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# Domain Decomposition Methods for a System of Coupled Acoustic and Elastic Helmholtz Equations

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## Introduction

This paper develops some parallelizable non–overlapping domain decomposition iterative methods for a system of coupled acoustic and elastic Helmholtz equations which describes the interaction of an inviscid fluid and an elastic solid in the frequency domain. Two classes of iterative methods are proposed for decoupling the whole domain problem into fluid and solid subdomain problems. The crux of the methods is to replace the physical interface conditions with equivalent relaxation conditions as the transmission conditions. The utility of these methods is established by showing their strong convergence in the energy norm of the underlying fluid–solid interaction problem. Numerical experiments are provided to validate the analysis and to show the effectiveness of the methods.

The problems of wave propagation in composite media have long been subjects of both theoretical and practical studies. Important applications of such problems are found in inverse scattering, elastoacoustics, geosciences, and oceanography. For some recent developments on modeling, mathematical and numerical analysis, and computational simulations, we refer to [Bou87, DOAG91, FLW97a, FLW97b] and the references therein.

Because of existence of the physical interface, it is natural to use non–overlapping

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domain decompositions method to solve the fluid–solid interaction problem. In fact, non–overlapping domain decomposition methods have been successfully used to solve several coupled boundary value problems from scientific applications. See [QPV92] and the references therein. The non–overlapping domain decomposition methods developed in this paper are based on the idea of using convex combinations of the original physical interface conditions to transmit information between subdomains. See [BF97, Des91, Fen97, Lio90, SBG96] for expositions and discussions on this approach for homogeneous problems. It is more delicate to apply the idea to the heterogeneous fluid–solid interaction problem because using straightforward combinations of the original interface conditions as transmission conditions may lead to divergent iterative procedures. An implementation issue is also addressed in the paper (cf. [BF97]); we show that the difficulty of explicitly computing fluxes on the interface can be avoided through a simple modification.

The organization of this paper is as follows. In Section 2, the system of coupled acoustic Helmholtz equation and elastic Helmholtz equations is introduced as a special form of the fluid–solid interaction model proposed in [FLW97b]. In Section 3, two classes of non–overlapping domain decomposition algorithms are proposed for solving the coupled system. Strong convergence in the energy norm of the underlying fluid–solid interaction problem is established for the iterative methods. Finally, some numerical test results on the methods are presented in Section 4.

## Description of the problem

Wave propagation in a composite medium consisting of a pure (acoustic) fluid part and a pure (elastic) solid part is described by the following system of coupled acoustic and elastic wave equations (cf. [FLW97b])

$$\frac{1}{c^2}P_{tt} - \Delta P = G_f, \qquad \text{in } \Omega_f, \tag{1}$$

$$\rho_s \mathbf{U}_{tt} - \mathrm{div}(\sigma(\mathbf{U})) = \mathbf{G}_s, \qquad \text{in } \Omega_s, \tag{2}$$

$$\frac{\partial P}{\partial n_f} - \rho_f \mathbf{U}_{tt} \cdot n_s = 0, \qquad \text{on } \Gamma, \tag{3}$$

$$\sigma(\mathbf{U}) n_s - \frac{\partial P}{\partial n_f} = 0, \qquad \text{on } \Gamma, \tag{4}$$

$$\frac{1}{c}P_t + \frac{\partial P}{\partial n_f} = 0, \qquad \text{on } \Gamma_f, \tag{5}$$

$$\mathcal{A}_s \mathbf{U}_t + \sigma(\mathbf{U}) n_s = 0, \qquad \text{on } \Gamma_s, \tag{6}$$

$$P(x,0) = P_0(x), \quad P_t(x,0) = P_1(x), \qquad \text{in } \Omega_f, \tag{7}$$

$$\mathbf{U}(x,0) = \mathbf{U}_0(x), \quad \mathbf{U}_t(x,0) = \mathbf{U}_1(x), \qquad \text{in } \Omega_s, \tag{8}$$

where

$$\sigma(\mathbf{U}) = \lambda_s(\mathrm{div}\mathbf{U})I + 2\mu_s \epsilon(\mathbf{U}), \qquad \varepsilon(\mathbf{U}) = \frac{1}{2}[\nabla \mathbf{U} + (\nabla \mathbf{U})^T]. \tag{9}$$

In the system $\Omega_f$ and $\Omega_s$ denote the fluid and solid domains respectively, and $\Omega = \Omega_f \cup \Omega_s \subset \mathbf{R}^N$ $(N = 2, 3)$ denotes the domain of the whole composite medium. $\Gamma_f = \partial\Omega_f \setminus \Gamma$, $\Gamma_s = \partial\Omega_s \setminus \Gamma$ and $\Gamma = \partial\Omega_f \cap \partial\Omega_s$ is the interface between two media. $P$ is the pressure in $\Omega_f$, $\mathbf{U}$ is the displacement vector in $\Omega_s$, and $c$ is the wave speed in

the fluid medium. $\rho_i$ $(i = f, s)$ denotes the density of $\Omega_i$, $n_i$ $(i = f, s)$ denotes the unit outward normal to $\partial\Omega_i$. $\lambda_s > 0$ and $\mu_s \geq 0$ are the Lamé constants of $\Omega_s$. Equation (9) is the constitutive relation for $\Omega_s$. $I$ stands for the $N \times N$ identity matrix. The boundary conditions in (5) and (6) are the first order absorbing boundary conditions for the acoustic and elastic waves respectively. These boundary conditions are transparent to waves arriving normally at the boundary (cf. [EM79]). Finally, equations (3) and (4) are the interface conditions which describe the interaction between the fluid and the solid.

For many application problems, one is asked to find solutions of (1)–(9) of the form $\mathbf{U}(t, x) = \mathbf{u}(x)e^{i\omega t}$, $P(t, x) = p(x)e^{i\omega t}$ (i.e. time harmonic solutions) for some given time harmonic sources $G_f = g_f e^{i\omega t}$ and $\mathbf{G}_s = \mathbf{g}_s e^{i\omega t}$. Substituting $\mathbf{U}(t, x) = \mathbf{u}(x)e^{i\omega t}$ and $P(t, x) = p(x)e^{i\omega t}$ into (1)–(9) yields the following system of equations

$$-\tfrac{\omega^2}{c^2}p - \Delta p = g_f, \qquad \text{in } \Omega_f, \tag{10}$$

$$\tfrac{i\omega}{c}p + \tfrac{\partial p}{\partial n_f} = 0, \qquad \text{on } \Gamma_f, \tag{11}$$

$$-\omega^2\rho_s\mathbf{u} - \operatorname{div}(\sigma(\mathbf{u})) = \mathbf{g}_s, \quad \text{in } \Omega_s, \tag{12}$$

$$i\omega\mathcal{A}_s\mathbf{u} + \sigma(\mathbf{u})n_s = 0, \qquad \text{on } \Gamma_s, \tag{13}$$

$$\tfrac{\partial p}{\partial n_f} + \omega^2\rho_f\mathbf{u}\cdot n_s = 0, \qquad \text{on } \Gamma, \tag{14}$$

$$\sigma(\mathbf{u})n_s - pn_f = 0, \qquad \text{on } \Gamma. \tag{15}$$

Clearly, the above is the system of coupled scalar Helmholtz equation in the fluid subdomain and the (elastic) vector Helmholtz equations in the solid subdomain. The main objective of this paper is to develop some domain decomposition iterative methods for solving the system. Before introducing our domain decomposition methods, we first state the following existence and uniqueness results for the system.

**Theorem 1** *The boundary value problem (10)–(15) has a unique solution* $(p, \mathbf{u}) \in H^1(\Omega_f) \times \mathbf{H}^1(\Omega_s)$ *provided that* $g_f \in L^2(\Omega_f)$, $\mathbf{g}_s \in \mathbf{L}^2(\Omega_s)$.

The proof of Theorem 1 is based the Fredholm Alternative Principle. Notice that the coupled system is elliptic but not coercive. Due to space limitation, we will not give the proof here. We refer readers to ([MIB95]) to see a proof of similar type.

## Non–overlapping domain decomposition methods

Due to the existence of the physical interface and the heterogeneous nature of the problem, it is natural to solve the coupled system using a non–overlapping domain decomposition method. The goal of this section is to present some parallelizable iterative procedures for the problem based on non–overlapping domain decomposition. We show the utility of these iterative algorithms by establishing their convergence in the energy space of the underlying fluid–solid interaction problem.

As in the comparable methods of ([BF97, Des91, Fen97]) , the main idea here is to replace the original physical interface conditions (14)–(15) with the following equivalent Robin type interface conditions

$$\frac{\partial p}{\partial n_f} + \alpha p = -\omega^2 \rho_f \mathbf{u} \cdot n_s - \alpha \sigma(\mathbf{u}) n_s \cdot n_s, \quad \text{on } \Gamma, \tag{16}$$

$$\sigma(\mathbf{u}) n_s \cdot n_s + \beta \mathbf{u} \cdot n_s = -p - \frac{\beta}{\omega^2 \rho_f} \frac{\partial p}{\partial n_f}, \quad \text{on } \Gamma, \tag{17}$$

$$\sigma(\mathbf{u}) n_s \cdot \tau_s = 0, \quad \text{on } \Gamma, \tag{18}$$

for any constant $\alpha, \beta$ such that $\alpha\beta \neq \omega^2 \rho_f$.

Based on the new form of the interface conditions, we propose the following iterative algorithms. Algorithm 1 can be regarded as a Jacobi type algorithm, and Algorithm 2 as a Gauss-Seidel type algorithm.

**Algorithm 1**
Step 1 $\forall\, p^0 \in H^1(\Omega_f)$, $\mathbf{u}^0 \in \mathbf{H}^1(\Omega_s)$.
Step 2 For $k \geq 0$, define $(p^{k+1}, \mathbf{u}^{k+1})$ such that

$$\begin{aligned}
-\tfrac{\omega^2}{c^2} p^{k+1} - \Delta p^{k+1} &= g_f, && \text{in } \Omega_f,\\
\tfrac{i\omega}{c} p^{k+1} + \tfrac{\partial p^{k+1}}{\partial n_f} &= 0, && \text{on } \Gamma_f,\\
\tfrac{\partial p^{k+1}}{\partial n_f} + \alpha p^{k+1} &= -\omega^2 \rho_f \mathbf{u}^k \cdot n_s - \alpha \sigma(\mathbf{u}^k) n_s \cdot n_s, && \text{on } \Gamma;\\
-\omega^2 \rho_s \mathbf{u}^{k+1} - \operatorname{div}(\sigma(\mathbf{u}^{k+1})) &= \mathbf{g}_s, && \text{in } \Omega_s,\\
i\omega \mathcal{A}_s \mathbf{u}^{k+1} + \sigma(\mathbf{u}^{k+1}) n_s &= 0, && \text{on } \Gamma_s,\\
\sigma(\mathbf{u}^{k+1}) n_s \cdot n_s + \beta \mathbf{u}^{k+1} \cdot n_s &= -p^k - \tfrac{\beta}{\omega^2 \rho_f} \tfrac{\partial p^k}{\partial n_f}, && \text{on } \Gamma,\\
\sigma(\mathbf{u}^{k+1}) n_s \cdot \tau_s &= 0, && \text{on } \Gamma.
\end{aligned}$$

**Algorithm 2**
Step 1 $\forall\, \mathbf{u}^0 \in \mathbf{H}^1(\Omega_s)$.
Step 2 For $k \geq 0$, define $(p^k, \mathbf{u}^{k+1})$ such that

$$\begin{aligned}
-\tfrac{\omega^2}{c^2} p^k - \Delta p^k &= g_f, && \text{in } \Omega_f,\\
\tfrac{i\omega}{c} p^k + \tfrac{\partial p^k}{\partial n_f} &= 0, && \text{on } \Gamma_f,\\
\tfrac{\partial p^k}{\partial n_f} + \alpha p^k &= -\omega^2 \rho_f \mathbf{u}^k \cdot n_s - \alpha \sigma(\mathbf{u}^k) n_s \cdot n_s, && \text{on } \Gamma;\\
-\omega^2 \rho_s \mathbf{u}^{k+1} - \operatorname{div}(\sigma(\mathbf{u}^{k+1})) &= \mathbf{g}_s, && \text{in } \Omega_s,\\
i\omega \mathcal{A}_s \mathbf{u}^{k+1} + \sigma(\mathbf{u}^{k+1}) n_s &= 0, && \text{on } \Gamma_s,\\
\sigma(\mathbf{u}^{k+1}) n_s \cdot n_s + \beta \mathbf{u}^{k+1} \cdot n_s &= -p^k - \tfrac{\beta}{\omega^2 \rho_f} \tfrac{\partial p^k}{\partial n_f}, && \text{on } \Gamma,\\
\sigma(\mathbf{u}^{k+1}) n_s \cdot \tau_s &= 0, && \text{on } \Gamma.
\end{aligned}$$

The main result of this section is the following convergence theorem.

**Theorem 2** *Suppose $\alpha = i\omega\hat{\alpha}$, $\beta = i\omega\hat{\beta}$ where $\hat{\alpha} > 0$ and $\hat{\beta} > 0$. Then the sequence $\{(p^k, \mathbf{u}^k)\}$ generated by Algorithm 1 and Algorithm 2 satisfies*

*(1). $p^k \longrightarrow p$ strongly in $H^1(\Omega_f)$,* *(2). $\mathbf{u}^k \longrightarrow \mathbf{u}$ strongly in $\mathbf{H}^1(\Omega_s)$.*

**Proof**: Let $r^n = p - p^n$ and $\mathbf{e}^n = \mathbf{u} - \mathbf{u}^n$. The proof follows the same idea used in ([BF97, Des91, Fen97]). Here the "pseudo–energy" $E_n$ for Algorithm 1 is defined as

$$E_n = \left\| \frac{\partial r^n}{\partial n_f} + \alpha r^n \right\|^2_{L^2(\Gamma)} + \left( \frac{\omega^2 \rho_f}{\beta} \right)^2 \|\sigma(\mathbf{e}^n) n_s \cdot n_s + \beta \mathbf{e}^n \cdot n_s\|^2_{L^2(\Gamma)},$$

and for Algorithm 2 is given by

$$E_n = \left\| \frac{\partial r^{n-1}}{\partial n_f} + \alpha r^{n-1} \right\|^2_{L^2(\Gamma)} + \left( \frac{\omega^2 \rho_f}{\beta} \right)^2 \|\sigma(\mathbf{e}^n) n_s \cdot n_s + \beta \mathbf{e}^n \cdot n_s\|^2_{L^2(\Gamma)}.$$

It is not hard to show that the "pseudo–energy" $E_n$ is decreasing, and it decreases fast enough to guarantee the desired convergence of the iterates $\{(p^n, \mathbf{u}^n)\}$ when the relaxation parameters are chosen appropriately. See [BF97] for the details of a similar argument.

The above algorithms have a drawback with respect to implementation. For instance, solving for $(p^k, \mathbf{u}^{k+1})$ in Step 2 of Algorithm 2 requires the normal derivatives $\sigma(\mathbf{u}^k) n_s \cdot n_s$ and $\frac{\partial p^k}{\partial n_f}$ on the interface. Consequently, one is forced to use non–standard or hybrid finite element methods in order to implement Algorithm 1 and 2. This drawback can be easily avoided through a simple modification. For example, the modification to Algorithm 2 is given as follows.

**Algorithm 3**
Step 1 $\forall h_f^0 \in L^2(\Gamma)$.
Step 2 For $k \geq 0$, define $(p^k, \mathbf{u}^{k+1})$ such that

$$\begin{aligned}
-\tfrac{\omega^2}{c^2} p^k - \Delta p^k &= g_f, && \text{in } \Omega_f, \\
\tfrac{i\omega}{c} p^k + \tfrac{\partial p^k}{\partial n_f} &= 0, && \text{on } \Gamma_f, \\
\tfrac{\partial p^k}{\partial n_f} + \alpha p^k &= h_f^k, && \text{on } \Gamma, \\
h_s^k = (-1 + \tfrac{\alpha\beta}{\omega^2 \rho_f}) p^k - \tfrac{\beta}{\omega^2 \rho_f} h_f^k; & && \text{on } \Gamma; \\
-\omega^2 \rho_s \mathbf{u}^{k+1} - \operatorname{div}(\sigma(\mathbf{u}^{k+1})) &= \mathbf{g}_s, && \text{in } \Omega_s, \\
i\omega \mathcal{A}_s \mathbf{u}^{k+1} + \sigma(\mathbf{u}^{k+1}) n_s &= 0, && \text{on } \Gamma_s, \\
\sigma(\mathbf{u}^{k+1}) n_s \cdot n_s + \beta \mathbf{u}^{k+1} \cdot n_s &= h_s^k, && \text{on } \Gamma, \\
\sigma(\mathbf{u}^{k+1}) n_s \cdot \tau_s &= 0, && \text{on } \Gamma, \\
h_f^{k+1} = (-\omega^2 \rho_f + \alpha\beta) \mathbf{u}^{k+1} \cdot n_s - \alpha h_s^k, & && \text{on } \Gamma.
\end{aligned}$$

The equivalence of Algorithm 2 and 3 can be seen formally from the following identity

$$\begin{aligned}
\sigma(\mathbf{u}^{k+1}) n_s \cdot n_s + \beta \mathbf{u}^{k+1} \cdot n_s = h_s^k &= \left( -1 + \frac{\alpha\beta}{\omega^2 \rho_f} \right) p^k - \frac{\beta}{\omega^2 \rho_f} {h_f}^k \\
= \left( -1 + \frac{\alpha\beta}{\omega^2 \rho_f} \right) p^k - \frac{\beta}{\omega^2 \rho_f} \left( \frac{\partial p^k}{\partial n_f} + \alpha p^k \right) &= -p^k - \frac{\beta}{\omega^2 \rho_f} \frac{\partial p^k}{\partial n_f}.
\end{aligned}$$

The modification to Algorithm 1 can be constructed similarly. Following the proof of Theorem 2, it can be shown that the statement of Theorem 2 also holds for modified algorithms.

## Numerical Experiments

In this section we present the numerical results of two test problems in order to validate the theoretical analysis established in the previous section and to demonstrate the effectiveness of the proposed domain decomposition algorithms. To test the domain decomposition algorithm, we perform two sets of numerical experiments. In the first set, the true solution is known; in the second, the true solution is not known explicitly. In all experiments, $\Omega_f = [0,1] \times [0,1]$, $\Omega_s = [1,2] \times [0,1]$ and the mesh size is approximately 0.1. We use Algorithm 3 to generate all domain decomposition solutions, and the piecewise linear finite element method is employed as the discretization method in each experiment.

For the first set of experiments, we chose the following source functions

$$\mathbf{g}_s \equiv 0.$$
$$g_f = \sin^2(\pi x)\sin^2(\pi y)(4\pi^2 - \omega^2) - 2\pi^2[\sin^2(\pi x)\cos^2(\pi y) + \cos^2(\pi x)\sin^2(\pi y)].$$

Given the above sources, it is easy to show that the true solution is $\mathbf{u}(x,y) = 0$, $p(x,y) = \sin^2(\pi x)\sin^2(\pi y)$

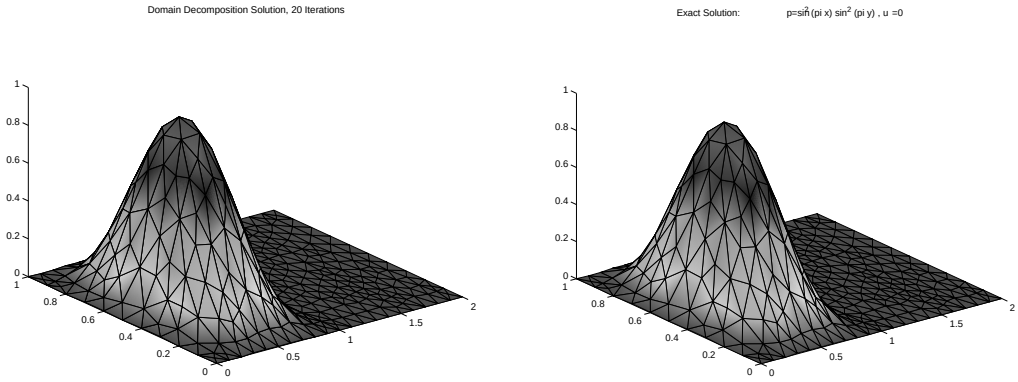


**Figure 1** Exact solution vs. domain decomposition solution

Figure 1 shows the true solution and the domain decomposition solution. Table 1 shows the $L^\infty$–norm and the $L^2$–norm of the error. In table 1, $(p, \mathbf{u})$ denotes the true solution of (10)–(15). The numerical results indicate that after about 5 iterations the domain decomposition solution already has the accuracy of the global finite element solution (see [FLW97a] ).

| n | $\|p^n - p\|_{L^2}$ | $\|p^n - p\|_{L^\infty}$ | $\|\mathbf{u}^n - \mathbf{u}\|_{L^2}$ | $\|\mathbf{u}^n - \mathbf{u}\|_{L^\infty}$ |
|---|---|---|---|---|
| 5 | .00774713 | .00150090 | .00737340 | .00184845 |
| 10 | .00750596 | .00151935 | .00731775 | .00172242 |
| 20 | .00750852 | .00152213 | .00731816 | .00172427 |

**Table 1** Errors of domain decomposition solution

In the second experiment, we choose the source functions

$$g_f(x, y) = x^2 e^y, \qquad \mathbf{g}_s(x, y) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$

In this test, because the true solution of the problem is not known explicitly and the whole domain finite element solution is not easily obtained, we tested the accuracy of our domain decomposition solutions by calculating the relative error of successive iterates. Figure 22 shows the domain decomposition solution after 30 iterations. Note that the graph shows the real part of the first coordinate of the solution $\mathbf{u}$. Table 2 shows the $L^2$–norm and the $L^\infty$–norm of the relative errors of the domain decomposition solutions after 5, 10, 20 and 30 iterations, respectively.

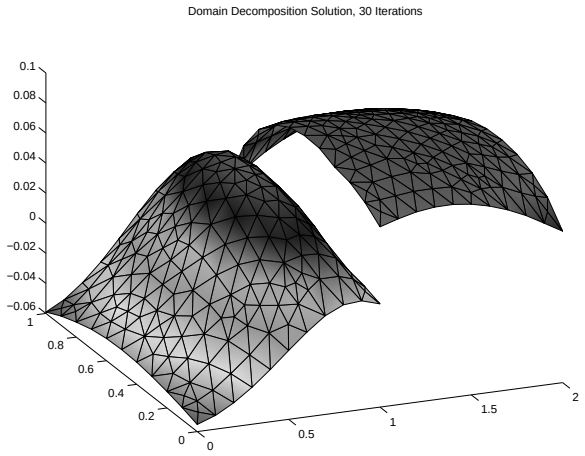


**Figure 2** Domain decomposition solution after 30 interactions

| n | $\frac{\left\|p^n - p^{n+1}\right\|_{L^2}}{\left\|p^{n+1}\right\|_{L^2}}$ | $\frac{\left\|p^n - p^{n+1}\right\|_{L^\infty}}{\left\|p^{n+1}\right\|_{L^\infty}}$ | $\frac{\left\|\mathbf{u}^n - \mathbf{u}^{n+1}\right\|_{L^2}}{\left\|\mathbf{u}^{n+1}\right\|_{L^2}}$ | $\frac{\left\|\mathbf{u}^n - \mathbf{u}^{n+1}\right\|_{L^\infty}}{\left\|\mathbf{u}^{n+1}\right\|_{L^\infty}}$ |
|---|---|---|---|---|
| 5 | .00762755 | .01312341 | .00133775 | .00313325 |
| 10 | 6.825e-05 | 2.292e-05 | 1.697e-05 | 7.146e-05 |
| 20 | 6.103e-08 | 2.175e-07 | 1.736e-08 | 7.198e-08 |
| 30 | 8.856e-11 | 3.640e-10 | 2.442e-11 | 9.257e-11 |

**Table 2** Relative errors of domain decomposition solution

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