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Additive Domain Decomposition Algorithms for a Class of Mixed Finite Element Methods

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1. Introduction

We discuss three different domain decomposition methods for saddle point problems with a penalty term. All of them are based on the overlapping additive Schwarz method. In particular, we present results for a mixed formulation from linear elasticity which is well suited for almost incompressible materials. The saddle point problems are discretized by mixed finite elements; this results in the solution of large, indefinite linear systems. These linear systems are solved by appropriate Krylov space methods in combination with domain decomposition preconditioners. First, we discuss an indefinite preconditioner which can be formulated as an overlapping Schwarz method analogous to the methods for symmetric positive definite problems proposed and analyzed by Dryja and Widlund [**3**]. The second approach can be interpreted as an inexact, overlapping additive Schwarz method, i.e. a domain decomposition method with inexact subdomain solves. This preconditioner is symmetric positive definite and can be analyzed as a block–diagonal preconditioner, cf. [**5**]. Our third method is based on a block–triangular formulation, cf. [**4, 7**], it uses an overlapping additive Schwarz method for each of the block solvers. Numerical results indicate that all of our methods are scalable. For brevity, for a list of references to other domain decomposition approaches for saddle point problems, we refer to Klawonn and Pavarino [**6**]. There are several other approaches to solve saddle point problems iteratively, for a list of references we refer to [**5**], [**4**], and [**8**].

The results in this paper have been obtained in joint work with Luca F. Pavarino from the University of Pavia, Italy.

The outline of this paper is as follows. In Section 2, we introduce the saddle point problem and a suitable finite element discretization. In Section 3, we present our preconditioner for saddle point problems with a penalty term. In Section 4, computational results are given.

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2. A mixed formulation of linear elasticity

Let $\Omega \subset \mathbf{R}^d, d = 2,3$ be a polygonal (resp. polyhedral) domain. We consider a linear elastic material and denote by λ and μ the Lamé constants. The linear strain tensor ε is defined by $\varepsilon_{ij} := \frac{1}{2}(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i})$. The material is assumed to be fixed along the part of the boundary $\Gamma_0 \subset \partial\Omega$, to be subject to a surface force f_1 along $\Gamma_1 := \partial\Omega \backslash \Gamma_0$, and to an external force f_0. Other parameters often used in the literature are Young's modulus E and the Poisson ratio ν. They are related to the Lamé constants by $\nu = \frac{\lambda}{2(\lambda+\mu)}$ and $E = \frac{\mu(3\lambda+2\mu)}{\lambda+\mu}$. It is known that the displacement method of linear elasticity in combination with low order conforming finite elements is not suitable for almost incompressible materials. These are materials where the Poisson ratio ν approaches $1/2$, or, in terms of the Lamé constants, where $\lambda \gg \mu$. This failure is called Poisson locking. One approach to avoid the locking effect is based on a mixed formulation, cf. Brezzi and Fortin [**2**] for a more detailed discussion. We consider

$$(1)\qquad \begin{array}{rcrcl} -2\mu \int_\Omega \varepsilon(u) : \varepsilon(v)\, dx & + & \int_\Omega \mathrm{div} v\, p\, dx & = & \int_\Omega f_0\, v\, dx + \int_{\Gamma_1} f_1\, v\, ds \\ \int_\Omega \mathrm{div} u\, q\, dx & - & \frac{1}{\lambda} \int_\Omega p\, q\, dx & = & 0 \end{array}$$

$\forall v \in V, \forall q \in M$, where $V := \{v \in \left(H^1(\Omega)\right)^d \; : \; v = 0 \text{ on } \Gamma_0\}$ and $M := L_2(\Omega)$. Here, u denotes the displacement vector and p the Lagrange multiplier or pressure.

Let us now consider a formal framework for saddle point problems with a penalty term. Let V and M be two Hilbert spaces with inner products $(\cdot,\cdot)_V$, $(\cdot,\cdot)_M$ and denote by $\|\cdot\|_V, \|\cdot\|_M$ the induced norms. Furthermore, let $a(\cdot,\cdot) \; : \; V \times V \to \mathbf{R}$, $b(\cdot,\cdot) \; : \; V \times M \to \mathbf{R}$, and $c(\cdot,\cdot) \; : \; M \times M \to$ be bilinear forms. Then, we consider the abstract problem

Find $(u,p) \in V \times M$, s.t.

$$(2)\qquad \begin{array}{rcrcll} a(u,v) & + & b(v,p) & = & < F, v > & \forall v \in V \\ b(u,q) & - & t^2\, c(p,q) & = & < G, q > & \forall q \in M, \quad t \in [0,1], \end{array}$$

where $F \in V'$ and $G \in M'$.

Problem (2) is well-posed under some assumptions on the bilinear forms. Let $a(\cdot,\cdot)$ be a continuous, symmetric, and $V-$elliptic bilinear form, i.e. $\exists \alpha > 0$, s.t. $a(v,v) \geq \alpha \|v\|_V^2 \;\; \forall v \in V$, let $b(\cdot,\cdot)$ be a continuous bilinear form satisfying an inf–sup condition, i.e. $\exists \beta_0 > 0$, s.t. $\inf_{q \in M} \sup_{v \in V} \frac{b(v,q)}{\|v\|_V \|q\|_M} \geq \beta_0$, and let $c(\cdot,\cdot)$ be a continuous, symmetric and positive semi–definite bilinear form. Under these assumptions, the operator associated with Problem (2) is uniformly bounded with respect to the penalty term t. Note that these are not the most general assumptions for (2) to be uniquely solvable. For a proof and further discussions, see e.g. Braess [**1**]. This result also holds for suitable mixed finite elements, cf. [**1**]. The mixed formulation of linear elasticity clearly satisfies the assumptions made in the abstract formulation. Thus, (1) is a well–posed problem.

We discretize the saddle point problem (1) by a variant of the Taylor–Hood element, i.e. the $P_1(h) - P_1(2h)$ element. This element uses continuous, piecewise linear functions on a triangular mesh τ_h with the typical meshsize h for the displacement u and continuous, linear functions on a triangular mesh τ_{2h} with the meshsize $2h$ for the pressure p. Thus, the finite element spaces $V^h \subset V$ and $M^h \subset M$ are given by $V^h := \{v_h \in (\mathcal{C}(\Omega))^d \cap V \;\; : \;\; v_h \in \mathcal{P}_1 \text{ on } T \in \tau_h\}$ and

$M^h := \{q_h \in \mathcal{C}(\Omega) \cap M \ : \ q_h \in \mathcal{P}_1 \text{ on } T \in \tau_{2h}\}$. This results in a stable finite element method, cf. Brezzi and Fortin [**2**] . Discretizing (2) by $P_1(h) - P_1(2h)$ elements, we obtain a linear system of algebraic equations

$$\mathcal{A}x = \mathcal{F}, \tag{3}$$

where

$$\mathcal{A} := \begin{pmatrix} A & B^t \\ B & -t^2 C \end{pmatrix}, \quad \mathcal{F} := \begin{pmatrix} F \\ G \end{pmatrix}.$$

3. Additive domain decomposition algorithms

We discuss three different additive domain decomposition methods. In order to keep our presentation simple, we consider for the rest of the paper the discrete problem using matrices and vectors instead of operators and functions. For simplicity, we also always have in mind the concrete problem of mixed linear elasticity.

Let τ_H be a coarse finite element triangulation of Ω into N subdomains Ω_i with the characteristic diameter H. By refinement of τ_H, we obtain a fine triangulation τ_h with the typical mesh size h. We denote by H/h the size of a subdomain without overlap. From the given overlapping domain decomposition $\{\Omega_i\}_{i=1}^N$, we construct an overlapping partition of Ω by extension of each Ω_i to a larger subregion $\Omega_i^{'}$ consisting of all elements of τ_h within a distance $\delta > 0$ from Ω_i.

An important ingredient for the construction of our preconditioners are restriction matrices $R_i, i = 1, \dots, N$ which, applied to a vector of the global space, return the degrees of freedom associated with the interior of $\Omega_i^{'}$. For the description of the coarse part, we need an extra restriction matrix R_0^t constructed by interpolation from the degrees of freedom of the coarse to the fine triangulation. With respect to the partition of our saddle point problem into displacement and Lagrange multiplier (or hydrostatic pressure) variables, we can always assume an associated partion of our restriction matrices, i.e. $R_i = (R_{i,u}\, R_{i,p}), i = 0, \dots, N$.

3.1. An exact overlapping additive Schwarz method. Our first method can be formulated as an additive Schwarz method in the general Schwarz framework now well–known for the construction of preconditioners for symmetric positive definite problems, cf. Smith, Bjørstad, and Gropp [**9**]. It has the form

$$\mathcal{B}^{-1} = R_0^t \mathcal{A}_0^{-1} R_0 \ + \ \sum_{i=1}^N R_i^t \mathcal{A}_i^{-1} R_i, \tag{4}$$

where the $\mathcal{A}_i$ are local problems associated with the subdomains and $\mathcal{A}_0$ is the coarse problem stemming from the coarse triangulation, cf. also Klawonn and Pavarino [**6**]. Schwarz methods can also be defined in terms of a space decomposition of $V^h \times M^h$ into a sum of local subspaces and a coarse space

$$V^h \times M^h = V_0^h \times M_0^h \ + \ \sum_{i=1}^N V_i^h \times M_i^h.$$

For the $P_1(h) - P_1(2h)$ finite elements, we define local problems with zero Dirichlet boundary conditions for both displacement and pressure variables on the internal subdomain boundaries $\partial\Omega_i^{'} \setminus \partial\Omega$. Additionally, we impose zero mean value for the pressure on each Ω_i'. Then, we obtain the subspaces

$$\begin{aligned} V_i^h &:= V^h \cap \left(H_0^1(\Omega_i')\right)^d \\ M_i^h &:= \{q_h \in M^h \cap L_0^2(\Omega_i') \,:\, q_h = 0 \text{ on } \Omega \setminus \Omega_i'\}. \end{aligned}$$

Since we use different mesh sizes for the displacement and pressure triangulation, we have a minimal overlap of one pressure node, i.e. $\delta = 2h$. Using the restriction matrices R_i, our local problems $\mathcal{A}_i$ are defined by $\mathcal{A}_i := R_i\,\mathcal{A}\,R_i^t$. The coarse problem $\mathcal{A}_0 := \mathcal{A}_H$ is associated with $V_0^h := V^H, M_0^h := M^H$ and R_0^t is the usual piecewise bilinear interpolation matrix between coarse and fine degrees of freedom.

Note that $\mathcal{B}^{-1}$ is an indefinite preconditioner, and that it is well–defined since $\mathcal{A}_0$ and $\mathcal{A}_i$ are regular matrices by construction. We are currently working on a theoretical analysis of this method.

3.2. A block–diagonal preconditioner. Our second method is given by

$$\mathcal{B}_D^{-1} := R_0^t\,\mathcal{D}_0^{-1}\,R_0 \;+\; \sum_{i=1}^{N} R_i^t\,\mathcal{D}_i^{-1}\,R_i, \tag{5}$$

where

$$\mathcal{D} := \begin{pmatrix} A & O \\ O & M_p \end{pmatrix},$$

and M_p denotes the pressure mass matrix associated with the fine triangulation τ_h. Here, we define our restriction matrices $R_i, i = 1, \ldots, N$, s.t. the local problems are defined with zero Dirichlet boundary conditions for both displacement and pressure variables. In this case, we do not need the local mean value of the pressure to be zero since we do not have to solve local saddle point problems in this preconditioner. Analogous to the first preconditioner, we define the local problems as $\mathcal{D}_i := R_i\,\mathcal{D}\,R_i^t$ and the coarse problem $\mathcal{D}_0 := \mathcal{D}_H$. This approach can be interpreted as an inexact additive Schwarz method, where the exact subdomain solves are replaced by appropriate matrices. It can also be written as a block–diagonal preconditioner

$$\mathcal{B}_D^{-1} = \begin{pmatrix} \hat{A}^{-1} & O \\ O & \hat{M}_p^{-1} \end{pmatrix},$$

with $\hat{A}^{-1} := R_{0,u}^t\,A_0^{-1}\,R_{0,u} \;+\; \sum_{i=1}^N R_{i,u}^t\,A_i^{-1}\,R_{i,u}$ and $\hat{M}_p^{-1} := R_{0,p}^t\,M_{0,p}^{-1}\,R_{0,p} \;+\; \sum_{i=1}^N R_{i,p}^t\,M_{i,p}^{-1}\,R_{i,p}$. Analogous to the first preconditioner, we define the coarse and local problems as $A_0 := A_H, M_{0,p} := M_{p,H}$ and $A_i := R_{i,u}\,A\,R_{i,u}^t, M_{i,p} := R_{i,p}\,M_p\,R_{i,p}^t$. The spaces $V_i^h, i = 1, \ldots, N$, V_0^h, and M_0^h are as in the previous subsection. Only the local spaces for the pressure are now of the form $M_i^h := \{q_h \in M^h \,:\, q_h = 0 \text{ on } \Omega \setminus \Omega_i'\}$. Note that this preconditioner is symmetric positive definite and can be used with the preconditioned conjugate residual method (PCR). It can be analyzed in the framework of block–diagonal preconditioners for saddle point problems with a penalty term, cf. Klawonn [**5**].

3.3. A block–triangular preconditioner. Our third preconditioner is of block–triangular form where the block solvers are constructed by using an overlapping additive Schwarz method. This method cannot be directly formulated in the

Schwarz terminology. The preconditioner has the form

$$\mathcal{B}_T^{-1} := \begin{pmatrix} \hat{A} & O \\ B & -\hat{M}_p \end{pmatrix}^{-1}, \tag{6}$$

where $\hat{A}$ and $\hat{M}_p$ are defined as in Section 3.2. This preconditioner is indefinite and can be analyzed as a block–triangular preconditioner for saddle point problems with a penalty term, cf. Klawonn [**4**] or Klawonn and Starke [**7**].

4. Numerical experiments

We apply our preconditioners to the problem of planar, linear elasticity, cf. Section 2. Without loss of generality, we use $E = 1$ as the value of Young's modulus. As domain, we consider the unit square, i.e. $\Omega = (0,1)^2$ and we use homogeneous Dirichlet boundary conditions for the displacements on the whole boundary. In this case, our problem reduces to

$$\begin{array}{lclclcl} -2\mu \int_\Omega \nabla u \, \nabla v \, dx & + & \int_\Omega \mathrm{div} v \, p \, dx & = & \int_\Omega f_0 \, v \, dx & & \forall v \in \left(H_0^1(\Omega)\right)^2 \\ \int_\Omega \mathrm{div} u \, q \, dx & - & \frac{1}{\lambda+\mu} \int_\Omega p \, q \, dx & = & 0 & & \forall q \in L_{2,0}(\Omega), \end{array} \tag{7}$$

cf. Brezzi and Fortin [**2**], Section VI.1.

We discretize this problem with the $P_1(h) - P_1(2h)$ elements from Section 2. The domain decomposition of Ω is constructed by dividing the unit square into smaller squares with the side length H. All computations were carried out using MATLAB. As Krylov space methods, we consider GMRES in combination with the preconditioners $\mathcal{B}^{-1}$ and $\mathcal{B}_T^{-1}$, and we use the preconditioned conjugate residual method (PCR) with the preconditioner $\mathcal{B}_D^{-1}$. The initial guess is always zero and as a stopping criterion, we use $\|r_k\|_2/\|r_0\|_2 \leq 10^{-6}$, where r_k is the $k-th$ residual of the respective iterative method. In all of our experiments, f is a uniformly distributed random vector and we use the minimal overlap $\delta = 2h$.

To see if our domain decomposition methods are scalable, we carried out some experiments with constant subdomain size $H/h = 8$, appropriately refined mesh size h, and increased number of subdomains $N = 1/H^2$. Our experiments indicate that all three domain decomposition methods give a scaled speedup, cf. Figures 1, 2, i.e. the number of iterations seems to be bounded independently of h and N. In Figure 3, we show a comparison of the iteration numbers of the different preconditioners for the incompressible limit case. Since we are using two different Krylov space methods, in order to have a unified stopping criterion, we ran the experiments presented in Figure 3 using the reduction of the relative error $\|e_k\|_2/\|e_0\|_2 \leq 10^{-6}$ as a stopping criterion. Here, e_k is computed by comparing the $k-$th iterate with the solution obtained by Gaussian elimination. To have a comparison between our domain decomposition methods and the best possible block–diagonal and block–triangular preconditioners (based on inexact solvers for A and M_p), we also include a set of experiments with these preconditioners using exact block–solvers for A and M_p, cf. Figure 3. From these results, we see that our exact additive Schwarz preconditioner has a convergence rate which is comparable to the one of the exact block–triangular preconditioner.

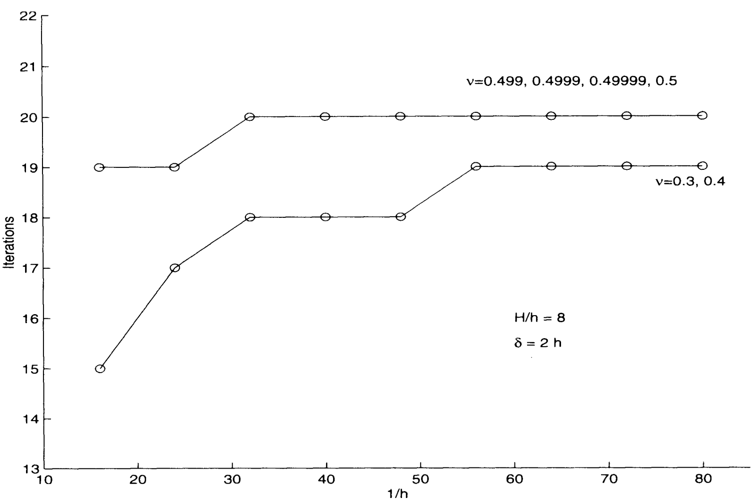

FIGURE 1. Elasticity problem with $P_1(h) - P_1(2h)$ finite elements: iteration counts for GMRES with overlapping additive Schwarz preconditioner $\mathcal{B}^{-1}$

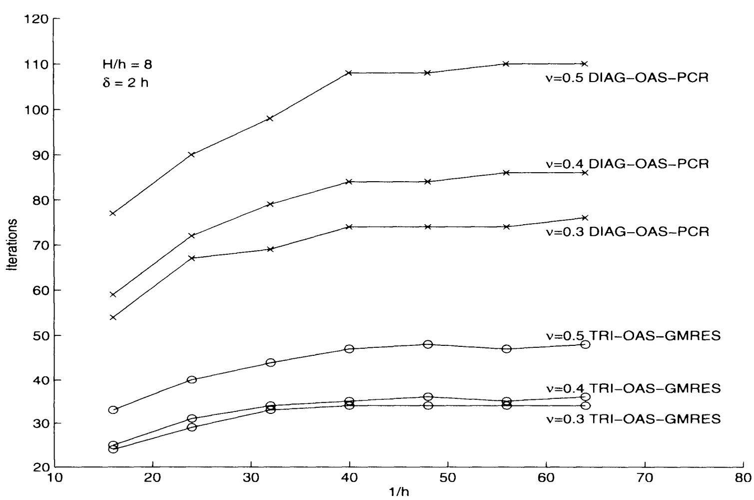

FIGURE 2. Elasticity problem with $P_1(h) - P_1(2h)$ finite elements: iteration counts for PCR with block-diagonal (DIAG-OAS-PCR) and GMRES with block-triangular (TRI-OAS-GMRES) preconditioners using the overlapping additive Schwarz preconditioner as block solver.

5. Conclusions

From our numerical results we see that the overlapping additive Schwarz approach is also a powerful way to construct preconditioners for saddle point problems. There is strong indication that scalability which is known to hold for symmetric positive definite problems also carries over to saddle point problems. Moreover, the exact overlapping additive Schwarz method gives results that are comparable to

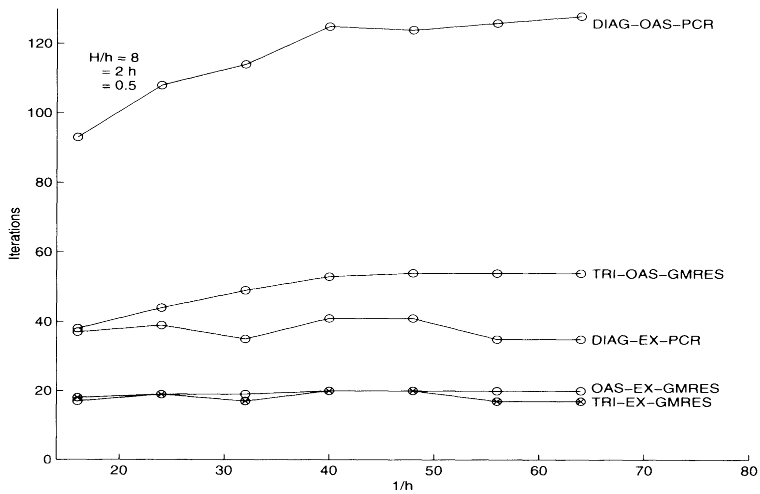

FIGURE 3. Elasticity problem with $P_1(h) - P_1(2h)$ finite elements: iteration counts for different preconditioners; OAS-EX-GMRES=GMRES with overlapping additive Schwarz and exact subdomain solvers, DIAG-OAS-PCR=PCR with block-diagonal preconditioner and overlapping additive Schwarz as block solvers, TRI-OAS-GMRES=GMRES with block-triangular preconditioner and overlapping additive Schwarz as subdomain solvers, DIAG-EX-PCR=PCR with block-diagonal preconditioner and exact block solvers, and TRI-EX-GMRES=GMRES with block-triangular preconditioner and exact block solvers.

those obtained under best of circumstances by the block–triangular preconditioner. The convergence rates of the exact additive Schwarz method are significantly faster than those obtained by the domain decomposition methods based on the block–diagonal and block–triangular approaches. We are currently working with Luca F. Pavarino on a more detailed comparison, taking into account also the complexity of the different preconditioners.

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