with $\frac{\partial u}{\partial x}(1,y) = 0, u(0,y) = 1,\ 0 \le y \le 1$ and $\frac{\partial u}{\partial y}(x,1) = 0, u(x,0) = 1,\ 0 \le x \le 1$. The convection velocity is $a = y,\ b = 0$. In that case, the solution is constant in all the domain : $u = 1$ in $[0,1]^2$. Table 5 justify the choice of the preconditioner M2. In fact, in that case the field λ associated to the solution on the interfaces is in the coarse grid space of preconditioner M2.

2. We consider the problem: $L(u) = 0,\ \ 0 \le x \le 1,\ 0 \le y \le 1$ with $\frac{\partial u}{\partial x}(1,y) = u(0,y) = 0,\ 0 \le y \le 1$ and $\frac{\partial u}{\partial y}(x,1) = 0, u(x,0) = 1,\ 0 \le x \le 1$,

TABLE 5. Number of iterations, 8×1 subdomains $a = y$, $b = 0, \nu = 1.d-2$, $CFL = 1.d9$, $h = \frac{1}{129}$, $\log_{10}(Error) < 1.d-6$

	without preconditioner	preconditioner M1	preconditioner M2
OO2	15	17	1

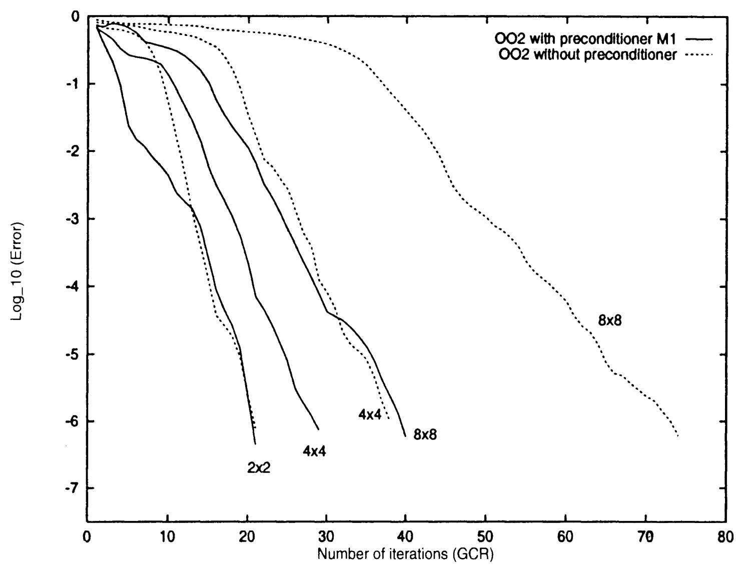

FIGURE 2. Preconditioner M1: Decomposition in $N \times N$ subdomains ($N = 2, 4, 8$); rotating velocity, $\nu = 1.d-2$, $CFL = 1.d9$, $h = \frac{1}{241}$

with a rotating convection velocity: $a = -\sin(\pi(y - \frac{1}{2}))\cos(\pi(x - \frac{1}{2}))$ and $b = \cos(\pi(y - \frac{1}{2}))\sin(\pi(x - \frac{1}{2}))$. Different methods have been developed to solve this problem (see for example [**17**]). Here we want to observe the behavior of the preconditioner on this problem. Figure 3 shows that the convergence of the OO2 method with the preconditioner M2 is nearly independent of the number of subdomains. The convergence is better with preconditioner M2 than preconditioner M1 (figure 2).

4. Conclusion

The OO2 method appears to be a very efficient method, applied to convection-diffusion problems. With the coarse grid preconditioner, the convergence ratio is numerically nearly independent of the number of subdomains.

References

1. C. Borel and M. Bredif, *High performance parallelized implicit Euler solver for the analysis of unsteady aerodynamic flows*, First European CFD Conference, Brussels (1992).
2. C. Carlenzoli and A. Quarteroni, *Adaptive domain decomposition methods for advection-diffusion problems*, Modeling, Mesh Generation, and Adaptive Numerical Methods for Partial Differential Equations (I. Babuska and Al., eds.), The IMA Volumes in Mathematics and its applications, no. 75, Springer-Verlag, 1995, pp. 165–187.

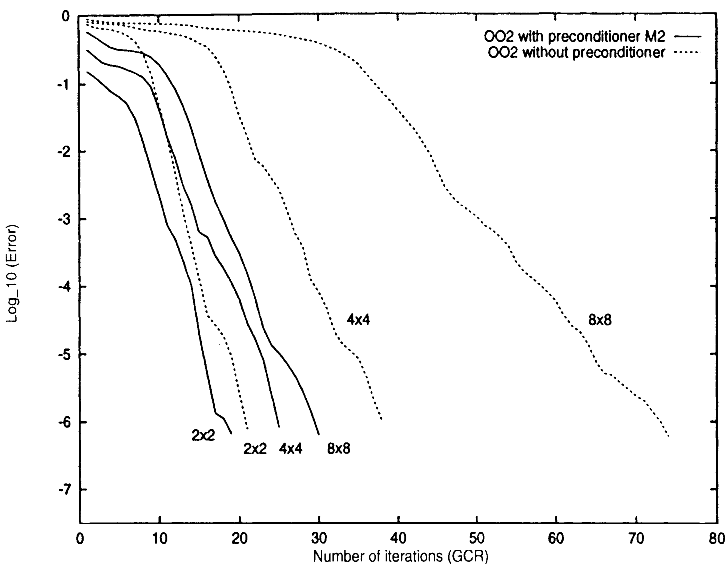

FIGURE 3. Preconditioner M2: Decomposition in $N \times N$ subdomains ($N = 2, 4, 8$); rotating velocity, $\nu = 1.d-2$, $CFL = 1.d9$, $h = \frac{1}{241}$

3. P. Chevalier and F. Nataf, *Une méthode de décomposition de domaine avec des conditions d'interface optimisées d'ordre 2 (OO2) pour l'équation d'Helmholtz*, note CRAS (1997), (To appear).
4. B. Despres, *Décomposition de domaine et problème de Helmholtz*, C.R. Acad. Sci., Paris **311** (1990), 313–316.
5. C. Farhat, P.S. Chen, J. Mandel, and F.X. Roux, *The two-level FETI method - Part II: Extension to shell problems, parallel implementation and performance results*, Computer Methods in Applied Mechanics and Engineering, (in press).
6. C. Farhat and J. Mandel, *The two-level FETI method for static and dynamic plate problems - Part I: An optimal iterative solver for biharmonic systems*, Computer Methods in Applied Mechanics and Engineering, (in press).
7. C. Farhat, J. Mandel, and F.X. Roux, *Optimal convergence properties of the FETI domain decomposition method*, Computer Methods in Applied Mechanics and Engineering **115** (1994), 367–388.
8. L. Halpern, *Artificial boundary conditions for the advection-diffusion equations*, Math. Comp. **174** (1986), 425–438.
9. C. Japhet, *Optimized Krylov-Ventcell method. Application to convection-diffusion problems*, Nine International Conference on Domain Decomposition Methods for Partial Differential Equations (M. Espedal P. Bjorstad and D. Keyes, eds.), John Wiley & Sons Ltd, (in press).
10. ———, *Conditions aux limites artificielles et décomposition de domaine : Méthode OO2 (Optimisé Ordre 2). Application à la résolution de problèmes en mécanique des fluides*, Rapport interne 373, CMAP, Ecole Polytechnique, October 1997.
11. P. L. Lions, *On the Schwarz alternating method III: A variant for nonoverlapping subdomains*, Third International Symposium on Domain Decomposition Methods for Partial Differential Equations, SIAM, 1989, pp. 202–223.
12. F. Nataf and F. Nier, *Convergence rate of some domain decomposition methods for overlapping and nonoverlapping subdomains*, Numerische Mathematik **75** (1997), 357–377.
13. F. Nataf and F. Rogier, *Factorisation of the convection-diffusion operator and the Schwarz algorithm*, M^3AS **5** (1995), no. 1, 67–93.
14. F. Nataf, F. Rogier, and E. de Sturler, *Domain decomposition methods for fluid dynamics*, Navier-Stokes equations on related non linear analysis (A. Sequeira, ed.), Plenum Press Corporation, 1995, pp. 307–377.

15. F.X. Roux and C Farhat, *Parallel implementation of the two-level FETI method*, Nine International Conference on Domain Decomposition Methods for Partial Differential Equations (M. Espedal P. Bjorstad and D. Keyes, eds.), John Wiley & Sons Ltd, (in press).
16. Y. Saad, *Iterative methods for sparse linear systems*, PWS Publishing Compagny, 1996.
17. W.P. Tang, *Numerical solution of a turning point problem*, Fifth International Conference on Domain Decomposition Methods for Partial Differential Equations (T. Chan, D. Keyes, G. Meurant, S. Scroggs, and R. Voigt, eds.), Norfolk, 1991.

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