Intergrid Transfer Operators for Biharmonic Problems Using Nonconforming Plate Elements on Nonnested Meshes

Zhongci Shi and Zhenghui Xie

1. Introduction

The aim of this paper is to construct a preconditioner for biharmonic problems using nonconforming plate element on nonnested meshes by additive Schwarz methods. The success of the methods depends heavily on the existence of a uniformly, or nearly uniformly, bounded decomposition of a function space in which the problem is defined, and intergrid transfer operators with certain stable approximation properties play an important role in the decomposition [1, 2, 4, 5, 7, 8, 3]. For the case when coarse and fine spaces are all nonconforming, a natural intergrid operator seems to be one defined by taking averages of the nodal parameters. We define an intergrid transfer operator for nonconforming plate elements in this natural way, discuss its stable approximation properties, and obtain the stable factor $(H/h)^{3/2}$. It is also shown that the stable factor cannot be improved. However, to get an optimal preconditioner, we need in general the stability with a factor $C$ independent of mesh parameters $H$ and $h$. Therefore, it cannot be used for that purpose. To obtain an optimal preconditioner for biharmonic problems using nonconforming plate elements on nonnested meshes by additive Schwarz methods, we define an intergrid transfer operator, prove certain stable approximation properties, construct a uniformly bounded decomposition for the finite element space, and then get optimal convergence properties with a not necessarily shape regular subdomain partitioning. Here the fine mesh may not be quasi-uniform.

2. A sharp estimate

Let $\Omega$ be a bounded polygonal domain in $\mathbb{R}^2$ with boundary $\partial \Omega$. We consider the following biharmonic Dirichlet problem:

$$\Delta^2 u = f \text{ in } \Omega, \quad u = \frac{\partial u}{\partial n} = 0 \text{ on } \partial \Omega.$$  

1991 Mathematics Subject Classification. Primary 65F10; Secondary 65N30, 65N55.
Key words and phrases. Intergrid Transfer Operator, Additive Schwarz Method, Biharmonic Equation, Nonconforming Plate Elements, Nonnested Meshes.
This work was supported by Chinese National Key Project of Fundamental Research: Methods and Theories in Large-scale Scientific and Engineering Computing.
The variational form of the problem (1) is: to find \( u \in H_0^4(\Omega) \) such that

\[
a(u, v) = (f, v), \forall v \in H_0^4(\Omega),
\]

where

\[
a(u, v) = \int_{\Omega} \{ \Delta u \Delta v + (1 - \sigma)(2 \partial_{12} u \partial_{12} v - \partial_{11} u \partial_{22} v - \partial_{22} u \partial_{11} v) \} \, dx,
\]

\[
(f, v) = \int_{\Omega} f v \, dx, \quad \sigma \in (0, 0.5) \text{ is the Poisson ratio}.
\]

The unique solvability of the problem (2) for \( f \in L^2(\Omega) \) follows from the continuity and coerciveness of the bilinear form in \( H_0^2(\Omega) \) and Lax-Milgram theorem.

Let \( J_h \) be a triangulation of \( \Omega \), and \( V_h \) a nonconforming plate element space. The corresponding finite element discrete equation for problem (2) is: to find \( u_h \in V_h \) such that

\[
a_h(u_h, v_h) = (f, v_h), \quad \forall v \in V_h,
\]

where

\[
a_h(u_h, v_h) = \sum_{\tau \in J_h} \int_{\tau} \{ \Delta u_h \Delta v_h + (1 - \sigma)(2 \partial_{12} u_h \partial_{12} v_h - \partial_{11} u_h \partial_{22} v_h - \partial_{22} u_h \partial_{11} v_h) \} \, dx.
\]

To obtain an optimal preconditioner of problem (3) by additive Schwarz methods, the intergrid transfer operator with certain stable approximation properties plays an important role. For the case when coarse and fine spaces are all nonconforming, a natural intergrid transfer operator seems to be one defined by taking the average of the values of the nodal parameters. We define an intergrid transfer operator in this natural way. However, it can be shown that the intergrid transfer operator is not suitable for obtaining an optimal preconditioner.

We take Morley element as an example. In this section, let \( J_H \) and \( J_h \) be two quasi-uniform triangulations of \( \Omega \) and \( V_H \) the associated Morley element spaces. We assume that \( J_h \) is a refinement of \( J_H \). Note that \( V_H \not\subseteq V_h \).

The intergrid transfer operator \( I_H^h : V_H \rightarrow V_h \) is defined as follows.

For \( v \in V_H \), \( I_H^h v \in V_h \) is defined so that

\( a ) \) if \( p \) is a vertex of \( J_h \), which is also a vertex of \( J_H \) or in the interior of \( \tau \in J_H \), \( (I_H^h v)(p) = v(p) \); for other vertices \( p \) of \( J_h \), \( v \) may have a jump at \( p \) and \( I_H^h v \) takes the averages of \( v \) at \( p \);

\( b ) \) if \( m \) is a midpoint of an edge of \( J_h \) which is in the interior of \( \tau \in J_H \), \( \frac{\partial(I_H^h v)}{\partial n}(m) = \frac{\partial v}{\partial n}(m) \); for one of other edge midpoints \( m \) associated with \( J_h \), \( \frac{\partial v}{\partial n} \) may have several jumps and \( \frac{\partial(I_H^h v)}{\partial m}(m) \) takes the arithmetic average value of \( \frac{\partial v}{\partial m} \) at \( m \).

About the operator \( I_H^h \) we have the following sharp estimates.

**THEOREM 1.** For \( v \in V_H \), we have

\[
|I_H^h v - v|_{0, h, \Omega} \leq C(H^3 h)^{1/2} |v|_{2, H, \Omega},
\]

\[
|I_H^h v - v|_{1, h, \Omega} \leq C \left( \frac{H^3}{h} \right)^{1/2} |v|_{2, H, \Omega},
\]

and

\[
|I_H^h v|_{2, h, \Omega} \leq C \left( \frac{H}{h} \right)^{3/2} |v|_{2, H, \Omega}.
\]

Furthermore, the estimates (5) and (6) are sharp.
The proof of the theorem can be found in Shi and Xie [6]. For other nonconforming plate elements, we can get similar results.

To get an optimal preconditioner, we need in general the stability with a factor $C$ independent of mesh parameters $H$ and $h$. Therefore, it cannot be used for obtaining an optimal preconditioner.

3. An intergrid transfer operator for nonconforming plate element on nonnested meshes

We now define another intergrid transfer operator, discuss its stable approximation properties, and construct an optimal preconditioner for problem (2) using nonconforming plate elements on nonnested meshes.

3.1. Stable approximation properties. Let $J_{H_c}$ be a quasi-uniform triangulation of $\Omega$. $J_{H_c}$ will be referred to as the coarse grid. Here $H_c$ is the maximum diameter of this coarse triangulation. Let $J_h$ be a triangulation of $\Omega$ that satisfies the minimal angle condition in this section. In general, $J_h$ is not a subdivision of $J_{H_c}$. We assume that each fine triangle intersects with at most $n_0$ coarse triangles, $n_0 \leq C$,

\[
\begin{cases}
    h \leq CH_c, \quad \text{and} \\
    |\tau| \leq C|k|, \quad \text{if } \bar{k} \cap \bar{\tau} \neq \phi, \tau \in J_h, k \in J_{H_c},
\end{cases}
\]

where $| \cdot |$ means the area in $R^2$.

Let $V_{H_c}$ be the Morley element spaces associated with meshes $H_c$, nodal parameters of which vanish on $\partial \Omega$. Note that $V_{H_c} \subset V_h$. For the space $V_{H_c}$, we take $W_{H_c} = AR^H_c(\Omega)$ to be its conforming relative, where $W_{H_c}$ is the $P_3$ Argyris element space \{ $w \in C^1(\Omega) : w|_T \in P_3(T), \forall T \in J_{H_c}, w = \partial_n w = 0$ on $\partial \Omega$ \}. The conforming interpolation operator $E_{H_c} : V_{H_c} \rightarrow W_{H_c}$ is defined as follows (cf. Brenner [1]):

\[
\begin{cases}
    (E_{H_c}v)(p) = v(p); \\
    (D^\alpha E_{H_c}v)(p) = \text{average of } (D^\alpha v_i)(p), |\alpha| = 1; \\
    (D^\alpha E_{H_c}v)(p) = 0, |\alpha| = 2; \\
    \partial_n E_{H_c}v(m) = \partial_n v(m);
\end{cases}
\]

where $p$ is vertex, $m$ is midpoint of sides, $v_i = v|_{T_i}$ and $T_i$ contains $p$ as a vertex. $E_{H_c} : V_{H_c} \rightarrow W_{H_c}$ can be defined similarly. We have [1]

\[
\| v - E_{H_c}v \|_{L^2(T)} + H_c\|v - E_{H_c}v\|_{1,T} + H^2_c\|E_{H_c}v\|_{2,T} \leq CH^2_c\|v\|_{2,T}, \forall v \in V_{H_c},
\]

where $T \in J_{H_c}$, and

\[
\| w - E_{H_c}w \|_{L^2(\tau)} + h_\tau\|w - E_{H_c}w\|_{1,\tau} + h^2_\tau\|E_{H_c}w\|_{2,\tau} \leq CH^2_c\|w\|_{2,T}, \forall w \in V_{H_c},
\]

where $\tau \in J_h, h_\tau$ is the diameter of $\tau$.

Define the nodal interpolation operator $\Pi_{H_c} : C^0(\Omega) \rightarrow V_{H_c}$ as follows:

\[
\begin{cases}
    \Pi_{H_c}v(p) = v(p), \\
    \partial_n \Pi_{H_c}v(m) = \partial_n v(m).
\end{cases}
\]

$\Pi_{H_c} : C^0(\Omega) \rightarrow V_{H_c}$ can be defined similarly. Then, it is easy to prove that

\[
\| v - \Pi_{H_c}v \|_{L^2(T)} + H_c\|v - \Pi_{H_c}v\|_{1,T} + H^2_c\|\Pi_{H_c}v\|_{2,T} \leq CH^2_c\|v\|_{2,T}, \forall v \in H^2(T),
\]

where $T \in J_{H_c}$. 

The intergrid transfer operator $I^h_{H_c}: V_{H_c} \rightarrow V_h$ is defined by $I^h_{H_c} = \Pi_h \cdot E_{H_c}$. For nested meshes, certain stable approximation properties of the intergrid transfer operator were discussed in Brenner [1]. However, for nonnested meshes, it can not be proved in the same way. We have the following theorem, which plays an important role in our analysis.

**Theorem 2.** There exists a constant $C > 0$, independent of $h, H_c$ such that for $u \in V_{H_c}$,

\begin{align}
|I^h_{H_c} u|_{2,H_c} &\leq C |u|_{2,H_c}, \\
|u - I^h_{H_c} u|_{0,\Omega} + H_c |E_{H_c} u - I^h_{H_c} u|_{1,H_c} &\leq C H^2 |u|_{2,H_c},
\end{align}

where $|u|_{i,H_c}^2 = \sum_{\tau \in J_{H_c}} |u|^2_{i,\tau}$.

**Proof.** We first prove (12). Let $\bar{u} = E_{H_c} u$. The essential step is to establish the estimate

\begin{align}
|I^h_{H_c} u|^2_{H^2(k)} &\leq C \sum_{\tau \cap k \neq \phi, \tau \in J_{H_c}} |\bar{u}|^2_{2,\infty,\tau} |k|, \forall u \in V_{H_c}, \text{ here } k \in J_h.
\end{align}

Let $\tau = \Delta p_1 p_2 p_3$, and $m_1, m_2, m_3$ be the midpoint of the edge $p_2 p_3, p_3 p_1$, and $p_1 p_2$ of $\tau$, respectively. If $k$ belongs completely to a single coarse element $\tau, \tau \in J_{H_c}$, then (14) is obviously true. We now prove (14) in the case that $k$ does not belong completely to arbitrary coarse element $\tau, \tau \in J_{H_c}$. We know that

\begin{align}
|I^h_{H_c} u|_{H^2(k)}^2 &\leq C \sum_{i=1}^3 (\partial_n(\bar{u} - \bar{u}_i)(m_i))^2.
\end{align}

where $\bar{u}_i$ means the linear interpolation of $\bar{u}$. Let $\bar{p}_2 m_1$ be the line segment connecting points $p_2$ and $m_1$. We assume that $\bar{p}_2 m_1$ is cut into $l$ pieces by the coarse triangles $\tau_{i}^{m_1}, \ldots, \tau_{l}^{m_1}$, and $u(\cdot)$ is a polynomial on each piece. By the assumption made at the beginning of this section, $l \leq C$. Therefore, by using the triangle inequality, we have

\begin{align}
|\partial_n(\bar{u} - \bar{u}_i)(m_1)|^2 &\leq 2 |\partial_n(\bar{u} - \bar{u}_i)(p_2)|^2 + 2 |\partial_n(\bar{u}(p_2) - \partial_n(\bar{u})(m_1)|^2 \equiv I + II.
\end{align}

Let $g = \bar{u} - \bar{u}_i$, then $g(p_1) = g(p_2) = g(p_3) = 0$, and there exists $\xi_1 \in \bar{p}_1 p_2, \xi_2 \in \bar{p}_2 p_3$ such that

\begin{align}
\partial_{\bar{p}_1 p_2} g(\xi_1) = 0, \quad \partial_{\bar{p}_2 p_3} g(\xi_2) = 0.
\end{align}

Hence using the triangle inequality and the mean value theorem, we obtain

\begin{align}
|\partial_{\bar{p}_1 p_2} g(p_2)|^2 &\leq |\partial_{\bar{p}_1 p_2} g(p_2) - \partial_{\bar{p}_1 p_2} g(\xi_2)|^2 \leq C \sum_{\tau \cap k \neq \phi, \tau \in J_{H_c}} |\bar{u}|^2_{2,\infty,\tau} |k|,
\end{align}

and

\begin{align}
|\partial_{\bar{p}_2 p_3} g(p_2)|^2 &\leq |\partial_{\bar{p}_2 p_3} g(p_2) - \partial_{\bar{p}_2 p_3} g(\xi_1)|^2 \leq C \sum_{\tau \cap k \neq \phi, \tau \in J_{H_c}} |u|^2_{2,\infty,\tau} |k|.
\end{align}

From (16), (18)-(19) we have

\begin{align}
I &\leq C \sum_{\tau \cap k \neq \phi, \tau \in J_{H_c}} |\bar{u}|^2_{2,\infty,\tau} |k|.
\end{align}
Similarly,

\[ II = 2|\partial_n \bar{u}(p_2) - \partial_n \bar{u}(m_1)| \leq C \sum_{m=1}^{3} |\bar{u}|^2_{2,\infty, \tau, \tau \neq m, k} |k| \leq C \sum_{\tau \cap k \neq \emptyset, \tau \in J_H, |k|} |\bar{u}|^2_{2, \infty, \tau} |k|. \]

For \( m_2, m_3 \), we can get the estimates similar to (16), (20) and (21). Therefore, from (16), (20) and (21) and their similar estimates for \( m_2 \) and \( m_3 \), we have

\[ \sum_{i=1}^{3} |\partial_n (\bar{u} - \bar{u}_i)(m_i)|^2 \leq C \sum_{\tau \cap k \neq \emptyset, \tau \in J_H, |k|} |\bar{u}|^2_{2, \infty, \tau} |k|. \]

(14) follows from (15) and (22).

For \( \tau \in J_{H_c} \), we denote by \( \tau_j, j = 1, \ldots, l_1 \), all the coarse triangles which share at least one of the fine triangles that intersects with \( \tau \) (i.e., this fine triangle intersects with both \( \tau \) and \( \tau_j \)). \( l_1 \leq C \).

For all \( k_i \in J_{H_c}, i = 1, \ldots, m' \), whose intersection with \( \tau \) is nonempty, we obtain from (14) that

\[ |I_{H_c}^h u|^2_{2, \infty, \tau} \leq C \sum_{j=1}^{l_1} |\bar{u}|^2_{2, \infty, \tau_j} |k_i|, \forall u \in V_{H_c}. \]

By summing (23) over all \( k_i, i = 1, \ldots, m' \), and from an elementwise inverse estimate we have

\[ \sum_{k \cap \tau \neq \emptyset} |I_{H_c}^h u|^2_{2, \infty, \tau} \leq C \sum_{i=1}^{l_1} \sum_{j=1}^{m'} |\bar{u}|^2_{2, \infty, \tau_j} |k_i| \leq C \sum_{j=1}^{l_1} |u|^2_{2, \infty, \tau_j} \sum_{i=1}^{m'} |k_i| \]

\[ \leq C \sum_{j=1}^{l_1} |\bar{u}|^2_{2, \infty, \tau_j} |\tau| \leq C \sum_{j=1}^{l_1} |\bar{u}|^2_{2, \infty, \tau_j}. \]

Here we used the fact that, for each \( \tau \), the sum of the areas of the fine triangle that intersects with \( \tau \) is less than \( C|\tau| \) because of the assumption \( h \leq C H_c \).

By summing (24) over all \( \tau \in J_{H_c} \), and noting that the number of repetitions for each \( \tau \), in the summation is finite, we have

\[ |I_{H_c}^h u|^2_{2, H, \Omega} \leq \sum_{\tau \in J_{H_c}, k \cap \tau \neq \emptyset} \sum_{\tau} |I_{H_c}^h u|^2_{2, \infty, \tau} \leq C |\bar{u}|^2_{2, \infty, \Omega}. \]

(12) follows from (25) and (9).

We now turn to the proof of (13). Let \( k \in J_H \) be a fine triangle and \( p \) one of its nodes, which implies that \( w(p) = 0, \partial_n w(m) = 0 \). Here \( w = u - I_{H_c}^h u \). We consider the integral

\[ \|w\|^2_{L^2(k)} = \int_k w^2(x) dx \leq 2 \int_k (w(p) - w(x) - Dw(p),(x-p))^2 dx + 2 \int_k |Dw(p),(x-p)|^2 dx \equiv I_1 + I_2. \]

Let \( \overline{xp} \) be the line segment connecting points \( x \) and \( p \). We assume \( \overline{xp} \) is cut into \( l_2 \) pieces by coarse triangles \( \tau^1, \ldots, \tau^k \). Using the triangle inequality and the mean
value theorem, we have

\begin{equation}
I_1 \leq 2 \int_k (w(p) - w(x) - Dw(p). (p - x))^2 \, dx \leq 2l_2 \sum_{m=1}^{l_2} |w|_{2, \infty, \tau_{n_k} \cap k}^2 h_k^4 |k|,
\end{equation}

where $h_k$ is the diameter of element $k$. For the three edge midpoints $m_i (i = 1, 2, 3)$ of $k$, and by using $\partial_n w(m_i) = 0$ and the arguments similar to (17)-(19) we have

\begin{equation}
I_2 \leq 2l_2 \sum_{m=1}^{l_2} |w|_{2, \infty, \tau_{n_k} \cap k}^2 h_k^4 |k|.
\end{equation}

It follows from (26)-(28) and an elementwise inverse estimate that

\begin{equation}
\left\| w \right\|_{L^2(k)}^2 \leq C \sum_{\tau \cap k \neq \emptyset, \tau \in V_{H_n}} |w|_{2, \infty, \tau \cap k}^2 h_k^4 |k| \leq C \sum_{\tau \cap k \neq \emptyset, \tau \in V_{H_n}} (|\bar{u}|_{2, \infty, \tau}^2 + |I_{H_n} u|^2_{2, \infty, k}) h_k^4 |k| \leq C \sum_{\tau \cap k \neq \emptyset, \tau \in V_{H_n}} |\bar{u}|_{2, \infty, \tau}^2 h_k^4 |k| + Cn_0 h^4 |\bar{u}|_{2, k}^2.
\end{equation}

For $\tau \in J_{H_n}$from (29) and by the same notation as (23) we have

\begin{equation}
\left\| w \right\|_{L^2(k)}^2 \leq C \sum_{j=1}^{l_1} |\bar{u}|_{2, \tau_j}^2 h^4 |k_1| + C |\bar{u}|_{2, k_1}^2 h^4.
\end{equation}

By summing (30) over all $k_i, i = 1, \ldots, m$, and using the argument similar to (24), and the fact that $\sum_{k \cap \tau \neq \emptyset} |k| \leq C |\tau|$ we have

\begin{equation}
\sum_{k \cap \tau \neq \emptyset, k \in J_{H_n}} \left\| w \right\|_{L^2(k)}^2 = \sum_{i=1}^{m'} \left\| w \right\|_{L^2(k_i)}^2 \leq C \sum_{i=1}^{m'} \sum_{j=1}^{l_1} |\bar{u}|_{2, \tau_j}^2 h^4 |k_1| + C \sum_{i=1}^{m'} |\bar{u}|_{2, k_1}^2 h^4 \leq C h^4 \sum_{j=1}^{l_1} |\bar{u}|_{2, \tau_j}^2 |k_1| + C h^4 \sum_{i=1}^{l_1} |\bar{u}|_{2, \tau_j}^2 \leq C h^4 \sum_{j=1}^{l_1} |\bar{u}|_{2, \tau_j}^2.
\end{equation}

By summing (31) over $\tau$ in $J_{H_n}$ and noting that the number of repetitions, for each $\tau$, in the summation is finite, we obtain

\begin{equation}
\left\| w \right\|_{L^2(\Omega)}^2 \leq C h^4 |\bar{u}|_{H^2(\Omega)}^2 \leq C h^4 |\bar{u}|_{H^2(\Omega)}^2,
\end{equation}

and by the similar argument we can get

\begin{equation}
|w|_{1, \Omega}^2 \leq |E_{H_n} u - I_{H_n} u|^2_{1, \Omega} \leq C h^2 |E_{H_n} u|^2_{2, \Omega} \leq C h^2 |u|^2_{2, \Omega}.
\end{equation}

Combining (32) and (33) completes the proof of the lemma.

We now partition $\Omega$ into nonoverlapping subdomains $\{\Omega_i\}$ such that no $\partial \Omega_i$ cuts through any elements $\tau, \tau \in J_{H_n}$, and $\Omega = \bigcup_{i=1}^{N} \Omega_i$. Note that we do not assume that $\{\Omega_i\}$ forms a regular finite element subdivision of $\Omega$, nor that the diameters of $\Omega_i$ are of the same order. To obtain an overlapping decomposition of $\Omega$, we extend each $\Omega_i$ to a larger subdomain $\Omega'_i \supset \Omega_i$, which is also assumed not to cut any fine mesh triangles, such that $\text{dist}(\partial \Omega'_i \cap \Omega, \partial \Omega_i \cap \Omega) \geq C \delta$ for a constant $C > 0$. Here $\delta > 0$ will be referred to as the overlapping size. We assume that there exists an integer $N_c$ independent of the mesh parameters $h, H_c$ and $\delta$ such that any point
in $\Omega$ can belong to at most $N_c$ subdomains $\Omega'_i$. For each $\Omega'_i$, $i = 1, \cdots, N$, we define a finite element space

$$V_i = \{v \in V_h; \text{ nodal parameters } = 0 \text{ at } \partial \Omega'_i \text{ and outside } \Omega'_i \}$$

On the basis of Theorem 2 and (9)-(11), we can prove the next theorem, which shows that the decomposition $V_h = V_0 + V_1 + \cdots + V_N$ exists and is uniformly bounded when $\delta = O(H_c)$.

**Theorem 3.** For any $v \in V_h$, there exist $v_0 \in V_{H_c}, v_i \in V_i, i = 1, \cdots, N$, such that

$$v = I^h_{H_c} v_0 + v_1 + \cdots + v_N,$$

and in addition, there exists a constant $C_0 > 0$, independent of the mesh parameters $h, H_c$ and $\delta$ such that

$$a_{H_c}(v_0, v_0) + \sum_{i=1}^N a_h(v_i, v_i) \leq C_0 N_c \left( 1 + \frac{H^2_c}{\delta^2} + \frac{H^4_c}{\delta^4} \right) a_h(v, v), \forall v \in V_h.$$

The proof can be found in Xie [7].

**3.2. An Additive Schwarz Method.** Define $A_h : V_h \to V_h, A_i : V_i \to V_i (1 \leq i \leq N)$, and $A_{H_c} : V_{H_c} \to V_{H_c}$ by

$$(A_h v, w) = a_h(v, w), \forall v, w \in V_h,$$

$$(A_i v, w) = a_h(v, w), \forall v, w \in V_i,$$

$$(A_{H_c} v, w) = a_{H_c}(v, w), \forall v, w \in V_{H_c},$$

respectively. The operator $Q_i : V_h \to V_i, 1 \leq i \leq N$, is defined by

$$(Q_i v, w) = (v, w), \forall v \in V_h, w \in V_i$$

The operator $P_i : V_h \to V_i, 1 \leq i \leq N$, is defined by

$$a_h(P_i v, w) = a_h(v, w), \forall v \in V_h, w \in V_i.$$

The operators $I^h_{H_c}, P^H_{H_c} : V_h \to V_{H_c}$ are defined by

$$(I^h_{H_c} v, w) = (v, I^h_{H_c} w), \forall v \in V_{H_c}, w \in V_h,$$

and

$$a(I^h_{H_c} v, w) = a_h(v, P^H_{H_c} w), \forall v \in V_{H_c}, w \in V_h.$$

respectively.

The two level additive Schwarz preconditioner $B : V_h \to V_h$ is defined by

$$B = I^h_{H_c} A_{H_c}^{-1} I^h_{H_c} + \sum_{i=1}^N A_i^{-1} Q_i.$$

It can be easily seen that the operator $P = BA_h = I^h_{H_c} P^H_{H_c} + \sum_{i=1}^N P_i$ is symmetric positive-definite with respect to $a_h(\cdot, \cdot)$.

On the basis of Theorem 2 and Theorem 3, we obtain the following theorem which shows that $P$ is uniformly bounded from both above and below when $\delta = O(H_c)$. 

THEOREM 4. The following estimate holds:
\[ \lambda_1 a_h(u, u) \leq a_h(Pu, u) \leq \lambda_2 a_h(u, u), \forall u \in V_h, \]

where
\[ \lambda_2/\lambda_1 \leq CN_c \left( 1 + \frac{H_f^2}{\delta^2} + \frac{H_B^2}{\delta^4} \right), \]

which is independent of the diameter of subdomains. This allows us to use subdomains of arbitrary shape.

Acknowledgment. The second author would like to thank Professors Lieheng Wang, Dehao Yu and Jingchao Xu for valuable discussions.

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