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The Coupling of Mixed and Conforming Finite Element Discretizations

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1. Introduction

In this paper, we introduce and analyze a special mortar finite element method. We restrict ourselves to the case of two disjoint subdomains, and use Raviart-Thomas finite elements in one subdomain and conforming finite elements in the other. In particular, this might be interesting for the coupling of different models and materials. Because of the different role of Dirichlet and Neumann boundary conditions a variational formulation without a Lagrange multiplier can be presented. It can be shown that no matching conditions for the discrete finite element spaces are necessary at the interface. Using static condensation, a coupling of conforming finite elements and enriched nonconforming Crouzeix-Raviart elements satisfying Dirichlet boundary conditions at the interface is obtained. Then the Dirichlet problem is extended to a variational problem on the whole nonconforming ansatz space. In this step a piecewise constant Lagrange multiplier comes into play. By eliminating the local cubic bubble functions, it can be shown that this is equivalent to a standard mortar coupling between conforming and Crouzeix-Raviart finite elements. Here the Lagrange multiplier lives on the side of the Crouzeix-Raviart elements. And in contrast to the standard mortar P1/P1 coupling the discrete ansatz space for the Lagrange multiplier consists of piecewise constant functions instead of continuous piecewise linear functions. We note that the piecewise constant Lagrange multiplier represents an approximation of the Neumann boundary condition at the interface. Finally, we present some numerical results and sketch the ideas of the algorithm. The arising saddle point problems is be solved by multigrid techniques with transforming smoothers.

The mortar methods have been introduced recently and a lot of work in this field has been done during the last few years; cf., e.g., [**1, 4, 5, 14, 15**]. For the construction of efficient iterative solvers we refer to [**2, 3, 20, 21**]. The concepts of a posteriori error estimators and adaptive refinement techniques have also been generalized to mortar methods on nonmatching grids; see e.g. [**13, 22, 25, 24**].

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Originally introduced for the coupling of spectral element methods and finite elements, this method has thus now been extended to a variety of special situations [**6, 7, 11, 12, 26**].

2. The continuous problem

We consider the following elliptic boundary value problem

$$
\begin{aligned}
Lu := -\operatorname{div}(a\nabla u) + b\,u &= f \quad \text{in } \Omega, \\
u &= 0 \quad \text{on } \Gamma := \partial\Omega
\end{aligned}
\tag{1}
$$

where Ω is a bounded, polygonal domain in $\mathbb{R}^2$ and $f \in L^2(\Omega)$. Furthermore, we assume $a = (a_{ij})_{i,j=1}^2$ to be a symmetric, uniformly positive definite matrix-valued function with $a_{ij} \in L^\infty(\Omega)$, $1 \le i, j \le 2$, and $0 \le b \in L^\infty(\Omega)$. The domain $\overline{\Omega}$ is decomposed into two nonoverlapping polyhedral subdomains $\overline{\Omega}_1 \cup \overline{\Omega}_2$, and we assume that $\operatorname{meas}(\partial\Omega_2 \cap \partial\Omega) \neq 0$. On Ω_1 we introduce a mixed formulation of the elliptic boundary value problem (1) with a Dirichlet boundary condition on $\Gamma := \partial\Omega_1 \cap \partial\Omega_2$, whereas on Ω_2 we use the standard variational formulation with a Neumann boundary condition on Γ. We denote by $\mathbf{n}$ the outer unit normal of Ω_1. The Dirichlet boundary condition on Γ will be given by the weak solution u_2 in Ω_2 and the Neumann boundary condition by the flux $\mathbf{j}_1$ in Ω_1. Then, the ansatz space for the solution $(\mathbf{j}_1, u_1)$ in Ω_1 is given by $H(\operatorname{div};\Omega_1) \times L^2(\Omega_1)$ and by $H^1_{0;\Gamma_2}(\Omega_2) := \{v \in H^1(\Omega_2) \mid v|_{\Gamma_2} = 0\}$, where $\Gamma_2 := \partial\Omega \cap \partial\Omega_2$ for the solution u_2 in Ω_2. We recall that no boundary condition on Γ has to be imposed on the ansatz spaces. In contrast to the standard case, Neumann boundary conditions are essential boundary conditions for the mixed formulation, i.e. they have to be enforced in the construction of the ansatz spaces. The coupling of the mixed and standard formulations leads to the following saddle point problem:
Find $(\mathbf{j}_1, u_1, u_2) \in H(\operatorname{div};\Omega_1) \times L^2(\Omega_1) \times H^1_{0;\Gamma_2}(\Omega_2)$ such that

$$
\begin{array}{lllcll}
a_1(\mathbf{j}_1, \mathbf{q}_1) & +\, b(\mathbf{q}_1, u_1) & -\, d(\mathbf{q}_1, u_2) & = & 0, & \mathbf{q}_1 \in H(\operatorname{div};\Omega_1), \\
b(\mathbf{j}_1, v_1) & -\, c(u_1, v_1) & & = & -(f, v_1)_{0;\Omega_1}, & v_1 \in L^2(\Omega_1), \\
-d(\mathbf{j}_1, v_2) & & -\, a_2(u_2, v_2) & = & -(f, v_2)_{0;\Omega_2}, & v_2 \in H^1_{0;\Gamma_2}(\Omega_2).
\end{array}
\tag{2}
$$

Here the bilinear forms $a_i(\cdot,\cdot)$, $1 \le i \le 2$, $b(\cdot,\cdot)$, $c(\cdot,\cdot)$ and $d(\cdot,\cdot)$ are given by

$$
\begin{array}{rcll}
a_2(w_2, v_2) & := & \int_{\Omega_2} (a\nabla v_2\, \nabla w_2 + b\, v_2\, w_2)\, dx, & v_2, w_2 \in H^1_{0;\Gamma_2}(\Omega_2), \\
a_1(\mathbf{p}_1, \mathbf{q}_1) & := & \int_{\Omega_1} a^{-1}\mathbf{p}_1 \cdot \mathbf{q}_1\, dx, & \mathbf{p}_1, \mathbf{q}_1 \in H(\operatorname{div};\Omega_1), \\
b(\mathbf{q}_1, v_1) & := & \int_{\Omega_1} \operatorname{div}\mathbf{q}_1\, v_1\, dx, & v_1 \in L^2(\Omega_1),\, \mathbf{q}_1 \in H(\operatorname{div};\Omega_1), \\
c(w_1, v_1) & := & \int_{\Omega_1} b\, w_1\, v_1\, dx, & v_1, w_1 \in L^2(\Omega_1), \\
d(\mathbf{q}_1, v_2) & := & \langle \mathbf{q}_1\mathbf{n}, v_2 \rangle, & \mathbf{q}_1 \in H(\operatorname{div};\Omega_1), v_2 \in H^1_{0;\Gamma_2}(\Omega_2),
\end{array}
$$

and $\langle\cdot,\cdot\rangle$, stands for the duality pairing of $H^{-1/2}(\Gamma)$ and $H^{1/2}(\Gamma)$. The kernel of the operator $B : H(\operatorname{div};\Omega_1) \times H^1_{0;\Gamma_2}(\Omega_2) \longrightarrow L^2(\Omega_1)$, which is associated with the linear form $b(\cdot, v_1)$ is $\operatorname{Ker}B := \{(\mathbf{q}_1, v_2) \in H(\operatorname{div};\Omega_1) \times H^1_{0;\Gamma_2}(\Omega_2) \mid \operatorname{div}\mathbf{q} = 0\}$. On $H(\operatorname{div};\Omega_1) \times H^1_{0;\Gamma_2}(\Omega_2)$, we introduce the nonsymmetric bilinear form $a(\sigma,\tau) := a_2(w_2, v_2) + d(\mathbf{p}_1, v_2) + a_1(\mathbf{p}_1, \mathbf{q}_1) - d(\mathbf{q}_1, w_2)$ where $\sigma := (\mathbf{q}_1, v_2)$, $\tau := (\mathbf{p}_1, w_2)$, and the norm $\|\cdot\|$ is given by $\|\sigma\|^2 := \|v_2\|^2_{1;\Omega_2} + \|\mathbf{q}_1\|^2_{\operatorname{div};\Omega_1}$. Taking the continuity of the bilinear forms, the Babuška-Brezzi condition, and the coercivity of $a(\cdot,\cdot)$ on $\operatorname{Ker}B$, $a(\sigma,\sigma) \ge \alpha\|\sigma\|^2$, $\sigma \in \operatorname{Ker}B$, into account, we obtain unique solvability of the saddle point problem (2); see e.g. [**19**].

3. Discretization and A Priori Estimates

We restrict ourselves to the case that simplicial triangulations $\mathcal{T}_{h_1}$ and $\mathcal{T}_{h_2}$ are given on both subdomains Ω_1 and Ω_2. However, our results can be easily extended to more general situations including polar grids. The sets of edges of the meshes are denoted by $\mathcal{E}_{h_1}$ and $\mathcal{E}_{h_2}$. We use the Raviart-Thomas space of order k_1, $RT_{k_1}(\Omega_1;\mathcal{T}_{h_1}) \subset H(\text{div};\Omega_1)$, $k_1 \geq 0$, for the approximation of the flux $\mathbf{j}_1$ in Ω_1, the space of piecewise polynomials of order k_1, $W_{k_1}(\Omega_1;\mathcal{T}_{h_1}) := \{v \in L^2(\Omega_1) \mid v|_T \in P_{k_1}(T),\, T \in \mathcal{T}_{h_1}\}$ for the approximation of the primal variable u_1 in Ω_1, and conforming P_{k_2} finite elements $S_{k_2}(\Omega_2;\mathcal{T}_{h_2}) \subset H^1_{0;\Gamma_2}(\Omega_2)$ in Ω_2. Associated with this discretization is the following discrete saddle point problem:
Find $(\mathbf{j}_{h_1}, u_{h_1}, u_{h_2}) \in RT_{k_1}(\Omega_1;\mathcal{T}_{h_1}) \times W_{k_1}(\Omega_1;\mathcal{T}_{h_1}) \times S_{k_2}(\Omega_2;\mathcal{T}_{h_2})$ such that

(3)
$$\begin{array}{llllll} a_1(\mathbf{j}_{h_1},\mathbf{q}_h) & + b(\mathbf{q}_h,u_{h_1}) & - d(\mathbf{q}_h,u_{h_2}) & = & 0, & \mathbf{q}_h \in RT_{k_1}(\Omega_1;\mathcal{T}_{h_1}), \\ b(\mathbf{j}_{h_1},w_h) & - c(u_{h_1},w_h) & & = & -(f,w_h)_{0;\Omega_1}, & w_h \in W_{k_1}(\Omega_1;\mathcal{T}_{h_1}), \\ -d(\mathbf{j}_{h_1},v_h) & & - a_2(u_{h_2},v_h) & = & (f,v_h)_{0;\Omega_2}, & v_h \in S_{k_2}(\Omega_2;\mathcal{T}_{h_2}). \end{array}$$

It can be easily seen that the discrete Babuška-Brezzi condition is satisfied with a constant independent of the refinement level. In addition, the kernel of the discrete operator B_{k_1} is a subspace of Ker$B$. Therefore, an upper bound for the discretization error is given by the best approximation, and we obtain the well known a priori estimate, see e.g. [**19**],

$$\begin{aligned} &\|\mathbf{j}-\mathbf{j}_{h_1}\|^2_{\text{div};\Omega_1} + \|u-u_{h_1}\|^2_{0;\Omega_1} + \|u-u_{h_2}\|^2_{1;\Omega_2} \\ &\leq C\left(h_1^{2(k_1+1)}(\|u\|^2_{k_1+1;\Omega_1} + \|\mathbf{j}\|^2_{k_1+1;\Omega_1} + \|f\|^2_{k_1+1;\Omega_1}) + h_2^{2k_2}\|u\|^2_{k_2+1;\Omega_2}\right) \end{aligned} \tag{4}$$

if the problem has a regular enough solution. In fact, the constant C is independent of the ratio of h_1 and h_2 and there is no matching condition for the triangulations $\mathcal{T}_{h_1}$ and $\mathcal{T}_{h_2}$ at the interface required.

4. An Equivalent Nonconforming Formulation

It is well known that mixed finite element techniques are equivalent to nonconforming ones [**8**]. Introducing interelement Lagrange multipliers, the flux variable as well as the primal variable can be evaluated locally and the resulting Schur complement system is the same as for the positive definite variational problem associated with a nonstandard nonconforming Crouzeix-Raviart discretization [**19**]. In addition, the mixed finite element solution can be obtained by a local postprocessing from these Crouzeix-Raviart finite element solution.

We now restrict ourselves to the lowest order Raviart-Thomas ansatz space ($k_1 = 0$). To obtain the equivalence, we consider the enriched Crouzeix-Raviart space $NC(\Omega_1;\mathcal{T}_{h_1}) := CR(\Omega_1;\mathcal{T}_{h_1}) + B_3(\Omega_1;\mathcal{T}_{h_1})$ where $CR(\Omega_1;\mathcal{T}_{h_1})$ is the Crouzeix-Raviart space of piecewise linear functions which are continuous at the midpoints of the triangulation $\mathcal{T}_{h_1}$ and equal to zero at the midpoints of any boundary edge $e \in \mathcal{E}_{h_1} \cap \partial\Omega$. $B_3(\Omega_1;\mathcal{T}_{h_1})$ is the space of piecewise cubic bubble functions which vanish on the boundary of the elements. Then, we can obtain equivalence between the saddle point problem

$$\begin{array}{lllll} a_1(\mathbf{j}_{h_1},\mathbf{q}_h) & + b(\mathbf{q}_h,u_{h_1}) & = & d(\mathbf{q}_h,u_{h_2}), & \mathbf{q}_h \in RT_0(\Omega_1;\mathcal{T}_{h_1}) \\ b(\mathbf{j}_{h_1},w_h) & - c(u_{h_1},w_h) & = & -(f,w_h)_{0;\Omega_1}, & w_h \in W_0(\Omega_1;\mathcal{T}_{h_1}) \end{array} \tag{5}$$

and the positive definite problem: Find $\Psi_{h_1} \in NC^{u_{h_2}}(\Omega_1;\mathcal{T}_{h_1})$ such that

$$(6) \qquad a_{NC}(\Psi_{h_1},\psi_h) = (f,\Pi_0\psi_h)_{0;\Omega_1}, \quad \psi_h \in NC^0(\Omega_1;\mathcal{T}_{h_1}).$$

Here, $a_{NC}(\phi_h,\psi_h) := \sum_{T\in\mathcal{T}_{h_1}} \int_T P_{a^{-1}}(a\nabla\phi_h)\nabla\psi_h + b\Pi_0\phi_h\,\Pi_0\psi_h\,dx$, and Π_0 stands for the L^2-projection onto $W_0(\Omega_1;\mathcal{T}_{h_1})$. $P_{a^{-1}}$ is the weighted L^2-projection, with weight a^{-1}, onto the three dimensional local Raviart-Thomas space of lowest order, and $NC^g(\Omega_1;\mathcal{T}_{h_1}) := \{\psi_h \in NC(\Omega_1;\mathcal{T}_{h_1}) \mid \int_e \Psi_{h_1}\,d\sigma = \int_e g\,d\sigma,\ e\in\mathcal{E}_{h_1}\cap\Gamma\}$.

Using the equivalence of (5) and (6) in (3), we get:
Find $(\Psi_{h_1},u_{h_2}) \in NC^{u_{h_2}}(\Omega_1;\mathcal{T}_{h_1}) \times S_{k_2}(\Omega_2;\mathcal{T}_{h_2})$ such that

(7)
$$\begin{array}{rcll} a_{NC}(\Psi_{h_1},\psi_h) & = & (f,\Pi_0\psi_h)_{0;\Omega_1}, & \psi_h \in NC^0(\Omega_1;\mathcal{T}_{h_1}),\\ a_2(u_{h_2},v_h) + d(P_{a^{-1}}(a\nabla\Psi_{h_1}),v_h) & = & (f,v_h)_{0;\Omega_2}, & v_h \in S_{k_2}(\Omega_2;\mathcal{T}_{h_2}). \end{array}$$

Note that the ansatz space on Ω_1 depends on the solution in Ω_2.

For the numerical solution, we transfer (7) into a saddle point problem where no boundary condition has to be imposed on the ansatz spaces at the interface. It can be shown that the Dirichlet problem (6) can be extended to a variational problem on the whole space $NC(\Omega_1;\mathcal{T}_{h_1})$. In fact, we obtain

$$(8) \quad a_{NC}(\Psi_{h_1},\psi_h) - d(P_{a^{-1}}(a\nabla\Psi_{h_1}),\psi_h) = (f,\Pi_0\psi_h)_{0;\Omega_1}, \quad \psi_h \in NC(\Omega_1;\mathcal{T}_{h_1}).$$

Let $M(\Gamma;\mathcal{E}_{h_1}) := \{\mu \in L^2(\Gamma) \mid \mu|_e \in P_0(e),\ e\in\mathcal{E}_{h_1}\cap\Gamma\}$ be the space of piecewise constant Lagrange multipliers associated with the 1D triangulation of Γ inherited from $\mathcal{T}_{h_1}$. Then, the condition $\Psi_{h_1} \in NC^{u_{h_2}}(\Omega_1;\mathcal{T}_{h_1})$ is nothing else than $\Psi_{h_1} \in NC(\Omega_1;\mathcal{T}_{h_1})$ and

$$(9) \qquad \int_\Gamma \mu(\Psi_{h_1} - u_{h_2})\,d\sigma = 0, \quad \mu \in M(\Gamma;\mathcal{E}_{h_1}).$$

THEOREM 1. *Let $(\Psi_{h_1},u_{h_2}) \in NC^{u_{h_2}}(\Omega_1;\mathcal{T}_{h_1}) \times S_{k_2}(\Omega_2;\mathcal{T}_{h_2})$ be the solution of (7). Then, $u_M := (\Psi_{h_1},u_{h_2})$ and $\lambda_M := P_{a^{-1}}(a\nabla\Psi_{h_1})|_\Gamma$ is the unique solution of the following saddle point problem: Find $(u_M,\lambda_M) \in (NC(\Omega_1;\mathcal{T}_{h_1}) \times S_{k_2}(\Omega_2;\mathcal{T}_{h_2})) \times M(\Gamma;\mathcal{E}_{h_1})$ such that*

$$(10) \qquad \begin{array}{rcll} a(u_M,v) - \hat{d}(\lambda_M,v) & = & f(v), & v \in NC(\Omega_1;\mathcal{T}_{h_1}) \times S_{k_2}(\Omega_2;\mathcal{T}_{h_2}),\\ \hat{d}(\mu,u_M) & = & 0, & \mu \in M(\Gamma;\mathcal{E}_{h_1}). \end{array}$$

Here the bilinear and linear forms are given by:

$$\begin{array}{ll} a(w,v) := a_2(w,v) + a_{NC}(w,v), & v,\,w \in NC(\Omega_1;\mathcal{T}_{h_1}) \times S_{k_2}(\Omega_2;\mathcal{T}_{h_2}),\\ \hat{d}(\mu,v) := \int_\Gamma \mu(v|_{\Omega_1} - v|_{\Omega_2})\,d\sigma, & \mu \in M(\Gamma;\mathcal{E}_{h_1}),\\ f(v) := (f,v)_{0;\Omega_2} + (f,\Pi_0 v)_{0;\Omega_1}. & \end{array}$$

Taking (8) and (9) into account, the assertion is an easy consequence of (7).

Theorem 1 states the equivalence of (3) and (10) in the case $k_1 = 0$ with $\mathbf{j}_{h_1} = P_{a^{-1}}(a\nabla u_M|_{\Omega_1})$, $u_{h_1} = \Pi_0 u_M|_{\Omega_1}$ and $u_{h_2} = u_M|_{\Omega_2}$. In fact, (10) is a mortar finite element coupling between the conforming and nonconforming ansatz spaces. The Lagrange multiplier $\lambda_M = \mathbf{j}_{h_1}\mathbf{n}|_\Gamma$ is associated with the side of the nonconforming discretization, and it gives an approximation of the Neumann boundary condition on the interface Γ.

REMARK 2. For the numerical solution, we will eliminate locally the cubic bubble functions in (10). In particular, for the special case $b = 0$ and the diffusion coefficient a is piecewise constant, we obtain the standard variational problem for

Crouzeix-Raviart elements where the right hand side f is replaced by $\Pi_0 f$. Then, the nonconforming solution Ψ_{h_1} is given by

$$\Psi_{h_1}|_T = u_{h_1}|_T + \frac{5}{12}\sum_{i=1}^{3} h_{e_i}^2 \, \Pi_0 f|_T(\lambda_1\lambda_2\lambda_3), \quad T \in \mathcal{T}_{h_1}$$

where λ_i, $1 \le i \le 3$ are the barycentric coordinates, and h_{e_i} is the length of the edge $e_i \subset \partial T$, $1 \le i \le 3$. Here, u_{h_1} stands for the Crouzeix-Raviart part of the mortar finite element solution of (10) restricted on $(CR(\Omega_1;\mathcal{T}_{h_1}) \times S_{k_2}(\Omega_2;\mathcal{T}_{h_2})) \times M(\Gamma;\mathcal{E}_{h_1})$.

5. Numerical algorithm

The numerical approximation of (2) is based on the equivalence between mixed and nonconforming finite elements. Thus, we use the variational problem given in Theorem 1, where additional Lagrange multipliers at the interface are required. We recall that the cubic bubble functions in the saddle point problem (10) can be locally eliminated. Thus, we obtain finally a reduced saddle point problem defined on $(CR(\Omega_1;\mathcal{T}_{h_1}) \times S_{k_2}(\Omega_2;\mathcal{T}_{h_2})) \times M(\Gamma;\mathcal{E}_{h_1})$.

The construction of efficient iterative solvers for this type of saddle point problems has often been based on domain decomposition ideas; see e. g. [**2, 3, 21**]. This approach involves a preconditioner for the exact Schur complement. Here, we apply standard multigrid methods with transforming smoothers. The analysis of transforming smoothers for mortar finite elements is similar to the analysis for the Stokes problem given in [**16, 23**]. The technical details for the mortar case will be presented in a forthcoming paper.

In contrast to the Stokes problem, the Schur complement for mortar elements is of smaller dimension. It is associated with the one dimensional interface. Thus the Schur complement of the constructed smoother can be assembled exactly without loosing the optimal complexity of the algorithm. In addition, the condition number of the Schur complement of the smoother is bounded independent of the meshsize. And our numerical results indicate that optimal order convergence also can be obtained with an approximate Schur complement.

We present two numerical examples implemented using the software toolbox *UG* [**9, 10**] and its finite element library. Two different model problems are considered. We consider $-\mathrm{div}\,(a\nabla u) = 1$ on $(0;2) \times (0;1)$ with homogeneous Dirichlet boundary conditions and a discontinuous coefficient a. On subdomain $\Omega_2 := (0;1) \times (0;1)$, a is equal to 1, and on subdomain $\Omega_1 := (1;2) \times (0;1)$, a is equal to 0.001. This example shows the effect of nonmatching grids with different stepsizes and a piecewise constant discontinuous diffusion coefficient. In addition, the triangulation on Ω_2 is slightly distorted. Whereas the second example is a simple model for a rotating geometry with two circles which can occur for time dependent problems. Here, we solve $-\Delta u = \sin(x) + \exp(y)$ with homogeneous Dirichlet boundary conditions on the unit circle, and Ω_1 is the interior circle (see Figure 1).

To apply multigrid algorithms to mortar finite elements, we consider a hierarchy $X_0, X_1, X_2, ..., X_k$ of finite element spaces, where

$$X_l := CR(\Omega_1, \mathcal{T}_{h_1^{(l)}}) \times S_1(\Omega_2, \mathcal{T}_{h_2^{(l)}}) \times M(\Gamma;\mathcal{E}_{h_1^{(l)}})$$

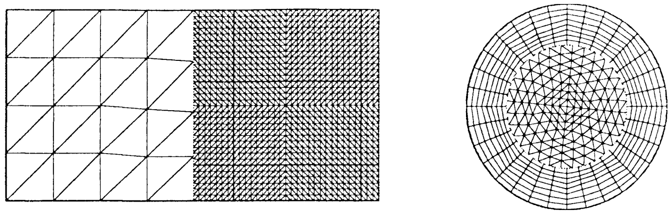

FIGURE 1. Triangulation for example 1 (left) and example 2 (right)

and $h_1^{(l-1)} = 2h_1^{(l)}$, $h_2^{(l-1)} = 2h_2^{(l)}$. Note, that the finite element spaces are nonnested in the first component.

The fine grid problem is of the form

$$K_k z_k = f_k, \qquad K_k = \begin{pmatrix} A_k & B_k \\ B_k & 0 \end{pmatrix}, \tag{11}$$

where the matrix A_k corresponds to the bilinear form a of Theorem 1, and the matrix B_k describes the mortar finite element coupling corresponding to the bilinear form $\hat{d}$. Starting with a vector z_k^0, the multigrid iteration for the solution of (11) consists of preconditioning steps

$$z_k^{i+1} = z_k^i + M_k(d_k^i) \quad \text{with} \quad d_k^i = f_k - K_k z_k^i,$$

where the multigrid corrections $c_k^i = M_k(d_k^i)$ are defined recursively. On the coarser levels, $l = 0, \dots, k-1$, we need the stiffness matrix K_l and the restricted defect d_l. Appropriate grid transfer and smoothing matrices are required:

The prolongation matrix P_l corresponding to the grid transfer $V_{l-1} \longrightarrow V_l$ can be constructed as a block diagonal matrix for the three ansatz spaces. On $S_1(\Omega_2, \mathcal{T}_{h_2})$ we choose piecewise linear interpolation. In case of problem 2, we replace the piecewise linear elements on Ω_2 by piecewise bilinear elements, and use the bilinear interpolation operator. On $CR(\Omega_1, \mathcal{T}_{h_1})$, we use the averaged interpolation introduced by Brenner [**18**], and for $M(\Gamma; \mathcal{E}_{h_1})$ a piecewise constant interpolation. The restriction matrix is the transposed operator $R_l = P_l^T$.

For the smoothing process of the correction vector

$$c_l^{m+1} = c_l^m + \tilde{K}_l^{-1}(d_l - K_l c_l^m), \tag{12}$$

we consider a smoothing matrix of the following form

$$\tilde{K}_l := \begin{pmatrix} \tilde{A}_l & B_l^T \\ B_l & 0 \end{pmatrix} = \begin{pmatrix} \tilde{A}_l & 0 \\ B_l & -\tilde{S}_l \end{pmatrix} \begin{pmatrix} 1 & \tilde{A}_l^{-1} B_l^T \\ 0 & 1 \end{pmatrix},$$

where $\tilde{A}_l := (\mathrm{diag}(A_l) - \mathrm{lower}(A_l))\mathrm{diag}(A_l)^{-1}(\mathrm{diag}(A_l) - \mathrm{upper}(A_l))$ is the symmetric Gauß-Seidel decomposition of A_l. The construction of the approximated Schur complement $\tilde{S}_l$ is motivated by the following Lemma. In case of the Stokes problem, it is given in [**16**], (Lemma 3.2).

LEMMA 3. *If $\tilde{S}_l := B_l\, \tilde{A}_l^{-1}\, B_l^T$, then the iteration (12) satisfies the smoothing property*

$$\|K_l(c_l - c_l^m)\| \leq c \frac{||\tilde{A}_l||}{m}\, ||c_l||$$

where c_l^0 is the start iterate and $c_l := K_l^{-1} d_l$ denotes the exact correction for the given defect d_l on level l.

TABLE 1. Asymptotic convergence rates (average over a defect reduction of 10^{-10})

Example 1			Example 2		
number of elements			number of elements		
Ω_1	Ω_2	conv. rate	Ω_1	Ω_2	conv. rate
2048	32	0.24	1024	1024	0.22
8192	128	0.23	4096	4096	0.22
32768	512	0.19	16384	16384	0.21
131072	2048	0.21	65536	65536	0.19

For H^2-regular domains, a L^2-estimate can be derived and similar to [**17**] the approximation property can be obtained. This proves W-cycle convergence for the multigrid method. Note, that due to the nonconforming part in the discretization the Galerkin property does not hold ($K_{l-1} \neq R_l K_l P_l$) and standard methods for the proof of V-cycle convergence fail.

For our numerical experiments, the Schur complement $B_l \ \tilde{A}_l^{-1} \ B_l^T$ of the smoother $\tilde{K}_l$ is replaced by the damped symmetric Gauß-Seidel decomposition of the approximate Schur complement

$$\hat{S}_l = B_l \ \mathrm{diag}(A_l)^{-1} B_l^T.$$

Now, de can define the multigrid V-cycle $c_l = M_k(d_l)$.

For $l = 0$,
set $c_0 = K_0^{-1} d_0$.
For $l > 0$,
set $c_l^0 = 0$,
define $c_l^1, ..., c_l^{\nu_0}$ by the smoothing iteration (12),
restrict the defect $d_{l-1} = R_l(d_l - K_l c_l^{\nu_0})$,
apply the coarse grid correction $c_l^{\nu_0+1} = c_l^{\nu_0} + P_l M_{l-1}(d_{l-1})$,
and post smoothing steps $c_l^{\nu_0+1}, ..., c_l^{\nu_0+1+\nu_1}$;
set $c_l = c_l^{\nu_0+1+\nu_1}$.

In the computations, we use $\nu_0 = \nu_1 = 2$.

The convergence rates are given in Table 1. In both cases robust results and asymptotic convergence rates independent of the refinement level are obtained. Multigrid V-cycles with transforming smoothers are efficient iterative solvers for saddle point problems arising from mortar finite element discretizations.

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