http://dx.doi.org/10.1090/conm/218/03011

Contemporary Mathematics
Volume **218**, 1998
B 0-8218-0988-1-03011-9

Physical and Computational Domain Decompositions for Modeling Subsurface Flows

Mary F. Wheeler and Ivan Yotov

1. Introduction

Modeling of multiphase flow in permeable media plays a central role in subsurface environmental remediation as well as in problems associated with production of hydrocarbon energy from existing oil and gas fields. Numerical simulation is essential for risk assessment, cost reduction, and rational and efficient use of resources.

The contamination of groundwater is one of the most serious environmental problems facing the world. For example, more than 50% of drinking water in the Unites States comes from groundwater. More than 10,000 active military installations and over 6,200 closed installations in the United States require subsurface remediation. The process is difficult and extremely expensive and only now is technology emerging to cope with this severe and widespread problem. Hydrocarbons contribute almost two-thirds of the nation's energy supply. Moreover, recoverable reserves are being increased twice as fast by enhanced oil recovery techniques as by exploration.

Features that make the above problems difficult for numerical simulation include: multiple phases and chemical components, multi-scale heterogeneities, stiff gradients, irregular geometries with internal boundaries such as faults and layers, and multi-physics. Because of the uncertainty in the data, one frequently assumes stochastic coefficients and thus is forced to multiple realizations; therefore both computational efficiency and accuracy are crucial in the simulations. For efficiency, the future lies in developing parallel simulators which utilize domain decomposition algorithms.

One may ask what are the important aspects of parallel computation for these complex physical models. First, in all cases, one must be able to partition dynamically the geological domain based upon the physics of the model. Second, efficient distribution of the computations must be performed. Critical issues here are load balancing and minimal communication overhead. It is important to note that the two decompositions may be different.

1991 *Mathematics Subject Classification.* Primary 65N55; Secondary 76S05.

This work was supported in part by the U.S. Department of Energy and the National Science Foundation.

In this paper we will discuss a novel numerical methodology for subsurface modeling based on multiblock domain decomposition formulations. Multiblock discretizations involve the introduction of special approximating spaces (mortars) on interfaces of adjacent subdomains. This paradigm is consistent with a physical/engineering description of the mathematical equations: that is, the equations hold with their usual meaning on the sub-domains, which have physically meaningful interface boundary conditions between them. The following features make the multiblock approach computationally attractive.

In many cases geometrically highly irregular domains can be described as unions of relatively simple blocks. Each block is independently covered by a relatively simple (e.g. logically rectangular) grid. The grids do not have to match on the interfaces between blocks. The local grid structure allows for more efficient and accurate discretization techniques to be employed. (For example, mixed finite element/finite volume methods are more accurate and efficient on structured than unstructured grids). Moreover, structured and unstructured grids could be coupled, if the geometry of a given block is very irregular.

Since the numerical grids may be non-matching across interfaces, they can be constructed to follow large scale geological features such as faults, heterogeneous layers, and other internal boundaries. This is critical for the accuracy of the numerical methods.

The multiblock approach allows for rigorous coupling of different physical processes, mathematical models, or discretization methods in different parts of the simulation domain (e.g., coupling underground with surface flow or coupling mixed finite element with standard finite element methods).

Dynamic grid adaptivity can be performed locally on each block. This is very convenient for the fast reconstruction of grids and calculation of stiffness matrices in time-dependent problems. Mortar degrees of freedom may also vary, providing an additional degree of adaptivity. For complex problems with multiscale heterogeneities and behavior, this approach provides a new mechanism for upscaling by computing an effective flow field without having to compute effective permeabilities. Moreover, the jump in fluxes along interfaces is a good indicator for the magnitude of the local discretization error.

The multiblock structure of the discrete systems of equations allows for efficient parallel domain decomposition solvers and preconditioners, which maximize data and computation locality, to be designed and applied. In addition, weighted space filling curve techniques provide efficient tools for distributing computations among processors. Near optimal load balancing and minimal communication overhead can be achieved, even for unstructured or dynamically adapted grids and computationally rough problems (problems with a nonuniform computational load) [**26**].

Mortar finite elements have been successfully applied for standard finite element and spectral finite element discretizations on non-matching grids (see, e.g. [**9, 8**]).

We have demonstrated in recent work that mortar domain decomposition is a viable approach for modeling subsurface flow and transport. Physical and mathematical considerations lead us to emphasize locally mass conservative schemes, in particular mixed finite element (finite volume) methods for subdomain discretizations. Theoretical and numerical results for single phase flow indicate multiblock mixed finite element methods are highly accurate (superconvergent) for both pressure and velocity [**27, 1, 5, 7, 29**]. A parallel non-overlapping domain decomposition implementation, based on a method originally proposed by Glowinski and

Wheeler [**16, 13, 12**], provides an efficient scalable solution technique [**27**]. Some efficient preconditioners have also been developed [**18**]. An extension of the method to a degenerate parabolic equation arising in two phase flow is presented in [**28**], where optimal convergence is shown.

In this paper we present a nonlinear domain decomposition algorithm for multiphase flow in porous media, based on mortar mixed finite element discretizations. The global discrete nonlinear system of equations is reduced to a nonlinear interface problem in the mortar space. The results demonstrate that this approach works very well for systems of transient highly non-linear differential equations.

The rest of the paper is organized as follows. In the next section we present a multiblock formulation and discretization for a two phase flow model. The domain decomposition algorithm is described in Section 3. Computational results, including some results on mortar adaptivity and upscaling are given in Section 4. We close in Section 5 with remarks on possible extensions and conclusions.

2. Multiblock formulation and discretization

To illustrate the numerical technique, we consider a two-phase flow model. In a multiblock formulation the domain $\Omega \subset \mathbf{R}^3$ is decomposed into a series of subdomains Ω_k, $k = 1, ..., n_b$. Let $\Gamma_{kl} = \partial\Omega_k \cap \partial\Omega_l$ be the interface between Ω_k and Ω_l. We note that Γ_{kl} does not have to coincide with an edge (face) of either subdomain.

The governing mass conservation equations are imposed on each subdomain Ω_k:

$$\frac{\partial(\phi\rho_\alpha S_\alpha)}{\partial t} + \nabla \cdot \mathbf{U}_\alpha = q_\alpha, \tag{1}$$

where $\alpha = w$ (wetting), n (non-wetting) denotes the phase, S_α is the phase saturation, $\rho_\alpha = \rho_\alpha(P_\alpha)$ is the phase density, ϕ is the porosity, q_α is the source term, and

$$\mathbf{U}_\alpha = -\frac{k_\alpha(S_\alpha)K}{\mu_\alpha}\rho_\alpha(\nabla P_\alpha - \rho_\alpha g \nabla D) \tag{2}$$

is the Darcy velocity. Here P_α is the phase pressure, $k_\alpha(S_\alpha)$ is the phase relative permeability, μ_α is the phase viscosity, K is the rock permeability tensor, g is the gravitational constant, and D is the depth. On each interface Γ_{kl} the following physically meaningful continuity conditions are imposed:

$$P_\alpha|_{\Omega_k} = P_\alpha|_{\Omega_l}, \tag{3}$$

$$[\mathbf{U}_\alpha \cdot \nu]_{kl} \equiv \mathbf{U}_\alpha|_{\Omega_k} \cdot \nu_k + \mathbf{U}_\alpha|_{\Omega_l} \cdot \nu_l = 0, \tag{4}$$

where ν_k denotes the outward unit normal vector on $\partial\Omega_k$. The above equations are coupled via the volume balance equation and the capillary pressure relation

$$S_w + S_n = 1, \quad p_c(S_w) = P_n - P_w, \tag{5}$$

which are imposed on each Ω_k and Γ_{kl}. We assume that no flow $\mathbf{U}_\alpha \cdot \nu = 0$ is imposed on $\partial\Omega$, although more general types of boundary conditions can also be treated.

2.1. Discretization spaces. It is important to choose properly the subdomain and interface discretization spaces in order to obtain a stable and accurate scheme. A variant of the mixed method, the expanded mixed method, has been developed for accurate and efficient treatment of irregular domains. The implementation and analysis of the method for single phase flow have been described in several previous works (see [**6, 2, 3**] for single block and [**27, 5, 29**] for multiblock domains). The original problem is transformed into a problem on a union of regular computational (reference) grids. The permeability after the mapping is usually a full tensor (except in some trivial cases). The mixed method could then be approximated by cell-centered finite differences for the pressure, which is an efficient and highly accurate scheme [**6**].

To simplify the presentation we will only describe here the rectangular reference case. For a definition of the spaces on logically rectangular and triangular grids, we refer to [**2**] (also see [**24, 10**]). Let us denote the rectangular partition of Ω_k by $\mathcal{T}_{h_k}$, where h_k is associated with the size of the elements. The lowest order Raviart-Thomas spaces RT_0 [**23**] are defined on $\mathcal{T}_{h_k}$ by

$$\begin{aligned}
\tilde{\mathbf{V}}_{h_k} &= \big\{\mathbf{v} = (v_1, v_2, v_3) : \mathbf{v}|_E = (\alpha_1 x_1 + \beta_1, \alpha_2 x_2 + \beta_2, \alpha_3 x_3 + \beta_3)^T : \\
&\qquad \alpha_l, \beta_l \in \mathbf{R} \text{ for all } E \in \mathcal{T}_{h_k}, \\
&\qquad \text{and each } v_l \text{ is continuous in the } l\text{th coordinate direction}\big\}, \\
\mathbf{V}_{h_k} &= \big\{\mathbf{v} \in \tilde{\mathbf{V}}_{h_k} : \mathbf{v} \cdot \nu_k = 0 \text{ on } \partial\Omega_k \cap \partial\Omega\big\} \\
W_{h_k} &= \big\{w : w|_E = \alpha : \alpha \in \mathbf{R} \text{ for all } E \in \mathcal{T}_{h_k}\big\}.
\end{aligned}$$

To impose the interface matching condition (3)–(4) we introduce a Lagrange multiplier or mortar finite element space $M_{h_{kl}}$ defined on a rectangular grid $\mathcal{T}_{h_{kl}}$ on Γ_{kl}, where h_{kl} is associated with the size of the elements in $\mathcal{T}_{h_{kl}}$. In this space we approximate the interface pressures and saturations, and impose weakly normal continuity of fluxes.

If the subdomain grids adjacent to Γ_{kl} match, we take $\mathcal{T}_{h_{kl}}$ to be the trace of the subdomain grids and define the matching mortar space by

$$M^m_{h_{kl}} = \{\mu : \mu|_e = \alpha : \alpha \in \mathbf{R}, \text{ for all } e \in \mathcal{T}_{h_{kl}}\}.$$

If the grids adjacent to Γ_{kl} are non-matching, the interface grid need not match either of them. Later we impose a mild condition on $\mathcal{T}_{h_{kl}}$ to guarantee solvability of the numerical scheme. We define our non-matching mortar space on an element $e \in \mathcal{T}_{h_{kl}}$ by

$$M^n_h(e) = \big\{\alpha\xi_1\xi_2 + \beta\xi_1 + \gamma\xi_2 + \delta : \alpha, \beta, \gamma, \delta \in \mathbf{R}\big\},$$

where ξ_l are the coordinate variables on e. Then, for each Γ_{kl}, we give two possibilities for the non-matching mortar space, a discontinuous and a continuous version, as

$$\begin{aligned}
M^{n,d}_{h_{kl}} &= \big\{\mu : \mu|_e \in M^n_h(e) \text{ for all } e \in \mathcal{T}_{h_{kl}}\big\}, \\
M^{n,c}_{h_{kl}} &= \big\{\mu : \mu|_e \in M^n_h(e) \text{ for all } e \in \mathcal{T}_{h_{kl}},\ \mu \text{ is continuous on } \Gamma_{kl}\big\}.
\end{aligned}$$

We denote by $M_{h_{kl}}$ any choice of $M^{n,d}_{h_{kl}}$, $M^{n,c}_{h_{kl}}$, or $M^m_{h_{kl}}$ (on matching interfaces).

REMARK 1. The usual piece-wise constant Lagrange multiplier space for RT_0 is not a good choice in the case of non-matching grids, since it only provides $O(1)$ approximation on the interfaces and a suboptimal global convergence. With the

above choice for mortar space, optimal convergence and, in some cases, superconvergence is recovered for both pressure and velocity (see [**27, 1**] for single phase flow and [**28**] for two phase flow).

2.2. The expanded mortar mixed finite element method. Following [**6**], let, for $\alpha = w, n$,

$$\tilde{\mathbf{U}}_\alpha = -\nabla P_\alpha.$$

Then

$$\mathbf{U}_\alpha = -\frac{k_\alpha(S_\alpha)K}{\mu_\alpha}\rho_\alpha(\tilde{\mathbf{U}}_\alpha - \rho_\alpha g \nabla D).$$

Before formulating the method, we note that two of the unknowns can be eliminated using relations (5). Therefore the primary variables can be chosen to be one pressure and one saturation which we denote by P and S.

Let $0 = t_0 < t_1 < t_2 < ...$, let $\Delta t^n = t_n - t_{n-1}$, and let $f^n = f(t_n)$.

In the backward Euler multiblock expanded mixed finite element approximation of (1)-(5) we seek, for $1 \le k < l \le n_b$ and $n = 1, 2, 3...$, $\mathbf{U}^n_{h,\alpha}|_{\Omega_k} \in \mathbf{V}_{h_k}$, $\tilde{\mathbf{U}}^n_{h,\alpha}|_{\Omega_k} \in \tilde{\mathbf{V}}_{h_k}$, $P^n_h|_{\Omega_k} \in W_{h_k}$, $S^n_h|_{\Omega_k} \in W_{h_k}$, $\bar{P}^n_h|_{\Gamma_{kl}} \in M_{h_{kl}}$, and $\bar{S}^n_h|_{\Gamma_{kl}} \in M_{h_{kl}}$ such that, for $\alpha = w$ and n,

$$\int_{\Omega_k} \frac{S^n_{h,\alpha} - S^{n-1}_{h,\alpha}}{\Delta t^n} w\,dx + \int_{\Omega_k} \nabla \cdot \mathbf{U}^n_{h,\alpha}\, w\,dx = \int_{\Omega_k} q_\alpha\, w\,dx,\ w \in W_{h_k}, \tag{6}$$

$$\int_{\Omega_k} \tilde{\mathbf{U}}^n_{h,\alpha} \cdot \mathbf{v}\,dx = \int_{\Omega_k} P^n_{h,\alpha} \nabla \cdot \mathbf{v}\,dx - \int_{\partial\Omega_k \setminus \partial\Omega} \bar{P}^n_{h,\alpha} \mathbf{v} \cdot \nu_k\,d\sigma,\ \mathbf{v} \in \mathbf{V}_{h_k}, \tag{7}$$

$$\int_{\Omega_k} \mathbf{U}^n_{h,\alpha} \cdot \tilde{\mathbf{v}}\,dx = \int_{\Omega_k} \frac{k^n_{h,\alpha} K}{\mu_{h,\alpha}} \rho^n_{h,\alpha}(\tilde{\mathbf{U}}^n_{h,\alpha} - \rho^n_{h,\alpha} g \nabla D) \cdot \tilde{\mathbf{v}}\,dx,\ \tilde{\mathbf{v}} \in \tilde{\mathbf{V}}_{h_k}, \tag{8}$$

$$\int_{\Gamma_{kl}} [\mathbf{U}^n_{h,\alpha} \cdot \nu]_{kl}\, \mu\,d\sigma = 0,\ \mu \in M_{h_{kl}}. \tag{9}$$

Here $k^n_{h,\alpha}$ and $\rho^n_{h,\alpha} \in W_{h_k}$ are given functions of the subdomain primary variables P^n_h and S^n_h. The mortar functions $\bar{P}^n_{h,\alpha}$ can be computed using(5), given the mortar primary variables $\bar{P}^n_h$ and $\bar{S}^n_h$.

REMARK 2. Introducing the pressure gradients $\tilde{\mathbf{U}}_\alpha$ in the expanded mixed method allows for proper handling of the degenerate (for $S_\alpha = 0$) relative permeability $k_\alpha(S_\alpha)$ in (7)–(8). It also allows, even for a full permeability tensor K, to accurately approximate the mixed method on each subdomain by cell-centered finite differences for P_h and S_h. This is achieved by approximating the vector integrals in (7) and (8) by a trapezoidal quadrature rule and eliminating $\tilde{\mathbf{U}}_{h,\alpha}$ and $\mathbf{U}_{h,\alpha}$ from the system [**6, 2, 3**].

REMARK 3. A necessary condition for solvability of the scheme is that, for any $\phi \in M_{h_{kl}}$,

$$Q_{h,k}\phi = Q_{h,l}\phi = 0 \Rightarrow \phi = 0, \tag{10}$$

where $Q_{h,k}$ is the L^2-projection onto $\mathbf{V}_{h_k} \cdot \nu_k$. This is not a very restrictive condition and requires that the mortar grid is not too fine compared to the subdomain grids. One choice that satisfies this condition for both continuous and discontinuous mortars is to take the trace of either subdomain grid and coarsen it by two in each direction (see [**27, 1**] for details).

3. Domain decomposition

To solve the discrete system (6)–(9) on each time step, we reduce it to an interface problem in the mortar space. This approach is based on a domain decomposition algorithm for single phase flow developed originally for conforming grids [**16**], and later generalized to non-matching grids coupled with mortars [**27**].

3.1. Interface formulation. Let

$$M_h = \bigoplus_{1\le k<l\le n_b} M_{h_{kl}}$$

denote the mortar space on $\Gamma = \cup_{1\le k<l\le n_b}\Gamma_{kl}$ and let $\mathbf{M}_h = M_h \times M_h$. We define a non-linear interface functional $B^n : \mathbf{M}_h \times \mathbf{M}_h \to \mathbf{R}$ as follows. For $\psi = (\bar{P}_h^n, \bar{S}_h^n)^T \in \mathbf{M}_h$ and $\mu = (\mu_w, \mu_n) \in \mathbf{M}_h$, let

$$B^n(\psi,\mu) = \sum_{1<k<l<n_b} \int_{\Gamma_{kl}} ([\mathbf{U}_{h,w}^n(\psi)\cdot\nu]_{kl}\,\mu_w + [\mathbf{U}_{h,n}^n(\psi)\cdot\nu]_{kl}\,\mu_n)d\sigma,$$

where $(S_h^n(\psi), \mathbf{U}_{h,\alpha}^n(\psi))$ are solutions to the series of subdomain problems (6)–(8) with boundary data $\bar{P}_{h,\alpha}^n(\psi)$.

Define a non-linear interface operator $\mathcal{B}^n : \mathbf{M}_h \to \mathbf{M}_h$ by

$$\langle \mathcal{B}^n\psi, \mu\rangle = B^n(\psi,\mu), \quad \forall \mu \in \mathbf{M}_h,$$

where $\langle\cdot,\cdot\rangle$ is the L^2-inner product in $\mathbf{M}_h$. It is now easy to see that the solution to (6)–(9) equals $(\psi, S_h^n(\psi), \mathbf{U}_{h,\alpha}^n(\psi))$, where $\psi \in \mathbf{M}_h$ solves

$$\mathcal{B}^n(\psi) = 0. \tag{11}$$

3.2. Iterative solution of the interface problem. We solve the system of nonlinear equations on the interface (11) by an inexact Newton method. Each Newton step s is computed by a forward difference GMRES iteration for solving $\mathcal{B}'(\psi)s = -\mathcal{B}(\psi)$ (we omit superscript n for simplicity). On each GMRES iteration the action of the Jacobian $\mathcal{B}'(\psi)$ on a vector μ is approximated by the forward difference

$$D_\delta\mathcal{B}(\psi:\mu) = \begin{cases} 0, & \mu = 0, \\ ||\mu||\frac{\mathcal{B}(\psi+\delta||\psi||\mu/||\mu||)-\mathcal{B}(\psi)}{\delta||\psi||}, & \mu\neq 0,\ \psi\neq 0, \\ ||\mu||\frac{\mathcal{B}(\delta\mu/||\mu||)-\mathcal{B}(\psi)}{\delta}, & \mu\neq 0,\ \psi = 0. \end{cases}$$

We take $\delta = \sqrt{\epsilon}$, where ϵ is the nonlinear tolerance for evaluation of $\mathcal{B}$. The inexact Newton-GMRES algorithm is described in details in [**17**].

Note that each GMRES iteration only requires one evaluation of the nonlinear operator $\mathcal{B}$. The evaluation of $\mathcal{B}$ involves solving subdomain problems (6)–(8) in parallel and two inexpensive projection steps - from the mortar grid onto the local subdomain grids and from the local grids onto the mortar grid. Since each block can be distributed among a number of processors, the subdomain solvers are parallel themselves. This two level parallelism is needed to account for both the physical and the computational domain decomposition. The subdomain problems are also nonlinear and are solved by a preconditioned Newton-Krylov solver (see [**14**] for a detailed description). We must note that, since the perturbation δ is very small, the subdomain solution with boundary data ψ is a very good initial guess for solving

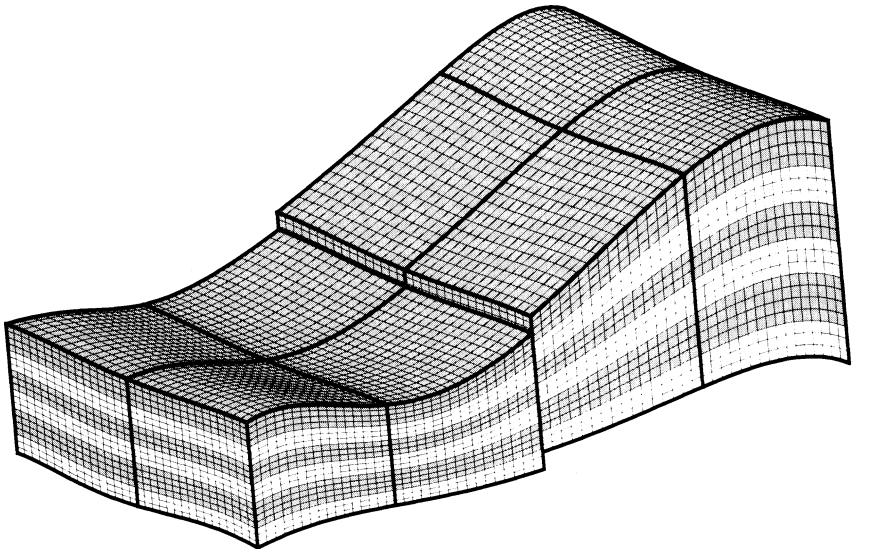


FIGURE 1. Geological layers and numerical grids. The dark layers (400 md) are eight times more permeable than the light layers.

subdomain problems with boundary data $\psi + \delta||\psi||\mu/||\mu||$. As a result it usually takes only one nonlinear subdomain iteration to evaluate $\mathcal{B}(\psi + \delta||\psi||\mu/||\mu||)$.

4. Computational results

In this section we present numerical results illustrating the application of the method described in the previous two sections to modeling two phase subsurface flow. We also give some results on adapting mortar degrees of freedom and its relation to upscaling in the case of single phase flow.

4.1. A two phase flow simulation. The methodology described above has been implemented in a parallel implicit multiblock two phase flow simulator UT-MB [**25, 21**]. The simulator is built on top of an object oriented parallel computational infrastructure [**22**], which is based on DAGH (Distributed Adaptive Grid Hierarchy) library [**20**].

In this example we present the results of a two phase oil-water flow simulation in a faulted heterogeneous irregularly shaped multiblock reservoir. A fault cuts through the middle of the domain and divides it into two blocks. The curvilinear numerical grids follow the geological layers and are non-matching across the fault (see Figure 1). Each block is covered by a $32\times32\times20$ grid. The simulation was done on eight processors on IBM SP2, each block distributed among four processors. Oil concentration contours after 281 days of displacement (water is injected at the right front corner and producer is placed at the left back corner) are given on Figure 2.

4.2. Mortar adaptivity and upscaling. Adapting mortar degrees of freedom may result in substantial reduction of the cost for solving the interface problem. Note that solvability condition (10) does not prevent from using mortar grids much coarser that the subdomain grids. One must expect, however, certain loss of accuracy with coarsening the interface grids. In the following example we study how

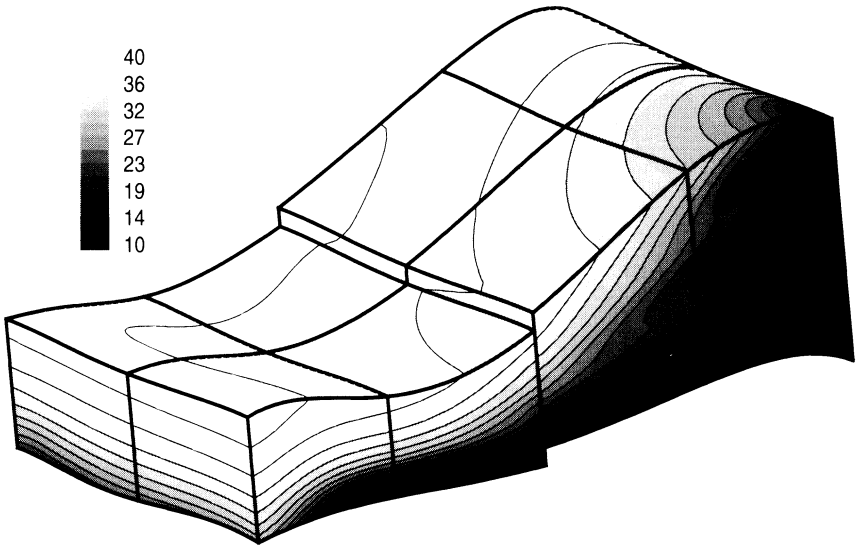


FIGURE 2. Oil concentration contours at 281 days.

reduction of mortar degrees of freedom affects the number of interface iterations and the flux discretization error on the interface. Similar ideas have been explored by Dorr in [**15**]. We solve a single phase flow problem on a $32 \times 32 \times 32$ domain with a highly correlated log-normal permeability field and one injection and three production wells at the corners. A $2 \times 2 \times 2$ domain decomposition is employed. This example suites well the purpose of our study, due to the large heterogeneities and substantial flow through all interfaces. The results of the experiment are shown in Figure 3. The traces of subdomain grids on each interface are 16×16 and having 256 mortar degrees of freedom is equivalent to exact matching of the fluxes. We report the number of conjugate gradient iterations (no preconditioning) and relative flux L^2-error on the interface for several levels of coarsening the mortar grids and for three different types of mortars. We first note that the error for the piecewise constant mortars grows very rapidly and indicates that this is not a good choice. This is consistent with our theoretical results (see Remark 1). The two bilinear cases behave similarly, although the continuous case performs somewhat better. We observe that in this case, the number of mortar degrees of freedom, and consequently the number of interface iterations, can be reduced by a factor of two, with the relative flux error still being under ten percent. Moreover, the global relative error is even smaller, as the solution away from the interfaces is not affected as much.

The reduction of mortar degrees of freedom can be viewed as an upscaling procedure. Standard upscaling techniques compute effective permeabilities on coarse grids. It is usually difficult to estimate the error associated with the upscaling process. Here we compute, in a sense, an effective flow field and the flux jump is a good indication for the numerical error.

If only a single bilinear mortar is used on each interface, we have a two scale problem, where the solution is computed locally on the fine scale and fluxes match on the coarse (subdomain) scale. One can view the solution as a sum of a coarse

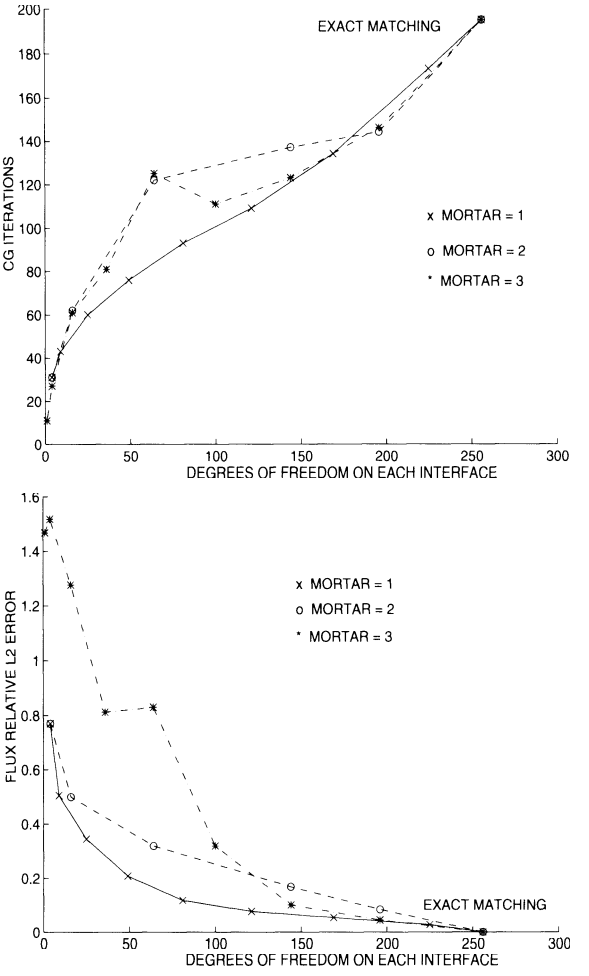
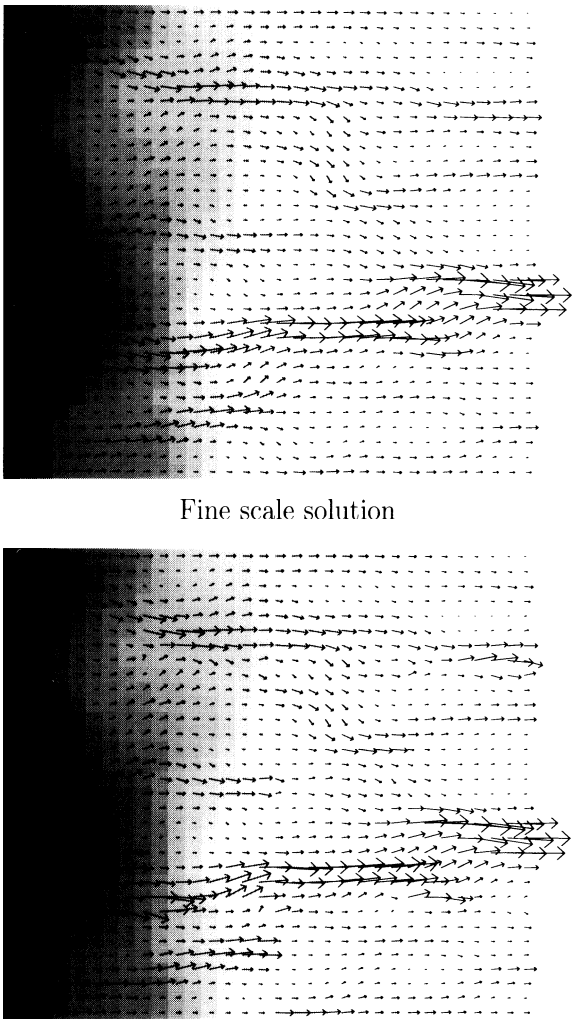


FIGURE 3. Dependence of interface iterations and error on number of interface degrees of freedom; mortar 1–continuous piecewise bilinears, mortar 2–discontinuous piecewise bilinears, mortar 3–piecewise constants.

grid solution and a local fine grid correction, which is similar to the approaches taken in [**4, 19**]. In the following example, also considered in [**4**], we solve the single phase flow equation with a log-normal permeability field originally presented in [**11**]. As can be seen in Figure 4, the solution on a fine 32×32 grid is very similar to the solution obtained by matching fluxes on a coarse 4×4 grid using a single linear mortar on each interface. We should note that a similar procedure using constant instead of linear mortars produced highly inaccurate results.

5. Conclusions

In this paper we considered two levels of domain decompositions - physical and computational. It is important to first decompose the physical problem with appropriate hierarchical models (geometry, geology, chemistry/physics) and then efficiently decompose the computations on a parallel machine.



Fine scale solution

"Upscaled" solution

FIGURE 4. Computed pressure (shade) and velocity (arrows) field for the two scale example.

We have introduced new mortar spaces which provide an accurate and efficient basis for discretizations on non-matching grids, hierarchical domain decomposition, and solvers. In addition, this approach allows the coupling of multiphysics, multinumerics, and multiscales.

We have demonstrated the applicability of these mortar space decompositions to two phase flow in permeable media. Further computational experiments have shown the computational cost can be reduced substantially by interface adaptivity, which is related to upscaling.

Our current research involves extensions of these techniques to three flowing phases and multiple solid phases, as well as coupling of fully implicit and various time-splitting schemes, as shown in Figure 5.

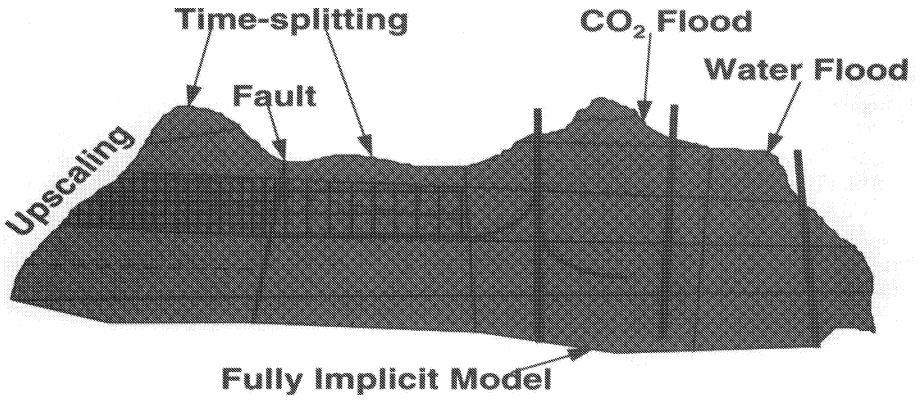


FIGURE 5. Multiphysics, multi-numerical models, complex geology, and upscaling.

References

1. T. Arbogast, L. C. Cowsar, M. F. Wheeler, and I. Yotov, *Mixed finite element methods on non-matching multiblock grids*, Tech. Report TICAM 96-50, Texas Inst. Comp. Appl. Math., University of Texas at Austin, 1996, submitted to SIAM J. Num. Anal.
2. T. Arbogast, C. N. Dawson, P. T. Keenan, M. F. Wheeler, and I. Yotov, *Enhanced cell-centered finite differences for elliptic equations on general geometry*, SIAM J. Sci. Comp. **19** (1998), no. 2, 404–425.
3. T. Arbogast, P. T. Keenan, M. F. Wheeler, and I. Yotov, *Logically rectangular mixed methods for Darcy flow on general geometry*, Thirteenth SPE Symposium on Reservoir Simulation, San Antonio, Texas, Society of Petroleum Engineers, Feb. 1995, SPE 29099, pp. 51–59.
4. T. Arbogast, S. Minkoff, and P. Keenan, *An operator-based approach to upscaling the pressure equation*, in preparation.
5. T. Arbogast, M. F. Wheeler, and I. Yotov, *Logically rectangular mixed methods for flow in irregular, heterogeneous domains*, Computational Methods in Water Resources XI (A. A. Aldama et al., eds.), Computational Mechanics Publications, Southampton, 1996, pp. 621–628.
6. ———, *Mixed finite elements for elliptic problems with tensor coefficients as cell-centered finite differences*, SIAM J. Numer. Anal. **34** (1997), no. 2, 828–852.
7. T. Arbogast and I. Yotov, *A non-mortar mixed finite element method for elliptic problems on non-matching multiblock grids*, Comput. Meth. Appl. Mech. Eng. **149** (1997), 255–265.
8. F. Ben Belgacem and Y. Maday, *The mortar element method for three-dimensional finite elements*, RAIRO Mod. Math. Anal. Num. **31** (1997), no. 2, 289–302.
9. C. Bernardi, Y. Maday, and A. T. Patera, *A new nonconforming approach to domain decomposition: the mortar element method*, Nonlinear partial differential equations and their applications (H. Brezis and J. L. Lions, eds.), Longman Scientific & Technical, UK, 1994.
10. F. Brezzi and M. Fortin, *Mixed and hybrid finite element methods*, Springer-Verlag, New York, 1991.
11. M. A. Christie, M. Mansfield, P. R. King, J. W. Barker, and I. D. Culverwell, *A renormalization-based upscaling technique for WAG floods in heterogeneous reservoirs*, Expanded Abstracts, Society of Petroleum Engineers, 1995, SPE 29127, pp. 353–361.
12. L. C. Cowsar, J. Mandel, and M. F. Wheeler, *Balancing domain decomposition for mixed finite elements*, Math. Comp. **64** (1995), 989–1015.

13. L. C. Cowsar and M. F. Wheeler, *Parallel domain decomposition method for mixed finite elements for elliptic partial differential equations*, Fourth International Symposium on Domain Decomposition Methods for Partial Differential Equations (R. Glowinski, Y. Kuznetsov, G. Meurant, J. Periaux, and O. Widlund, eds.), SIAM, Philadelphia, 1991.
14. C. N. Dawson, H. Klie, C. San Soucie, and M. F. Wheeler, *A parallel, implicit, cell-centered method for two-phase flow with a preconditioned Newton-Krylov solver*, Comp. Geosciences (1998).
15. M. R. Dorr, *On the discretization of interdomain coupling in elliptic boundary value problems*, Second International Symposium on Domain Decomposition Methods (T. F. Chan et al., eds.), SIAM, Philadelphia, 1989, pp. 17–37.
16. R. Glowinski and M. F. Wheeler, *Domain decomposition and mixed finite element methods for elliptic problems*, First International Symposium on Domain Decomposition Methods for Partial Differential Equations (R. Glowinski, G. H. Golub, G. A. Meurant, and J. Periaux, eds.), SIAM, Philadelphia, 1988, pp. 144–172.
17. C. T. Kelley, *Iterative methods for linear and nonlinear equations*, SIAM, Philadelphia, 1995.
18. Y. A. Kuznetsov and M. F. Wheeler, *Optimal order substructuring preconditioners for mixed finite element methods on non-matching grids*, East-West J. Numer. Math. **3** (1995), no. 2, 127–143.
19. N. Moes, J. T. Oden, and K. Vemaganti, *A two-scale strategy and a posteriori error estimation for modeling heterogeneous structures*, On new advances in adaptive computational methods in mechanics, Elsevier, 1998, To appear.
20. M. Parashar and J. C. Browne, *An infrastructure for parallel adaptive mesh-refinement techniques*, Tech. report, Department of Computer Science, University of Texas at Austin, 1995.
21. M. Parashar, J. A. Wheeler, G. Pope, K. Wang, and P. Wang, *A new generation eos compositional reservoir simulator. part II: Framework and multiprocessing*, Fourteenth SPE Symposium on Reservoir Simulation, Dalas, Texas, Society of Petroleum Engineers, June 1997, pp. 31–38.
22. M. Parashar and I. Yotov, *Hybrid parallelization for multi-block multi-physics applications*, in preparation.
23. R. A. Raviart and J. M. Thomas, *A mixed finite element method for 2nd order elliptic problems*, Mathematical Aspects of the Finite Element Method, Lecture Notes in Mathematics, vol. 606, Springer-Verlag, New York, 1977, pp. 292–315.
24. J. M. Thomas, *These de doctorat d'etat*, 'a l'Universite Pierre et Marie Curie, 1977.
25. P. Wang, I. Yotov, M. F. Wheeler, T. Arbogast, C. N. Dawson, M. Parashar, and K. Sepehrnoori, *A new generation eos compositional reservoir simulator. part I: Formulation and discretization*, Fourteenth SPE Symposium on Reservoir Simulation, Dalas, Texas, Society of Petroleum Engineers, June 1997, pp. 55–64.
26. M. F. Wheeler, C. N. Dawson, S. Chippada, H. C. Edwards, and M. L. Martinez, *Surface flow modeling of bays, estuaries and coastal oceans*, Parallel Computing Research **4** (1996), no. 3, 8–9.
27. I. Yotov, *Mixed finite element methods for flow in porous media*, Ph.D. thesis, Rice University. Houston, Texas, 1996, TR96-09, Dept. Comp. Appl. Math., Rice University and TICAM report 96-23, University of Texas at Austin.
28. ———, *A mixed finite element discretization on non-matching multiblock grids for a degenerate parabolic equation arising in porous media flow*, East-West J. Numer. Math. **5** (1997), no. 3, 211–230.
29. ———, *Mortar mixed finite element methods on irregular multiblock domains*, Third IMACS International Symposium on Iterative Methods in Scientific Computation (B. Chen, T. Mathew, and J. Wang, eds.), Academic Press, 1997, To appear.

Texas Institute for Computational and Applied Mathematics, Department of Aerospace Engineering & Engineering Mechanics, and Department of Petroleum and Geosystems Engineering, The University of Texas at Austin, Austin, TX 78712

E-mail address: mfw@ticam.utexas.edu

Texas Institute for Computational and Applied Mathematics , The University of Texas at Austin, Austin, TX 78712

E-mail address: yotov@ticam.utexas.edu